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TWO-VARIABLE RELATIVISTIC TENSOR HARMONICS

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Релятивистские тензорные гармоники
в случае двух переменных

Рассмотрено обобщение релятивистских сферических тензоров на случай двух переменных. В сферическом и спиральном базисах в системе отсчета Брейта построены лоренцевские коварианты, зависящие от трех переменных. Изучены их свойства: четность, ортонормальность на сфере, вложенной в трехмерное Евклидово пространство. Получено выражение для скалярного произведения в четырехмерном пространстве и условие независимости релятивистских тензорных гармоник. Переход к нерелятивистским ковариантам является очевидным. Построенные релятивистские тензоры могут быть полезны при изучении реакций с числом частиц в конечном состоянии больше двух.

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Two-Variable Relativistic Tensor Harmonics

Three bases in the Hilbert space of tensor fields on the unit spheres associated with two independent vectors are discussed: the tensor spherical harmonics and the symmetric and unsymmetric tensor helicity harmonics. Under the conditions which we specify they form complete sets of independent Lorentz covariants which may serve the purpose of the analysis of reactions with several particles in the final state.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

Recently, a new interest in the intermediate-energy ($\approx 10^2$ MeV) lepton- and meson-induced processes on nucleons and nuclei has arisen in connection with the increasing flow of the high-quality data coming from the SIN, TRIUMF and LAMPF meson facilities. The Lorentz covariants written in the Cartesian basis have been the traditional theoretical tool to analyse such reactions. Though manifestly covariant, they frequently do not fit the purposes of the physical investigations, e.g.,already the basic problem of selecting the independent covariants may prove to be practically insoluble in many cases.

It is our experience that the tensor harmonics in the spherical and helicity bases constitute a convenient, highly flexible framework for the construction of the Lorentz covariants: The most important mathematical properties of the tensor harmonics follow directly from the well-known formulas of the angular-momentum algebra, the construction of the independent sets of covariants is straightforward. Orthogonality properties of the tensor bases make the calculations of rates and other physical quantities much easier than with the cumbersome Cartesian techniques. Besides these technical advantages two gratifying properties of the new formalism constitute its main merit and should be mentioned. First, the relativistic tensor harmonics allow a natural unification of the theoretical treatment of a big class of different physical processes. Second, the formalism, though fully equivalent to the covariant Cartesian expressions is actually very much similar to the familiar nonrelativistic multipole-expansion formulas. Therefore, the physical results may always be easily interpreted by a direct extrapolation to the domain of classical nuclear physics. Using the tensor harmonics we need not to perform any "nonrelativistic reduction" which is normally done, e.g., via the Foldy-Wouthuysen transformation. Namely this is the step which frequently makes the treatment of physical processes unwieldy and brings in the approximations which are usually difficult to control. The trick here is indeed in choosing the appropriate reference frame. It is the Breit frame which being fully appropriate physically, gives simultaneously an enormous simplification of the formulas.

The formalism of the relativistic s-th order tensor spherical harmonics has been presented recently by Daumens and Minnaert /1/. For the corresponding analysis performed in the helicity basis we refer the reader to the paper by Akyeampong /2/. As a matter of fact the method was first introduced by Stech and Schülke /3/ who have considered, however, only the specific case of nuclear beta-decay. Recently, Delorme /4/ presented the application of the relativistic spherical tensor harmonics in the context of the so-called elementary-particle theory of nuclear currents. The treatment in Refs. /1-4/ is always limited to the one-variable harmonics which correspond to the case of binary reactions.

Here we shall present our results concerning the two-variable Lorentz-covariant tensor harmonics in the spherical and helicity bases. It will be shown below that they provide actually the most general description of the multi-variable tensor fields, which may be needed in the analysis of any reaction of the type $a+b \longrightarrow 1+2+\cdots n$.

