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BOSON-FERMION REPRESENTATIONS OF LIE SUPERALGEBRAS: AN EXAMPLE OF osp(1,2)

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Бозон-фермионные представления суперaлгебр Ли: пример osp(1,2)

Предложен метод построения бесконечномерных представлений суперaлгебр Ли при помощи бозонных представлений их подалгебр Ли. В качестве примера рассмотрена суперaлгебра osp(1,2); в явном виде найдены выражения для ее генераторов. Они выражаются через одну пару бозонных и (по крайней мере) одну пару фермионных операторов и зависят от одного параметра, при этом оператор Казимира кратен единичному оператору. Сужения некоторых из полученных представлений на вещественные формы osp(1,2) являются кососимметричными в парной части и представляют собой естественное обобщение кососимметрических представлений вещественных алгебр Ли. Отмечены также некоторые другие свойства предложенного метода.

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Boson-Fermion Representations of Lie Superalgebras: an Example of osp(1,2)

A method for constructing infinite-dimensional representations of Lie superalgebras employing boson representations of their Lie subalgebras is outlined. As an example the osp(1,2) superalgebra is considered; explicit formulae for its generators in terms of one pair of boson and at most one pair of fermion operators (depending on at most one parameter) are obtained, the Casimir operator being represented by a multiple of unity. The restrictions of some of the obtained representations to the real forms of osp(1,2) are skew-symmetric in the even part and represent a natural generalization of skew-symmetric (or star) representations of real Lie algebras. Some other aspects of the presented construction are also briefly discussed.

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1. Introduction

1.1 The physical concept of supersymmetry has attracted attention to the mathematical structure of Lie superalgebras and their representations. The finite-dimensional Lie superalgebras can be classified in a manner analogous to the Lie algebra theory. A complete classification is known for the complex simple Lie superalgebras /1-6/ as well as for real simple Lie superalgebras /1/ ; it allows one to describe also all semisimple Lie superalgebras, however, in a more complicated way than in the usual Lie theory. Various results are known concerning finite-dimensional irreducible representations of simple Lie superalgebras /1,8,11-13/ ; the situation is much less satisfactory for the infinite-dimensional representations. In this note we propose a method for constructing such representations, illustrating it on the simplest non-trivial example of the osp(1,2) superalgebra (also called B(0,1) , (sp(2),2) in terminology of Ref.5 or osp(2,1) in Ref.8).

1.2 There exists a wide family of the so-called boson representations for real classical simple Lie algebras. They are obtained with the help of canonical realizations of these algebras (cf.Ref.7 and references quoted therein) ; all of them are Schurean, i.e., such that every Casimir operator is represented by a multiple of the unit operator. They are, moreover, skew-symmetric. Our aim is to use some of these boson representations in order to construct Schurean representations of Lie superalgebras, especially such which have the even generators skew-symmetric.

1.3 The construction for a Lie superalgebra $L = L_0 + L_1$ will proceed as follows : in a suitably chosen boson representation of the Lie algebra L_0 we interchange the numerical parameter (or parameters) by an operator (operators) on some "fermion" Hilbert space $\mathcal{H}_F$. Then we look for generators of the odd part L_1 in the form of linear combinations of operators $T_B \otimes T_F$, where T_B depends on a certain number of pairs of boson operators on

$\mathcal{H}_B$ (representation space of the used boson representation), and analogously T_F is a function of one or more pairs of operators on $\mathcal{H}_F$ which obey the canonical anticommutation rules (CAR). Since in general we obtain a family of representations depending on some parameters, we can look in this set for those which have desired properties, e.g., by evaluating the Casimir operators we can select the Schurean representations, etc.

2. Schurean representations of the osp(1,2) superalgebra

We shall now realize in detail the construction sketched in the introduction for the particular case of osp(1,2) .

