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Описание ширин и спектров связанных состояний двух релятивистских фермионов

Целью работы является построение формализма для релятивистского описания системы двух частиц со спином 1/2. Используется двухчастичное трехмерное ковариантное уравнение, полученное в квазипотенциальном подходе. Квазипотенциальное уравнение в релятивистском конфигурационном представлении сведено в приближении OBEP к системе разностных парциальных уравнений, которая является релятивистским аналогом нерелятивистской системы Хамады-Джонстона. Методом ВКБ решена задача о спектре масс связанных состояний. С использованием точного релятивистского кулоновского решения двухчастичной задачи получены выражения для ширин лептонных распадов в кварковой модели.

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Descriptions of Width and Spectra of Two Relativistic Fermions Bound States

The aim of this article is the construction of the formalism for the relativistic description of two particles with spin 1/2. We use the two-particle three-dimensional equation, obtained by quasipotential approach. Quasipotential equation in the relativistic configurational space with OBEP potential is reduced to the system of partial equations which is the analog of nonrelativistic Hamada-Jonston system. WKB approach is used to calculate mass spectra and leptonic width of mesons in quark model.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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I. Introduction

The present paper may be treated as a sequel to papers[1-3] on the three-dimensional relativistic description of two-particle system with spin 1/2 in the quantum field theory. Principal formulae, allowing the practical application of our mathematical technique are obtained herein. It will be applicable to such attractive problems as the calculation of levels and width of systems like positronium, muon and baryon as well as bound states in a quark - antiquark system, that is old ρ, ω, φ and new J/ψ, Υ -mesons. Lately a nonrelativistic quark model based on the Schrödinger equation is widly used for the mass spectrum description of hard vector mesons of the J/ψ -particle type. For this purpose a confining potential is chosen and added by the Breit-Fermi potential accounting for spin effects.

$$V(r) = V_{conf}(r) + V_{spin}(r). \tag{1.1}$$

The Breit-Fermi potential is known to represent v^2/c^2 - expansion of Feynman matrix element of one-boson (OBEP) or one-gluon exchange. However, such a quasirelativistic approach applied to ordinary light vector ρ, ω, φ mesons is not self-consistent. It was shown in paper[4] that in this case the contribution of relativistic corrections appears to be of the same order as the contribution of the nonrelativistic Hamiltonian, which was initially taken as the principal one. Therefore an essentially relativistic approach is needed here.

In the present paper for the mass spectrum description of a system consisting of two fermions (for example, quark and antiquark), we use the equation of quasipotential approach /5/. As in previous papers we use the relativistic three-dimensional two-particle quasipotential equation, obtained by Kadyshevsky, on the basis of the covariant Hamiltonian formulation of field theory and later in papers /6/ on the basis of the covariant technique of equating times in the two-particle wave function of Bethe-Salpiter equation. In the case of equal masses $m_1 = m_2 = m$ this equation, written in the centre-of-mass-system ($\vec{p}_1 = -\vec{p}_2 = \vec{p}$) has the form

$$(2E_q - 2E_p)\Psi_{q\sigma_1\sigma_2}(\vec{p}) = (2\pi)^{-3}\int\frac{d\vec{k}}{E_k}\sum_{\sigma_1'\sigma_2'} V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{p},\vec{k};E_q)\Psi_{q\sigma_1'\sigma_2'}(\vec{k}). \qquad (1.2)$$

Now consider in more detail the part of quasipotential (1.1), containing spin effects. In the case of vector-boson exchange the quasipotential in r.h.s. of equation (1.1) represents the Feynman matrix element /1/

$$\langle\vec{p},\vec{\sigma}_1;-\vec{p},\vec{\sigma}_2|T_V^{(2)}|\vec{k},\vec{\sigma}_1;-\vec{k},\vec{\sigma}_2\rangle = g_V^2\,\frac{\bar{u}^{\sigma_1}(\vec{p})\gamma_\mu u^{\sigma_1'}(\vec{k})\,\bar{u}^{\sigma_2}(-\vec{p})\gamma_\mu u^{\sigma_2'}(-\vec{k})}{\mu^2-(\vec{p}-\vec{k})^2} =$$

$$= \sum_{\sigma_{1p}\sigma_{2p}} \xi^*_{\sigma_{1p}}\xi^*_{\sigma_{2p}}\,T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p})\,\xi^{\sigma_{1p}\sigma_{2p}}, \qquad (1.3)$$

μ is the mass of a boson, ξ^{σ} are two-component spinors.