In Sec. 2 we define the spherical tetrads and build up the second order tensor spherical basis. Section 3 is devoted to the (scalar) spherical harmonics in two variables. There we display the reduction formula which permits an easy elimination of those

harmonics which can be expressed as linear combinations (with scalar coefficients) of the harmonics which form the basic set. In Sec. 4 the second order tensor spherical harmonics in two variables are introduced and their most important properties are listed. In Sec. 5 two different forms of tensor harmonics in the helicity bases are deduced from the two-variable tensor spherical harmonics constructed in the preceding section. Finally, in Sec. 6 we indicate, using a particular example of the reaction with three particles in the final state, how the formalism of the present paper may be applied and indicate some of its merits in comparison with the Cartesian expressions.

2. Tensor Spherical Basis

First we have to introduce a set of orthogonal 4-vectors on which to define the projections of the tensor fields. Following Daumens and Minnaert /1/ we choose three space-like vectors e_μ^{1n} ($n=\pm1,0$) on the unit sphere $S^2(e)$ embedded in the subspace $E^3(e)$ orthogonal to the time-like vector $e_\mu^{00} \equiv e_\mu$. The complex vectors e_μ^{rn} satisfy the following conditions

$$(e_\mu^{rn})^* = (-1)^{r+n+1} e_\mu^{r-n} , \tag{1}$$

$$e_\mu^{rn} (e_\mu^{r'n'})^* = \delta_{rr'} \delta_{nn'} . \tag{2}$$

We use the Pauli metrics (i.e., $a_\mu b_\mu = ab = \vec{a}\cdot\vec{b} - a_0 b_0$) and the usual summation convention for repeated Greek indices ($\mu = 0,1,2,3$). Note that three vectors e_μ^{1n} ($n=\pm1,0$) form the usual 3-dimensional spherical basis.

The spherical components of an arbitrary vector a_μ in the basis just introduced are given from the decomposition

$$a_\mu = \sum_m a^{1m} e_\mu^{1m*} + a^{00} e_\mu^{00*} . \tag{3}$$

This means

$$a^{00} = a_\mu e_\mu^{00} , \qquad a^{1m} = a_\mu e_\mu^{1m} . \tag{4}$$

The construction of the tensor spherical bases of an arbitrary order has been discussed in detail by Daumens and Minnaert /1/. We shall deal with the 2nd order tensor basis

$$t_{\mu\lambda}^{(r_1r_2)rn} = \sum_{n_1n_2} \begin{bmatrix} r_1 & r_2 & r \\ n_1 & n_2 & n \end{bmatrix} e_\mu^{r_1n_1} e_\lambda^{r_2n_2} \tag{5}$$

only, where the symbol $[\,\vdots\vdots\,]$ denotes the Clebsch-Gordan coefficient. The parity and orthogonality properties of basis (5) read as follows:

$$P: \; t_{\mu\lambda}^{(r_1r_2)rn} = (-1)^{r_1+r_2} t_{\mu\lambda}^{(r_1r_2)rn}, \tag{6}$$

$$t_{\mu\lambda}^{(r_1r_2)rn} \cdot t_{\mu\lambda}^{(r_1'r_2')r'n'} = (-1)^{r+n} \delta_{r_1r_1'} \delta_{r_2r_2'} \delta_{rr'} \delta_{nn'} . \tag{7}$$

3. Spherical Harmonics in Two Variables

The relativistic spherical harmonics may be introduced by taking the projections of an arbitrary unit 4-vector u_μ on the basis e_μ^{rn}. The constructions as performed, e.g., in Ref. /1/

$$Y_m^\ell(u) = (-1)^\ell \left(\frac{(2\ell+1)!!}{4\pi\, \ell!}\right)^{\frac{1}{2}} (u_{\mu_1} \otimes u_{\mu_2} \cdots \otimes u_{\mu_\ell}) \sum_{m_i n_i} \begin{bmatrix} 1 & 1 & 2 \\ n_1 & n_2 & m_2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ m_2 & n_3 & m_3 \end{bmatrix} \cdots \begin{bmatrix} \ell\text{-}1 & 1 & \ell \\ m_{\ell-1} & n_\ell & m \end{bmatrix} e_{\mu_1}^{1n_1} e_{\mu_2}^{1n_2} \cdots e_{\mu_\ell}^{1n_\ell} \tag{8}$$