2.1 We shall use the basis introduced in Ref.8 : the even generators Q_3, $Q_\pm$ and the odd ones $V_\pm$ fulfil the following relations

$$[Q_3,Q_\pm] = \pm Q_\pm \quad , \quad [Q_+,Q_-] = 2Q_3 \quad , \tag{1a}$$

$$[Q_3,V_\pm] = \pm \frac{1}{2} V_\pm \quad , \tag{1b}$$

$$[Q_+,V_+] = 0 \quad , \quad [Q_+,V_-] = V_+ \quad , \tag{1c}$$

$$[Q_-,V_-] = 0 \quad , \quad [Q_-,V_+] = V_- \quad , \tag{1d}$$

$$\{V_+,V_+\} = \frac{1}{2} Q_+ \quad , \quad \{V_-,V_-\} = -\frac{1}{2} Q_- \quad , \tag{1e}$$

$$\{V_+,V_-\} = -\frac{1}{2} Q_3 \quad . \tag{1f}$$

The Casimir operator is expressed as

$$K_2 = Q_3^2 + \frac{1}{2}\{Q_+,Q_-\} + [V_+,V_-] \quad . \tag{2}$$

2.2 The even part of the osp(1,2) is the Lie algebra $sp(2,\mathbb{C}) \simeq sl(2,\mathbb{C})$. A suitable boson representation of this algebra can be constructed with the help of one pair of boson operators b , b^+ which obey the CCR : $[b,b^+] = I_B$; equivalently, one can use the linear combinations $p = 2^{-1/2}(b - b^+)$, $q = 2^{-1/2}(b+b^+)$ fulfilling

$$[p,q] = I_B \quad . \tag{3}$$

We choose the operators p,q on $\mathcal{H}_B$ in such a way that p is skew-symmetric and q is symmetric. This choice corresponds to the standard relation $b^+ \subset b^*$, where b^* denotes the adjoint of b . One can verify easily that the operators

$$Q_+^B = iq^2 \quad , \quad Q_3^B = \frac{1}{2}(qp + \frac{1}{2}I_B) \quad , \quad Q_-^B = \frac{i}{4}p^2 + i\gamma q^{-2} \qquad (4)$$

fulfil for each $\gamma \in \mathbb{C}$ the relations (1a) and generate therefore a representation of $sp(2)$ which is Schurean as can be easily seen. Moreover, if $\gamma \in \mathbb{R}$, then this representation considered as a representation of the real non-compact form $sp(2,\mathbb{R})$ is skew--symmetric because the operators (4) are skew-symmetric.

2.3 The relations (1a) are understood in the sense commonly accepted in the representation theory (cf.,e.g., Ref.9, Chap.11) : there exists a domain D dense in $\mathcal{H}_B$ such that the operators (4) map D into D . One has, of course, to specify a representation of the CCR (3) suitable for this purpose. We shall use in the following the Schrödinger representation in $\mathcal{H}_B = L^2(\mathbb{R})$: $(pf)(x) = f'(x)$, $(qf)(x) = xf(x)$. Its standard domain $\mathcal{S}(\mathbb{R})$ has to be, however, restricted for purposes of the representation (4), since $\mathcal{S}(\mathbb{R})$ is not contained in the domain of q^{-2} . We may take, e.g., $D = \mathcal{S}_0(\mathbb{R}) \equiv \{ f \in \mathcal{S}(\mathbb{R}) : f^{(k)}(0) = 0 \ , \ k = 0,1,2,\dots \}$ which can be easily seen to be dense in $L^2(\mathbb{R})$ and to have the other needed properties.

2.4 Remarks : (a) The Schrödinger representation is not, of course, the only possible. Alternatively, one can put $\mathcal{H}_B = L^2([\alpha,\beta])$ $\beta - \alpha < \infty$, $0 \notin [\alpha,\beta]$ with p,q acting in the same way as above. A suitable domain D consists now of all infinitely differentiable functions $f \in \mathcal{H}_B$ such that $f^{(k)}(\alpha) = f^{(k)}(\beta) = 0$, $k = 0,1,2,\dots$; it could be denoted as $C_0^\infty([\alpha,\beta])$ – cf.Ref.10, Sec.V.4 . In contrast to the Schrödinger representation, the operators q^n , $n = \pm 1 , \pm 2 , \dots$ are now bounded. Another possibility is represented by formally the same p,q , now in $\mathcal{H}_B = L^2(\mathbb{R}_+)$, $\mathbb{R}_+ = [0,\infty)$, with $D = \mathcal{S}_0(\mathbb{R}_+) \equiv \{ f = g \restriction \mathbb{R}_+ : g \in \mathcal{S}_0(\mathbb{R}) \}$.