Applying to (1.3) the suggested transformation /1/ from the four-dimensional representation to the three-dimension one [1] and separating the Wigner matrix of spin rotation, we obtain:

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[1] This transformation (1) plays the same role as the Foldy-Wenthuysen one /8/, allowing one to pass to the three-dimensional description of the spin. However, in contrast to (8) the discussed in (1) transformation does not deal with the expansion of interaction terms in powers of v^2/c^2.

$$T_V^{(2)}(\vec{k}(-)\vec{p};\vec{p}) = -\frac{g_V^2}{\mu^2+4\vec{x}^2}\Big[4m^2+4(\vec{\sigma}_1\vec{x})(\vec{\sigma}_2\vec{x})-(\vec{\sigma}_1\vec{\sigma}_2)\,\vec{x}^2+$$

$$+8p_0 x_0\, i(\vec{\sigma}_1+\vec{\sigma}_2)[\vec{p}\times\vec{x}]+\frac{8}{m^2}\left(p_0^2 x_0^2+2p_0 x_0(\vec{p}\vec{x})-m^4\right)+ \qquad (1.4)$$

$$+\frac{8}{m^2}(\vec{\sigma}_1\vec{p})(\vec{\sigma}_2\vec{x})(\vec{\sigma}_2\vec{p})(\vec{\sigma}_1\vec{x})\Big].$$

Here the quantity $\vec{x}$ means a half-transfer of momentum, that can be expressed through the momentum transfer in the Lobachevsky space introduced into /7/:

$$\vec{\Delta} = \vec{k}(-)\vec{p} = \Lambda_p^{-1}\vec{k} = \vec{k} - \frac{\vec{p}}{m}\left(k_0 - \frac{\vec{k}\vec{p}}{p_0+m}\right) \qquad (1.5)$$

by the formula $\vec{x} = \vec{\Delta}\sqrt{m/2(\Delta_0+m)}$ /1/.

Expression (1.4) has a form of the geometrical generalization of the Breit-Fermi potential taken in the momentum representation.

It is convenient to solve equation (1.3) by transformation to the relativistic configurational representation (RCR), introduced initially in /7/. To obtain this, instead of the Fourier-transformation there is used an expansion over the complete and orthogonal system of functions in the Lobachevsky space /9/.

$$\xi(\vec{p};\vec{n},r) = \left(\frac{p_0-\vec{p}\vec{n}}{m}\right)^{-1-irm}$$

$$p_0 = \sqrt{m^2+\vec{p}^2}\,,\quad \vec{n}^2 = 1\,. \qquad (1.6)$$

For these functions in papers /6,12/ we found an expansion over spherical harmonics:

$$\xi(\vec{p};\vec{n},r) = \sqrt{\frac{\pi}{2\,\mathrm{sh}\chi}}\sum_{\ell=0}^{\infty}(2\ell+1)\frac{\Gamma(i\frac{r}{\chi}+\ell+1)}{\Gamma(i\frac{r}{\chi}+1)}P^{-\frac{1}{2}-\ell}_{-\frac{1}{2}+i\frac{r}{\chi}}(\mathrm{ch}\chi)\,P_\ell\left(\frac{\vec{p}\vec{n}}{p}\right) = \qquad (1.7)$$

$$= \sum_{\ell=0}^{\infty}(2\ell+1)\, i^\ell\, p_\ell(\mathrm{ch}\chi, r)\, P_\ell\left(\frac{\vec{p}\vec{n}}{p}\right)$$

as well as the condition of completeness and orthogonality for radial parts

$$\frac{2\,sh\chi\, sh\chi'}{\pi \lambda^3}\int_0^\infty r^2 dr\, \mathcal{P}_\ell(ch\chi, r)\, \mathcal{P}^*_\ell(ch\chi', r) = \delta(\chi-\chi')$$

$$\frac{2 r r'}{\pi \lambda^3}\int_0^\infty sh^2\chi\, d\chi\, \mathcal{P}_\ell(ch\chi, r)\, \mathcal{P}^*_\ell(ch\chi, r') = \delta(r-r'). \qquad (1.8)$$

In /2,3/ there was discussed a problem on the form of the second and third terms (1.4) in the relativistic configurational representation. There are transforms of all the five spin structures present in (1.4) in this paper.