holds for the purely space-like vectors (i.e. $ue = 0$) only. This condition, however, is not too restrictive for the applications, since it can always be met if one works in the rest (Breit) system which is fully appropriate in major physical situations. Indeed, taking the definition of the Breit system as $Q_\mu = (\vec{0}, iQ_0)$ and choosing the spherical basis in such a way that $e_\mu^{00} \equiv Q_\mu / \sqrt{-Q^2}$, we may always instead of a_μ consider the vector

$$\tilde{a}_\mu = a_\mu - \frac{Qa}{Q^2} Q_\mu , \tag{9}$$

which is orthogonal to e_μ:

$$\tilde{a}_\mu e_\mu^{00} = 0 \tag{10}$$

and then define the unit 4-vector $u_\mu = \tilde{a}_\mu / \sqrt{\tilde{a}^2}$. In this way since the time component of u_μ vanish, the spherical harmonics $Y_m^\ell(u)$ defined in (8) on $S^2(e)$ become identical with the usual spherical harmonics as defined, e.g., in Ref. /5/. In the simplest case $\ell = 1$ we have

$$u_\mu = \sqrt{\frac{4\pi}{3}} \sum_m Y_m^1(u)\, e_\mu^{1m\,*} . \tag{11}$$

Proceeding to the case of two variables u, and v we construct as usual the objects, which transform according to the irreducible representations of the rotation group:

$$\{(\ell_1\ell_2)\ell m\} = \{Y_{\ell_1}(u) Y_{\ell_2}(v)\}_{\ell m} = \sum_{m_1m_2} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix} Y_{m_1}^{\ell_1}(u)\, Y_{m_2}^{\ell_2}(v) . \tag{12}$$

They form a complete orthogonal basis. We would like to keep only independent terms of this infinite-dimensional basis. By independent we mean such terms which cannot be expressed through the remaining ones as linear combinations with scalar (i.e., depending on u^2 , v^2 , and uv and their powers) coefficients.

To separate the independent harmonics we apply the identity

$$\{Y_s(u) Y_s(v)\}_{00} \{Y_t(u) Y_{t'}(v)\}_{\ell m}$$
$$= \sum_{\ell_1\ell_2} \hat{\ell}_1 \hat{\ell}_2 \hat{\ell} \begin{Bmatrix} s & s & 0 \\ t & t' & \ell \\ \ell_1 & \ell_2 & \ell \end{Bmatrix} \alpha(st\ell_1)\; \alpha(st'\ell_2) \{Y_{\ell_1}(u) Y_{\ell_2}(v)\}_{\ell m} , \tag{13}$$

which represents actually the relation between the two coupling schemes of the four momenta s, t, s, and t' re-written using the expansion

$$Y_{m_s}^s(u)\, Y_{m_t}^t(u) = \sum_{\ell_1 m_1} \alpha(st\ell_1) \begin{bmatrix} s & t & \ell_1 \\ m_s & m_t & m_1 \end{bmatrix} Y_{m_1}^{\ell_1}(u),$$
$$\alpha(st\ell_1) = \frac{\hat{s}\hat{t}}{\sqrt{4\pi}\, \hat{\ell}_1} \begin{bmatrix} s & t & \ell_1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{14}$$

where $\hat{a} = \sqrt{2a+1}$, and $\{\,\vdots\vdots\vdots\,\}$ stands for the well-known 9j symbol. Using (13), any harmonics $\{(\ell_1'\ell_2')\ell m\}$ with $\ell_1' + \ell_2' > \ell + 1$ can be expressed through the harmonics $\{(\ell_1\ell_2)\ell m\}$ with $\ell_1 + \ell_2 < \ell_1' + \ell_2'$. It is easy to see that this process can be repeated as many times as necessary in order to reach the harmonics $\{(\ell_1\ell_2)\ell m\}$ with

$$\ell_1 + \ell_2 = \ell , \tag{15a}$$

$$\ell_1 + \ell_2 = \ell + 1 . \tag{15b}$$

The construction just described leaves us with $2\ell + 1$ orthogonal terms of the form (12), which are independent in the above sense. They cannot be further reduced in view of the triangular condition $\vec{\ell}_1 + \vec{\ell}_2 = \vec{\ell}$.