(b) In order to avoid formality in (1a) only one may take some larger set of functions as a common domain of p,q , for example $D_n = \{ f : f^{(k)} \in AC[\alpha,\beta] \ , \ f^{(k)}(\alpha) = f^{(k)}(\beta) = 0 \ , \ k = 0,1,\dots,n-1 \}$, $n \geq 4$, in the first of the above-mentioned examples (cf.Ref.10, Sec.VIII.2 for the definition of $AC[\alpha,\beta]$). However, domains of this type are not invariant under action of the operator p .

2.5 As mentioned above, we choose $\mathcal{H} = \mathcal{H}_B \otimes \mathcal{H}_F$ for the representation space of osp(1,2) , $\mathcal{H}_F$ being some "fermion" Hilbert space to be specified later. The expressions for the even generators are obtained by extending the boson representation (4) onto $\mathcal{H}$ as follows

$$\tilde{Q}_+ = Q_+^B \otimes I_F \quad , \quad \tilde{Q}_3 = Q_3^B \otimes I_F \quad ,$$
$$\tilde{Q}_- = \tfrac{i}{4}(p^2 \otimes I_F) + i(q^{-2} \otimes A) \quad ; \tag{5}$$

the commutation relations (1a) are obviously satisfied for an arbitrary operator A . If we demand these operators to be skew-symmetric, the operator A has to be symmetric.

2.6 We shall make an ansatz for the operators $\tilde{V}_\pm$ on $\mathcal{H}$ that represent $V_\pm$. Comparison of the relations

$$[Q_3^B , q] = \tfrac{1}{2} q \quad , \quad [Q_3^B , ap + bq^{-1}] = - \tfrac{1}{2}(ap + bq^{-1}) \quad ,$$

$a,b \in \mathbb{C}$, which follow easily from the CCR (3) , to (1b) suggests the following ansatz

$$\tilde{V}_+ = q \otimes C_1 \quad , \quad \tilde{V}_- = q^{-1} \otimes C_2 + p \otimes C_3 \quad , \tag{6}$$

where the operators C_r, r=1,2,3 on $\mathcal{H}_F$ are to be specified using (1c,e,f) ; the relations (1d) are then identically satisfied. Clearly, the first of the relations (1c) is also fulfilled for any C_r's . Next the relations which do not contain the operator A has to be considered. One checks easily that the first of the relations (1e) together with the second of (1c) imply

$$\{C_1, C_1\} = \tfrac{i}{2} I_F \quad , \quad C_3 = \tfrac{i}{2} C_1 \quad . \tag{7a}$$

Further, (1f) gives

$$\{C_1, C_2\} = 0 \quad . \tag{7b}$$

Finally from the second of the relations (1e) together with (7a,b) the following expression for A is obtained

$$A = 2\, C_1 C_2 + 4i\, C_2^2 \quad .$$

2.7 <u>Proposition</u> : Let the operators C_1, C_2 on $\mathcal{H}_F$ fulfil

$$C_1^2 = \frac{i}{4} I_F \quad , \quad \{C_1, C_2\} = 0 \tag{8}$$

then the operators

$$\begin{aligned}
\tilde{Q}_+ &= iq^2 \otimes I_F \quad , \quad \tilde{Q}_3 = \frac{1}{2}(qp + \frac{1}{2} I_B) \otimes I_F \quad , \\
\tilde{Q}_- &= \frac{i}{4} p^2 \otimes I_F + iq^{-2} \otimes (2 C_1 C_2 + 4i C_2^2) \quad , \\
\tilde{V}_+ &= q \otimes C_1 \quad , \quad \tilde{V}_- = \frac{i}{2} p \otimes C_1 + q^{-1} \otimes C_2
\end{aligned} \tag{9}$$

satisfy the relations (1a-f) and the Casimir operator (2) is expressed as

$$K_2 = -\frac{1}{16} - 4i I_B \otimes C_2^2 \quad . \tag{10}$$

2.8 Let us examine now how many different pairs C_1, C_2 obeying (8) can exist on a given finite-dimensional $\mathcal{H}_F$. In addition to the assumption (8) we shall require irreducibility of the set $\{C_1, C_2\}$: if $\{C_1, C_2\}$ is reducible, then the corresponding representation of osp(1,2) determined by the relations (9) is also reducible.