Equation (1.2) transformed into RCR is a finite-difference one and may be reduced to the system of partial equations, being a relativistic analog of the nonrelativistic Hamada-Jonston system /11/.

To solve partial finite-difference equations we use the suggested analog /12/ of WKB method for RCR.

Calculation of the electromagnetic decay width of mesons is another important problem. To determine lepton width of vector mesons we use the V.Royen-Weisscopff formula (with account of colour).

$$\Gamma_{V\to e^+e^-} = 16\pi\alpha^2 M_V^{-2} e_q^2 |\Psi_{n\ell}(0)|^2, \qquad (1.9)$$

and for the two-photon decay of pseudoscalar mesons we use formula /14/

$$\Gamma_{P\to\gamma\gamma} = 12\pi\alpha^2 m^{-2} e_q^4 |\Psi_{n\ell}(0)|^2. \qquad (1.10)$$

In these formulae M_V is the mass of a meson; m and e_q are the mass and charge of quarks, and $\Psi_{n\ell}(0)$ is WF of system of a quark and anti-quark. Usually instead of this WF the nonrelativistic WF is used in the Coulomb field neglecting the rest of the spin part of the OBEP. The width of states with $\ell \neq 0$ in terms of the relativistic theory is known to turn into zero due to the behaviour of the WF at the origin of coordinates as $\Psi^{nonrel}_\ell(r) \xrightarrow[r\to 0]{} r^\ell$

This behaviour is of a kinematical character and is connected with the behaviour of free solutions $j_\ell(\kappa r) \to (\kappa r)^\ell$ for $r \to 0$. In a relativistic equation there is used another complete system of functions (1.6) having an essentially different behaviour as $r \to 0$, so that $\Psi^{rel}_\ell(0) \neq 0$ at $\ell \neq 0$.

In section 2 a quasipotential equation will be transformed to the system of partial finite-difference equations, in the relativistic configurational representation; section 3 is devoted to the solution of unbounded equations of the system through the WKB method; section 4 gives formulae for the width of vector meson decays and pseudoscalar mesons, using the exact Coulomb solution of the quasipotential equation at $r=0$.

2. System of Partial Two-Particle Equations in the Relativistic Configurational Representation

For the quasipotential and WF the transformation into the relativistic configurational representation is performed by formulae:

$$V(r,\vec{n};\vec{p}) = (2\pi)^{-3}\int d\Omega_\Delta\, \xi^*(\vec{\Delta};\vec{r})\, V(\vec{\Delta};\vec{p}), \qquad (2.1)$$

$$\Psi_{G_1G_2}(r) = (2\pi)^{-3}\int d\Omega_p\, \xi(\vec{p};\vec{n},r)\, \Psi_{G_1G_2}(\vec{p}). \qquad (2.2)$$

In /15/ the physical meaning of the parameter r present in the function is discussed in detail.

Applying (2.1) to (1.4) we obtain:

$$V(r,\vec{n};\vec{p}) = V_1(r,p_0) + V_2(r,\vec{n};p_0,\vec{p}) \qquad (2.3)$$

$$V_1(r,p_0) = -g_V^2\Big\{(2p_0^2-1)\Big[V_{KUK}(r) + \frac{16\pi\,\delta(r^2+1)}{r}\,\delta(\vec{n})\Big] + \Big[\frac{\mu^2}{3}V_{KUK}(r) - \frac{8\pi}{3}\,\frac{\delta(r^2+1)}{r}\,\delta(\vec{n})\Big]\cdot\Big[(\vec{\sigma}_1\vec{\sigma}_2)-(\vec{S}\vec{p})^2+\vec{p}^{\,2}\vec{S}^{\,2}\Big]\Big\} \qquad (2.4)$$

$$V_2(r,\vec{n};p_0,\vec{p}) = -g_V^2\Big\{(\vec{S}\vec{L})\frac{4}{r}\big[p_0 A(r) + (\vec{p}\vec{n})\varphi(r)\big] - 4ip_0(\vec{p}\vec{n})A(r) + \Big[S_{1,2} - (\vec{S}\vec{L})^2\frac{4}{r^2}\Big]\varphi(r)\Big\}. \qquad (2.5)$$