Concerning the case of the spherical harmonics in n variables it should be noted, that further generalization of (12) is actually not needed. In physical applications which we have in mind we always deal with the 4-dimensional Minkowski space. In this space there are only four independent vectors, that is the time-like vector $Q_\mu = (\vec{0}, iQ_0)$ and three space-like vectors, e.g., u_μ, v_μ, and $\varepsilon_{\mu\alpha\beta\gamma} Q_\alpha u_\beta v_\gamma$. Any other vector can be obtained as their linear combination with scalar coefficients. Therefore, the prescriptions (12) and (15) are sufficient to obtain the multivariable spherical harmonics as well.

4. Tensor Spherical Harmonics

The 2nd order tensor spherical harmonics in two variables are defined as

$$T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}(u,v,Q) = \sum_{mn} \begin{bmatrix} \ell & r & J \\ m & n & M \end{bmatrix} \{Y_{\ell_1}(u)Y_{\ell_2}(v)\}_{\ell m}\, t^{(r_1r_2)rn}_{\mu\lambda}(Q). \qquad (16)$$

The generalization to the tensor harmonics of an arbitrary higher order is straightforward.

Using the properties (6) and (7) of the basis tensor $t^{(r_1r_2)rn}_{\mu\lambda}$ and those of the two-variable spherical harmonics (12) we may easily see that

(i) the tensors $T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}$ for $\ell_1+\ell_2 = \ell, \ell+1$ form a set of independent Lorentz covariants;

(ii) they satisfy the identity

$$T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}(-u,-v,Q) = (-1)^{\ell_1+\ell_2}\, T^{(\ell_1\ell_2)\ell}_{(r_1r_2)r}(u,v,Q). \qquad (17)$$

Now, recalling the definitions of tensor

$$V_{\mu\lambda}(u,v) \xrightarrow{P} (-1)^{\delta_{\mu 0}+\delta_{\lambda 0}}\, V_{\mu\lambda}(-u,-v)$$

and pseudotensor

$$A_{\mu\lambda}(u,v) \xrightarrow{P} (-1)^{\delta_{\mu 0}+\delta_{\lambda 0}+1}\, A_{\mu\lambda}(-u,-v)$$

operators, we can easily see that the tensor harmonics (16) transform under the parity operation P like tensors (pseudotensors) if the sum $\ell_1+\ell_2+r_1+r_2$ is even (odd).

(iii) $T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}$ are orthonormal on the unit sphere embedded into the space $E^3(e)$ orthogonal to the vector e^{00}_μ :

$$\int d\Omega_{\vec{u}}\, d\Omega_{\vec{v}}\, \left(T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}\right)^* T^{(\ell_1'\ell_2')\ell'\;J'M'}_{(r_1'r_2')r'\;\mu\lambda} \qquad (18)$$

$$= \delta_{JJ'}\,\delta_{MM'}\,\delta_{\ell_1\ell_1'}\,\delta_{\ell_2\ell_2'}\,\delta_{\ell\ell'}\,\delta_{r_1r_1'}\,\delta_{r_2r_2'}\,\delta_{rr'} .$$

(iv) The scalar product of $T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda}$ in the 4-dimensional space is

$$T^{(\ell_1\ell_2)\ell\;JM}_{(r_1r_2)r\;\mu\lambda} \cdot T^{(\ell_1'\ell_2')\ell'\;J'M'}_{(r_1'r_2')r'\;\mu\lambda}$$

$$= \sum_{\substack{psx \\ m_p m_s m_x}} (4\pi)^{-1}\, \hat{\ell}_1 \hat{\ell}_2 \hat{\ell} \hat{J} \hat{\ell}_1' \hat{\ell}_2' \hat{\ell}' \hat{J}'\, \delta_{r_1r_1'}\,\delta_{r_2r_2'}\,\delta_{rr'} \begin{bmatrix} \ell_1 & \ell_1' & p \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ell_2 & \ell_2' & s \\ 0 & 0 & 0 \end{bmatrix} \qquad (19)$$