__Lemma__ : The only finite-dimensional irreducible representations of the relations (8) , up to similarity transformations, are the following ones :

(i) $\dim \mathcal{H}_F = 1$

$$C_1 = \frac{1}{2} \exp(\frac{\pi i}{4}) \quad , \quad C_2 = 0 \quad , \tag{11}$$

(ii) $\dim \mathcal{H}_F = 2$

$$C_1 = \frac{1}{2} \exp(\frac{\pi i}{4}) \sigma_2 \quad , \quad C_2 = \frac{c}{2} \exp(\frac{\pi i}{4}) \sigma_1 \quad , \tag{12}$$

where c is a non-zero complex number and σ_i are the Pauli matrices.

__Proof__ : Any linear operator on a finite-dimensional $\mathcal{H}_F$ has eigenvectors. Let us denote by x an eigenvector of C_2 :

$$C_2 x = \frac{c}{2} \exp(\frac{\pi i}{4}) x \quad , \quad c \in \mathbb{C} \tag{*}$$

and

$$y = 2 \exp(-\frac{\pi i}{4}) C_1 x \quad . \tag{**}$$

One can check easily that the linear envelope of the set $\{x, y\}$

is invariant under C_1 and C_2, so that in an irreducible representation the equality $\{x,y\}_{lin} = \mathcal{H}_F$ must hold. Consequently, there are only two irreducible cases : $\dim \mathcal{H}_F = 1$ where the implication (8) $\Rightarrow$ (11) is straightforward, and $\dim \mathcal{H}_F = 2$ where x and y are linearly independent. In this case the relations (8) imply

$$C_1 y = 2\exp(-\tfrac{\pi i}{4})\, C_1^2 x = \tfrac{1}{2}\exp(\tfrac{\pi i}{4})\, y \quad ,$$

$$C_2 y = 2\exp(-\tfrac{\pi i}{4})\, C_2 C_1 x = -c\, C\, x = -\tfrac{c}{2}\exp(\tfrac{\pi i}{4})\, y \quad ;$$

these relations together with (*),(**) show that the matrix representation of C_1, C_2 in the basis $\{x,y\}$ is the following

$$C_1 = \tfrac{1}{2}\exp(\tfrac{\pi i}{4})\sigma_1 \quad , \quad C_2 = \tfrac{c}{2}\exp(\tfrac{\pi i}{4})\sigma_3 \quad .$$

Finally, the similarity transformation $C_r \rightarrow U C_r U^{-1}$, $U = \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$ gives (12) . ■

2.9 Before we shall proceed further we must make a small digression. The osp(1,2) superalgebra has two real forms (osp(1,2;p;R) in the notation of Ref.1, $p = 0,1$) ; the first one is just the real linear envelope of the basis (1), the other is generated by the elements Q_α, $W_\beta \equiv iV_\beta$. The existence of these real forms is closely connected to an adjoint operation on osp(1,2) . Generally, an adjoint operation on a given complex Lie superalgebra $L = L_0 + L_1$ means an (antilinear) involution "+" on L which conserves L_0, L_1 and obeys $(a,b)^+ = (b^+,a^+)$, where $(.,.)$ denotes the corresponding super-Lie product in L , for all $a,b \in L$ ./11/ For osp(1,2) the adjoint operation is determined by the relations

$$Q_\alpha^+ = -Q_\alpha \quad , \quad \alpha = 3, \pm \quad , \tag{13a}$$

$$V_\beta^+ = -iV_\beta \quad , \quad \beta = \pm \quad . \tag{13b}$$

As to the other possible generalization of the adjoint operation in Lie algebras, the grade adjoint operation/11/, one can check easily that there is no such mapping "+" on osp(1,2) consistent with the relation (13a).*)

*) In contrast to this, there exist grade adjoint operations but no adjoint operations if $Q_\alpha^+ = Q_\alpha$ is required – cf.Refs.8,11 .

Further we have to adapt the concept of (grade) star representations introduced in Ref.11 for finite-dimensional representations. Let L be a complex Lie superalgebra equipped by a (grade) adjoint operation "+" , then a graded representation of L by linear (possibly unbounded) operators on a graded Hilbert space $\mathcal{H}$, $\pi : L \to \mathcal{L}(\mathcal{H})$ is called <u>(grade) star representation</u> if it fulfils

$$\pi(a^+) \subset \pi(a)^* \tag{14}$$

for all $a \in L$. Here $\pi(a)^*$ is the adjoint operator to $\pi(a)$; especially for the bounded operators the inclusion may be replaced by equality.