The following notations are introduced here:

$$\vec{S}=\frac{\vec{\sigma}_1+\vec{\sigma}_2}{2};\ \vec{L}=[\vec{p}\times\vec{r}];\ S_{12}=3(\vec{\sigma}_1\vec{n})(\vec{\sigma}_2\vec{n})-(\vec{\sigma}_1\vec{\sigma}_2)\ . \tag{2.6}$$

$$A(r)=\frac{1}{r(r-i)}(r\,\mathrm{th}\,r\alpha\,\sin\alpha+\cos\alpha)V_{Yuk}(r) \tag{2.7}$$

$$\varphi(r)=\frac{r^2}{3(r-i)(r-2i)}\Big[\mu^2+3\frac{\mu}{r}\Big(1-\frac{\mu^2}{2}\Big)\frac{\mathrm{th}\,r\alpha}{\sqrt{1-\frac{\mu^2}{4}}}+ \tag{2.8}$$

$$+\frac{3-2\mu^2\big(1-\frac{\mu^2}{4}\big)-\frac{3}{2\,\mathrm{ch}\,r\alpha}}{1-\frac{\mu^2}{4}}\Big]V_{Yuk}(r)$$

$V_{Yuk}(r)$ - the Yukawa potential transform: $4/(\mu^2+4\vec{\varkappa}^2)$

$$V_{Yuk}(r)=\frac{1}{4\pi r}\frac{\mathrm{ch}\,r\alpha}{\mathrm{sh}\,r\pi},\ \alpha=\arccos\frac{\mu^2-2}{2}\ (\mu^2<4). \tag{2.9}$$

In formulae (2.3)-(.29) we assume the mass of interacting particles m to be equal to 1.

Formula (2.3) is splitted into two parts $V_1(r,p_0)$ independent of the single vector "relativistic coordinate" direction $\vec{n}$ and $V_2(r,\vec{n};p_0,\vec{p})$ containing the dependence on $\vec{n}$ through expressions of the type

$$(\vec{p}\vec{n})\ ,\ [\vec{p}\times\vec{n}]\ ,\ S_{12}\ . \tag{2.10}$$

The necessity of such a splitting becomes obvious in the course of the transformation of the complete equation (1.2) into RCR using formulae (2.1), (2.2).

$$(2E_q-2\hat{H})\Psi_{\sigma_1\sigma_2}(\vec{r})=\int d\Omega_p\,d\Omega_k\,d\vec{r}_1\,\xi(\vec{p};\vec{n},\varkappa)\times$$

$$\times\xi^*(\vec{k};\vec{n}_1,r_1)\sum_{\sigma_1'\sigma_2'}V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(\vec{\Delta},\vec{p},E)\Psi_{\sigma_1'\sigma_2'}(\vec{r}_1) \tag{2.11}$$

$$\hat{H}=\mathrm{ch}\Big(i\lambda\frac{d}{dr}\Big)+i\frac{\lambda}{r}\mathrm{sh}\Big(i\lambda\frac{d}{dr}\Big)+\frac{\Delta_{\theta\phi}}{(r/\lambda)^2}\exp\Big(i\lambda\frac{d}{dr}\Big) \tag{2.12}$$

$\Delta_{\theta\phi}$ - angular part of the Laplas operator, $\lambda=\frac{\hbar}{mc}$. Due to the character of the addition theorem of the relativistic plane waves [16]:

$$\xi(\vec{k};\vec{n},r)=\xi(\vec{k}(-)\vec{p};\vec{n}_{\Lambda_p},r)\,\xi(\vec{p};\vec{n},r) \tag{2.13}$$

$$\vec{n}_{\Lambda_p}=\Big[m\vec{n}-\vec{p}\Big(1-\frac{\vec{p}\vec{n}}{p_0+m}\Big)\Big]/(p_0-\vec{p}\vec{n}) \tag{2.14}$$