$$\times \begin{bmatrix} p & s & x \\ m_p & m_s & m_x \end{bmatrix} \begin{bmatrix} J & J' & x \\ M & M' & m_x \end{bmatrix} W(\ell r x J'; J \ell') \begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ \ell_1' & \ell_2' & \ell' \\ p & s & x \end{Bmatrix} Y^p_{m_p}(u)\, Y^s_{m_s}(v),$$

where $W(\dots;\dots)$ denotes the Racah coefficient, and

(v) the number $N_J^{(s)}$ of independent tensor harmonics of an arbitrary order s is

$$N_J^{(s)} = \sum_r n_r^{(s)} \sum_{\ell=|J-r|}^{J+r} (2\ell+1) = \sum_r n_r^{(s)} (2r+1)(2J+1), \qquad (20)$$

where $n_r^{(s)}$ is the statistical weight of the corresponding basis tensor. E.g. for the 2nd order $(s=2)$ spherical harmonics (19) with $t^{(r_1r_2)rn}_{\mu\lambda}$ we have $n_0^{(2)} = 2$, $n_1^{(2)} = 3$, and $n_2^{(2)} = 1$, therefore

$$N_J^{(2)} = 16\,(2J+1). \qquad (21)$$

Note that while it seems to be actually impossible to count the number of independent Lorentz covariants when working with the Cartesian forms, in the spherical basis the result (20) has been obtained in a very natural and elegant way.

5. Tensor Harmonics in the Helicity Basis

The helicity basis states analogous to those of Jacob and Wick[6] are introduced here in a slightly formal way, therefore the interpretation in terms of conserving quantities may be lost. Nevertheless, the technique has proved to be very helpful when a particular direction may be chosen according to the nature of the physical problem. We keep calling helicities the projections of the (internal) angular momentum vectors (vectors e^{1n}_{μ} of the basic tetrads) on the directions ϑ_1, φ_1 and ϑ_2, φ_2 connected with the scalar spherical harmonics (8)

$$Y^{\ell}_{m}(u) = \frac{\hat{\ell}}{\sqrt{4\pi}} D^{\ell\,*}_{m0}(\varphi_1, \vartheta_1, 0), \tag{22}$$

and

$$Y^{\ell}_{m}(v) = \frac{\hat{\ell}}{\sqrt{4\pi}} D^{\ell\,*}_{m0}(\varphi_2, \vartheta_2, 0) \tag{23}$$

considered above. The one-variable tensor harmonics in the helicity basis have been discussed in substantial detail by Akyeampong[2]. We shall follow his definitions and notation where possible.

In our case the second order helicity tensor basis may depend on three variables Q_λ, u_λ, and v_λ. (The dependence on Q_λ comes through e^{00}_{λ} just as in the case of the spherical basis (5)). Indeed, we can write[5]

$$e^{1n}_{\lambda} = \sum_{k} D^{1\,*}_{nk}(\varphi_1 \vartheta_1 0)\, \epsilon^{1k}_{\lambda}(u) = \sum_{k'} D^{1\,*}_{nk'}(\varphi_2 \vartheta_2 0)\, \epsilon^{1k'}_{\lambda}(v) \tag{24}$$

and consider any combination of the tetrads $\epsilon^{rk}_{\mu}(u)$ and $\epsilon^{rk'}_{\mu}(v)$. The particular choice of the tensor helicity basis certainly depends on the character of physical problem to be solved. Here we describe two cases, the symmetric $S^{j_1 r_1 k_1\ JM}_{j_2 r_2 k_2\ \mu\lambda}$ and unsymmetric $U^{j \ell_2\ \ JM}_{(r_1 r_2) rk\ \mu\lambda}$ tensor harmonics which are constructed using the tensor bases $\epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(v)$ and $\epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(u)$, respectively.