In the following we shall look for star representations (with respect to the adjoint operation (13)) among the obtained representations of osp(1,2) . On the other hand, it is clear that <u>there is no grade star representation of osp(1,2) with the even generators skew-symmetric</u>.

2.10 Let us return now to the construction of representations. We shall consider first the case $\dim \mathcal{H}_P = 1$. Substitution of (11) into (9),(10) yields

$$\tilde{Q}_+ = iq^2 \quad , \quad \tilde{Q}_3 = \tfrac{1}{2}(qp + \tfrac{1}{2}I) \quad , \quad \tilde{Q}_- = \tfrac{i}{4}p^2 \quad , \tag{15a}$$

$$\tilde{V}_+ = \tfrac{1}{2}\exp(\tfrac{\pi i}{4})\, q \quad , \quad \tilde{V}_- = \tfrac{i}{4}\exp(\tfrac{\pi i}{4})\, p \quad , \tag{15b}$$

$$K_2 = -\tfrac{1}{16} \quad .$$

As has been mentioned in 2.3 we assume the operators p,q to be defined on some dense invariant domain D in $L^2(\mathbb{R})$, say $\mathcal{S}_0(\mathbb{R})$, by $(pf)(x) = f'(x)$, $(qf)(x) = xf(x)$. We denote by $\mathcal{L}(D)$ the vector space of linear operators which are defined on D and map D into D . In order to obtain a representation of osp(1,2) in terms of operators from $\mathcal{L}(D)$ we have to introduce the standard graded Lie algebra structure in $\mathcal{L}(D)$. This can be done as follows : to the unitary operator R on $L^2(\mathbb{R})$, $(Rf)(x) = f(-x)$ the orthogonal projections $P_\pm \equiv \frac{1}{2}(I \pm R)$ correspond so that $D = D_+ \oplus D_-$ where $D_\pm \equiv P_\pm D$. We introduce

$$\mathcal{L}_0 \equiv \{ T \in \mathcal{L}(D) : TD_\pm \subset D_\pm \} \quad ,$$

$$\mathcal{L}_1 \equiv \{ T \in \mathcal{L}(D) : TD_\pm \subset D_\mp \} \quad ;$$

then $\mathcal{L}(D) = \mathcal{L}_0 \oplus \mathcal{L}_1$.

The operators (15a) are clearly in $\mathcal{L}_0$ (even), while the operators (15b) belong to $\mathcal{L}_1$ (odd), hence by (15a,b) a representation of $osp(1,2)$ on $L^2(\mathbb{R})$ is determined. Each element of $osp(1,2)$ is represented in terms of (one pair of) boson operators only, therefore this representation is called a <u>pure boson representation</u> (PBR) .

The operators $\tilde{Q}_\alpha$ are clearly skew-symmetric and

$$-i\tilde{V}_+ = \frac{1}{2}\exp(-\frac{\pi i}{4})\, q \subset (\frac{1}{2}\exp(\frac{\pi i}{4})\, q\,)^* = \tilde{V}_+^* \quad ,$$

$$-i\tilde{V}_- = \frac{1}{4}\exp(\frac{\pi i}{4})\, p \subset -\frac{1}{4}\exp(\frac{\pi i}{4})\, p^* = \tilde{V}_-^* \quad ,$$

so our PBR is a star representation with respect to the adjoint operation (13).

<u>Summary of 2.10</u> : If $\dim \mathcal{H}_F = 1$ then there exists a representation of $osp(1,2)$ given by the relations (15) which is called PBR . The Casimir operator is equal to $-(1/16)I$ so that the PBR is Schurean ; moreover, it is a star representation with respect to the adjoint operation (13).