To construct the S harmonics we substitute Eqs. (22)-(24) in (16) and perform the standard recoupling:

$$T^{(\ell_1 \ell_2)\ell\ \ JM}_{(r_1 r_2) r\ \mu\lambda} = \frac{\hat{\ell}_1 \hat{\ell}_2}{4\pi} \sum_{\substack{m\,n \\ m_i n_i k_i}} \begin{bmatrix} \ell & r & J \\ m & n & M \end{bmatrix} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r \\ n_1 & n_2 & n \end{bmatrix} D^{\ell_1 *}_{m_1 0}(\varphi_1 \vartheta_1 0)\, D^{\ell_2 *}_{m_2 0}(\varphi_2 \vartheta_2 0)$$

$$\times D^{r_1 *}_{n_1 k_1}(\varphi_1 \vartheta_1 0)\, D^{r_2 *}_{n_2 k_2}(\varphi_2 \vartheta_2 0)\, \epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(v) \tag{25}$$

$$= \hat{\ell}_1 \hat{\ell}_2 \hat{\ell} \hat{r} \sum_{j_1 k_1 j_2 k_2} \begin{Bmatrix} \ell_1 & \ell_2 & \ell \\ r_1 & r_2 & r \\ j_1 & j_2 & J \end{Bmatrix} \begin{bmatrix} \ell_1 & r_1 & j_1 \\ 0 & k_1 & k_1 \end{bmatrix} \begin{bmatrix} \ell_2 & r_2 & j_2 \\ 0 & k_2 & k_2 \end{bmatrix} S^{j_1 r_1 k_1\ JM}_{j_2 r_2 k_2\ \mu\lambda}(u, v, Q),$$

where

$$S^{j_1 r_1 k_1\ JM}_{j_2 r_2 k_2\ \mu\lambda}(u, v, Q)$$

$$= \frac{\hat{j}_1 \hat{j}_2}{4\pi} \sum_{m_1' m_2'} \begin{bmatrix} j_1 & j_2 & J \\ m_1' & m_2' & M \end{bmatrix} D^{j_1 *}_{m_1' k_1}(\varphi_1 \vartheta_1 0)\, D^{j_2 *}_{m_2' k_2}(\varphi_2 \vartheta_2 0)\, \epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(v) \tag{26}$$

are just the helicity-basis tensor harmonics we are looking for.

Using again (22)-(24) and the Clebsch-Gordan expansion of the D-functions we have alternatively

$$T^{(\ell_1 \ell_2)\ell\ \ JM}_{(r_1 r_2) r\ \mu\lambda} = \sum_{m_i n_i m n} \begin{bmatrix} \ell_1 & \ell_2 & \ell \\ m_1 & m_2 & m \end{bmatrix} \begin{bmatrix} r_1 & r_2 & r \\ n_1 & n_2 & n \end{bmatrix} \begin{bmatrix} \ell & r & J \\ m & n & M \end{bmatrix} \frac{\hat{\ell}_1 \hat{\ell}_2}{4\pi} D^{\ell_1 *}_{m_1 0}(\varphi_1 \vartheta_1 0)\, D^{\ell_2 *}_{m_2 0}(\varphi_2 \vartheta_2 0)$$

$$\times \sum_{r' n' k} \begin{bmatrix} r_1 & r_2 & r' \\ n_1 & n_2 & n' \end{bmatrix} D^{r' *}_{n' k}(\varphi_1 \vartheta_1 0) \begin{bmatrix} r_1 & r_2 & r' \\ k_1 & k_2 & k \end{bmatrix} \epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(u) \tag{27}$$

$$= \hat{\ell}_1 \hat{\ell} \sum_{jk} \begin{bmatrix} \ell_1 & r & j \\ 0 & k & k \end{bmatrix} W(r j \ell \ell_2; \ell_1 J)\, U^{j \ell_2\ \ JM}_{(r_1 r_2) rk\ \mu\lambda}(u, v, Q).$$

In this last expression the U harmonics are

$$U^{j \ell_2\ \ JM}_{(r_1 r_2) rk\ \mu\lambda}(u, v, Q) = \sum_{m' m_2} \begin{bmatrix} j & \ell_2 & J \\ m' & m_2 & M \end{bmatrix} D^{j *}_{m' k}(\varphi_1 \vartheta_1 0)\, D^{\ell_2 *}_{m_2 0}(\varphi_2 \vartheta_2 0)$$

$$\times \frac{\hat{j} \hat{\ell}_2}{4\pi} \sum_{k_1 k_2} \begin{bmatrix} r_1 & r_2 & r \\ k_1 & k_2 & k \end{bmatrix} \epsilon^{r_1 k_1}_{\mu}(u)\, \epsilon^{r_2 k_2}_{\lambda}(u). \tag{28}$$

Now we list some useful properties of S and U :

(i) The harmonics (26) and (28) form the sets of independent (see Sec. 3) Lorentz covariants if just $2J+1$ different pairs $j_1 j_2$ $(j \ell_2)$ are taken for each combination of $r_1 k_1 r_2 k_2$ $(r_1 r_2 r k)$. The reduction formula which allows us to fix these restrictions and thus to deal with the independent covariants only can be easily obtained in the same manner as Eq. (13). It reads

$$\frac{4\pi}{\hat{s}^2} \}(ss)00\} \sum_{\tau_1 \tau_2} \begin{bmatrix} t_1 & t_2 & J \\ \tau_1 & \tau_2 & M \end{bmatrix} D^{t_1 *}_{\tau_1 \beta_1}(\varphi_1 \vartheta_1 0)\, D^{t_2 *}_{\tau_2 \beta_2}(\varphi_2 \vartheta_2 0) \tag{29}$$

$$= \sum_{j_1 j_2} \hat{j}_1 \hat{j}_2 \hat{J} \begin{Bmatrix} s & s & 0 \\ t_1 & t_2 & J \\ j_1 & j_2 & J \end{Bmatrix} \begin{bmatrix} s & t_1 & j_1 \\ 0 & \beta_1 & \beta_1 \end{bmatrix} \begin{bmatrix} s & t_2 & j_2 \\ 0 & \beta_2 & \beta_2 \end{bmatrix} \sum_{m_1 m_2} \begin{bmatrix} j_1 & j_2 & J \\ m_1 & m_2 & M \end{bmatrix} D^{j_1 *}_{m_1 \beta_1}(\varphi_1 \vartheta_1 0)\, D^{j_2 *}_{m_2 \beta_2}(\varphi_2 \vartheta_2 0).$$

(ii) The parity operator acts on them according to

$$S^{j_1 r_1 k_1 \;\; JM}_{j_2 r_2 k_2 \;\; \mu\lambda}(-u, -v, Q) = (-1)^{j_1+j_2+r_1+r_2}\, S^{j_1 r_1 -k_1 \;\; JM}_{j_2 r_2 -k_2 \;\; \mu\lambda}(u, v, Q), \tag{30}$$

$$U^{j \ell_2 \;\; JM}_{(r_1 r_2) r k \;\; \mu\lambda}(-u, -v, Q) = (-1)^{j+\ell_2+r}\, U^{j \ell_2 \;\; JM}_{(r_1 r_2) r -k \;\; \mu\lambda}(u, v, Q). \tag{31}$$

(iii) The S and U harmonics are orthonormal if integrated with $d\Omega_i = \sin\vartheta_i\, d\vartheta_i\, d\varphi_i$; e.g.:

$$\int d\Omega_u d\Omega_v \left(S^{j_1 r_1 k_1 \;\; JM}_{j_2 r_2 k_2 \;\; \mu\lambda} \right)^* S^{j_1' r_1' k_1' \;\; J'M'}_{j_2' r_2' k_2' \;\; \mu\lambda} \tag{32}$$

$$= \delta_{JJ'}\, \delta_{MM'}\, \delta_{j_1 j_1'}\, \delta_{j_2 j_2'}\, \delta_{r_1 r_1'}\, \delta_{r_2 r_2'}\, \delta_{k_1 k_1'}\, \delta_{k_2 k_2'}.$$