2.11 Let us consider further the case $\dim \mathcal{H}_F = 2$. Instead of σ_1 , σ_2 we introduce the operators

$$a = \frac{1}{2}(\sigma_1 - i\sigma_2) \quad , \quad a^+ = \frac{1}{2}(\sigma_1 + i\sigma_2)$$

which satisfy the CAR

$$a^2 = (a^+)^2 = 0 \quad , \quad \{a, a^+\} = I_F$$

and

$$[a^+, a] = \sigma_3 \quad .$$

Substitution of (12) into (9),(10) leads to

$$\tilde{Q}_+ = iq^2 \otimes I_F \quad , \quad \tilde{Q}_3 = \frac{1}{2}(qp + \frac{1}{2}I_B) \otimes I_F \quad ,$$
$$\tilde{Q}_- = \frac{i}{4}(p^2 - 4c^2 q^{-2}) \otimes I_F + \frac{ic}{2}\, q^{-2} \otimes [a^+, a] \quad , \qquad (16a)$$

$$\tilde{V}_+ = \frac{i}{2}\exp(\frac{\pi i}{4})\, q \otimes (a - a^+) \quad ,$$
$$\tilde{V}_- = \frac{1}{4}\exp(\frac{\pi i}{4})\, p \otimes (a^+ - a) + \frac{c}{2}\exp(\frac{\pi i}{4})\, q^{-1} \otimes (a + a^+) \quad , \qquad (16b)$$

$$K_2 = (c^2 - \tfrac{1}{16}) I \quad . \tag{16c}$$

The Hilbert space $\mathcal{H} = \mathcal{H}_B \otimes \mathbb{C}^2$ decomposes as $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_B$, i.e., its elements can be realized by $\binom{x}{y}$, $x,y \in \mathcal{H}_B$. If A is a linear operator on $\mathcal{H}_B$ and B a linear operator on $\mathbb{C}^2$ represented by a matrix (b_{ij}), then $A \otimes B$ is represented by $(b_{ij}A)$. We denote by $D^{(2)}$ the linear envelope of the Cartesian product $D \times D$ and by $\mathcal{L}(D^{(2)})$ the space of linear operators which are defined on $D^{(2)}$ and map $D^{(2)}$ into $D^{(2)}$. The standard GLA structure in $\mathcal{L}(D^{(2)})$ can be introduced as follows :

$$D_0^{(2)} \equiv \left\{ \binom{x}{0} \in \mathcal{H} \; : \; x \in D \right\} \quad , \quad D_1^{(2)} \equiv \left\{ \binom{0}{y} \in \mathcal{H} \; : \; y \in D \right\} \quad ,$$

$$D^{(2)} = D_0^{(2)} \oplus D_1^{(2)} \quad ,$$

$$\mathcal{L}_0 \equiv \{ T \in \mathcal{L}(D^{(2)}) \; : \; TD_r^{(2)} \subset D_r^{(2)} \; , \; r = 0,1 \} \quad ,$$

$$\mathcal{L}_1 \equiv \{ T \in \mathcal{L}(D^{(2)}) \; : \; TD_0^{(2)} \subset D_1^{(2)} \; , \; TD_1^{(2)} \subset D_0^{(2)} \} \quad .$$

The operator $A \otimes B \in \mathcal{L}(D^{(2)})$ belongs to $\mathcal{L}_0$ iff $b_{12} = b_{21} = 0$ and to $\mathcal{L}_1$ iff $b_{11} = b_{22} = 0$. As I_F, $[a^+,a]$ are represented by diagonal matrices and $(a \pm a^+)$ by antidiagonal ones we see that the operators (16a) are elements of $\mathcal{L}_0$ (even), while the operators (16b) belong to $\mathcal{L}_1$ (odd). Thus the relations (16a,b) determine a representation of osp(1,2) which is expressed in terms of one pair of boson operators p,q and one pair of fermion operators a,a^+. We shall call these representations <u>boson-fermion representations</u> (BFR). If $c = 0$, then the BFR is equivalent to a direct sum of two pure boson representations as can be easily seen.

When does the BFR become a star representation with respect to the adjoint operation (13) ? Clearly $\tilde{Q}_+^{\star} \supset -\tilde{Q}_+$, $\tilde{Q}_3^{\star} \supset -\tilde{Q}_3$ and

$$\tilde{Q}_-^{\star} \supset -\tilde{Q}_- + i(\bar{c}^2 - c^2)\, q^{-2} \otimes I_F - \tfrac{i}{2}(\bar{c} - c)\, q^{-2} \otimes [a^+,a] \quad ;$$

since the operators q^2, qp, p^2, q^{-2} on D and I, σ_3 on $\mathbb{C}^2$ are linearly independent, the conditions (13a),(14) are fulfilled iff $\bar{c} = c$. For real c we further obtain $\tilde{V}_\beta^{\star} \supset -i\tilde{V}_\beta$, thus the BFR is a star representation in this case.