(iv) The scalar product is

$$S^{j_1 r_1 k_1 \;\; JM}_{j_2 r_2 k_2 \;\; \mu\lambda} \cdot S^{j_1' r_1' k_1' \;\; J'M'}_{j_2' r_2' k_2' \;\; \mu\lambda} = (4\pi)^{-1} \hat{j}_1 \hat{j}_2 \hat{j}_1' \hat{j}_2' \hat{J} \hat{J}' (-1)^{r_1+r_2+k_1+k_2}$$

$$\cdot\, \delta_{r_1 r_1'}\, \delta_{r_2 r_2'}\, \delta_{-k_1 k_1'}\, \delta_{-k_2 k_2'} \sum_{J_i M_i \varkappa m_\varkappa} \begin{bmatrix} J_1 & J_2 & \varkappa \\ M_1 & M_2 & m_\varkappa \end{bmatrix} \begin{bmatrix} J & J' & \varkappa \\ M & M' & m_\varkappa \end{bmatrix} \tag{33}$$

$$\cdot \begin{bmatrix} j_1 & j_1' & J_1 \\ k_1 & -k_1 & 0 \end{bmatrix} \begin{bmatrix} j_2 & j_2' & J_2 \\ k_2 & -k_2 & 0 \end{bmatrix} \begin{Bmatrix} j_1 & j_2 & J \\ j_1' & j_2' & J' \\ J_1 & J_2 & \varkappa \end{Bmatrix} Y^{J_1}_{M_1}(u)\, Y^{J_2}_{M_2}(v).$$

The scalar product of the U harmonics has a similar expression, we do not display it.

6. Conclusions

In the present paper an extension of the relativistic tensor harmonics to the case of two variables has been suggested. They appear to be rather compact and very flexible in comparison with the Cartesian forms which were usually employed in the applications. To give just one example, consider the reactions of the radiative muon capture on atomic nuclei /7,8/:

$$\mu^- + A(J_i, p_i) \longrightarrow \nu_\mu + \gamma(k) + B(J_f, p_f) \tag{34}$$

Introducing the momenta $Q_\lambda = (p_f + p_i)_\lambda$ and $q_\lambda = (p_f - p_i)_\lambda$ we can fix the general form of the corresponding weak hadronic matrix elements in the elegant form which is also easy to manipulate

$$T_{\mu\lambda} = \sum_{\substack{JM \\ \ell_1 \ell_2 \ell\, r_1 r_2 r}} \hat{J}_i^{-1} \begin{bmatrix} J_i & J & J_f \\ M_i & M & M_f \end{bmatrix} F^{(\ell_1 \ell_2)\ell\; J}_{(r_1 r_2) r}(k, q, Q)\, T^{(\ell_1 \ell_2)\ell\; JM}_{(r_1 r_2) r\; \mu\lambda}(k, q, Q), \tag{35}$$

where $F^{(\ell_1 \ell_2)\ell\; J}_{(r_1 r_2) r}$ are form factors. The structure of this tensor in the Cartesian covariants is considerably more complicated and for different values of J_i and J_f must be constructed individually. For example for the reaction $\mu^- + p \longrightarrow \nu_\mu + \gamma(k) + n$ one finds /8/

$$T_{\mu\lambda} = -\bar{u}(p_f) \{ F_1 \gamma_\mu \gamma_\lambda + \gamma_\mu \gamma_\nu k_\nu [k_\lambda F_2 + q_\lambda F_3 + Q_\lambda F_4]$$
$$+ \cdots$$
$$+ F_{35} \gamma_5 \gamma_\mu \gamma_\lambda + \gamma_5 \gamma_\mu \gamma_\nu k_\nu [k_\lambda F_{36} + q_\lambda F_{37} + Q_\lambda F_{38}]$$
$$+ \cdots \} u(p_i),$$

where 68 terms /9/ containing the Dirac matrices γ_ν and form factors $F_i(k, q, Q)$ appear. Applications of the tensor harmonics in the helicity basis for the reaction (34) are presently being studied.

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9. Note that the dependent covariants always may appear in the Cartesian basis. Comparing with Eq. (21) we can see that already in the simplest case of reaction (34) on a proton $(J_i = J_f = \frac{1}{2})$ there are four superfluous covariants. The corresponding coupling equations, when constructed with the help of the symbol manipulating program SCHOONSCHIP appeared to contain 26 terms each.

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