<u>Summary of 2.11</u> : If $\dim \mathcal{H}_F = 2$, then the relations (16) determine a family of representations of osp(1,2) specified by

one complex parameter c. They are called BFR : the generators depend on one pair of boson and one pair of fermion operators. The Casimir operator is given by $K_2 = (c^2 - 1/16)I$, i.e., these representations are Schurean. The BFR corresponding to $c=0$ is equivalent to a direct sum of two copies of the PBR (15) . The BFR is a star representation with respect to the adjoint operation (13) iff $c \in \mathbb{R}$.

3. Comments

3.1 The pure boson representation given by the relation (15) can be easily generalized for the osp(1,2n) superalgebra (denoted also by B(0,n) , (sp(2n),2n) or osp(2n,1)). To this purpose one has to take n pairs of boson operators p_r, q_r obeying the CCR, p_r being skew-symmetric and q_r symmetric. It is well-known that the operators

$$X_{rs} = \frac{1}{2}\{q_r, p_s\} = -X_{-s,-r} \quad ,$$
$$X_{r,-s} = -\frac{i}{2}\{q_r, q_s\} \quad , \quad X_{-r,s} = -\frac{i}{2}\{p_r, p_s\} \quad ,$$

$r,s = 1,2,\ldots,n$, generate a skew-symmetric representation of the Lie algebra $sp(2n,\mathbb{R})$. Let us introduce further

$$V_r = \exp(\tfrac{\pi i}{4})\, q_r \quad , \quad V_{-r} = \exp(-\tfrac{\pi i}{4}) p_r \ ,$$

$r = 1,2,\ldots,n$, then the following relations are valid

$$[X_{ij}, V_k] = \delta_{kj} V_i - \epsilon_i \epsilon_j \delta_{-i,k} V_{-j} \quad ,$$
$$\{V_k, V_l\} = 2\epsilon_l X_{k,-l} = -2\epsilon_k X_{l,-k} \quad ,$$

where $i,j,k = \pm 1, \pm 2, \ldots, \pm n$. Consequently, the operators X_{ij} , V_k generate a representation of osp(1,2n) which is a star representation with respect to the adjoint operation analogous to (13). One may hope therefore that the described method will allow us to construct boson-fermion representations also for some other, physically more interesting Lie superalgebras, starting with suitable boson representations of their Lie subalgebras.

3.2 Canonical realizations of Lie algebras /7/ represent an effective method for constructing representations. The possibility of application of an analogous procedure to the Lie superalgebras is attractive : notice that the above presented construction is essentially an algebraical one. For this purpose one has to choose a suitable generalization of the Weyl algebra W_{2n} . We can take the most straightforward generalization [14,15] omitting other possibilities [16] : the associative algebra $W_{2n;m}$ with unity generated by the elements q_i, p_i , $i = 1,2,\ldots,n$ and ξ_k , $k = 1,2,\ldots,m$, which obey the relations

$$[p_i, q_j] = \delta_{ij} 1 \quad , \quad [p_i, p_j] = [q_i, q_j] = 0 \quad ,$$
$$\{\xi_k, \xi_l\} = \delta_{kl} 1 \quad , \quad [p_i, \xi_k] = [q_i, \xi_k] = 0 \quad .$$

Notice that the (complex) algebra generated by n pairs of boson operators b_i, b_i^+ and m pairs of fermion operators a_k, a_k^+ coincides with $W_{2n;2m}$: it is sufficient to choose p_i, q_j similarly as in 2.2 and $\xi_k = 2^{-1/2}(a_k + a_k^+)$, $\xi_{m+k} = i2^{-1/2}(a_k - a_k^+)$. Then, e.g., the main problem of Sec.2 may be interpreted as a search for Schurean realizations of osp(1,2) in the Weyl superalgebra $W_{2;2}$.

3.3 Finally, let us remark that a pure boson representation akin to our PBR was obtained within a different framework in Ref.17 .

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