$V_2(r,\vec{n};p_0,\vec{p})$ enters into equation (2.12) in a nonlocal way:

$$(2E_q-2\hat{H})\Psi_{\sigma_1\sigma_2}(\vec{r})=\sum_{\sigma_1'\sigma_2'}V^{\sigma_1'\sigma_2'}_{1\,\sigma_1\sigma_2}(r,p)\Psi_{\sigma_1'\sigma_2'}(\vec{r})+ \tag{2.15}$$

$$+\int d\vec{r}_1\sum_{\sigma_1'\sigma_2'}d\Omega_p\,\xi(\vec{p};\vec{n},r)\xi^*(\vec{p};\vec{n}_1,r_1)V^{\sigma_1'\sigma_2'}_{\sigma_1\sigma_2}(r_1,\vec{n}_{1\Lambda_p};\vec{p})\Psi_{\sigma_1'\sigma_2'}(\vec{r}_1)$$

since substituting it into the equation the dependence on vector $\vec{n}_1$ turns to the dependence on vector $\vec{n}_{1\Lambda_p}$. The vector rotation $\vec{n}_1\to\vec{n}_{1\Lambda_p}$ taken into account gives the following dependence of expressions (2.10) on the vector $\vec{p}$ and p_0 :

$$(\vec{p}\,\vec{n}_{1\Lambda_p})=m^2/(p_0-\vec{p}\vec{n}_1)-p_0 \tag{2.16}$$

$$[\vec{p}\,\vec{n}_{1\Lambda_p}]=m[\vec{p}\times\vec{n}_1]/(p_0-\vec{p}\vec{n}_1) \tag{2.17}$$

$$(\vec{\sigma}_1\vec{n}_{1\Lambda_p})(\vec{\sigma}_2\vec{n}_{1\Lambda_p})=Z_1^T+Z_2^T \tag{2.18}$$

$$Z_1^T=m^2\cdot(\vec{\sigma}_1\vec{n}_1)(\vec{\sigma}_2\vec{n}_1)/(p_0-\vec{p}\vec{n}_1)^2$$

$$Z_2^T=\frac{m^2}{(p_0-\vec{p}\vec{n}_1)^2}\Big\{-\frac{1}{m}\big[(\vec{\sigma}_1\vec{n}_1)(\vec{\sigma}_2\vec{p})+(\vec{\sigma}_1\vec{p})(\vec{\sigma}_2\vec{n}_1)\big]\Big(1-\frac{\vec{p}\vec{n}_1}{p_0+m}\Big)+$$

$$+\frac{1}{m^2}(\vec{\sigma}_1\vec{p})(\vec{\sigma}_2\vec{p})\Big(1-\frac{\vec{p}\vec{n}_1}{p_0+m}\Big)^2\Big]$$

However, with the help of the correlation:

$$\exp\left(i\frac{1}{m}\frac{d}{dr_1}\right)\xi^*(\vec{p};\vec{n}_1,r_1)=\frac{m}{p_0-\vec{p}\vec{n}_1}\xi^*(\vec{p};\vec{n}_1,r_1) \qquad (2.19)$$

expressions (2.16), (2.17) and the first term Z_1^T from (2.18) can be localazed. Z_1^T is a relativistic generalization of the expression $(\vec{\sigma}_1\vec{n})(\vec{\sigma}_2\vec{n})$ present in the operator of tensor forces. Confining ourselves to Z_1^T in accounting tensor forces we obtain the r.h.s. of equation (2.15) to be of a local form:

$$(2E-2\hat{H})\Psi_{\sigma_1\sigma_2}(\vec{r})=\sum_{\sigma_1'\sigma_2'}\hat{V}_{\sigma_1\sigma_2}^{\sigma_1'\sigma_2'}(\vec{r},\vec{p})\Psi_{\sigma_1'\sigma_2'}(\vec{r}) \qquad (2.20)$$

The rotation $\vec{n}\to\vec{n}_{\Lambda p}$ in tensor forces taken into account gives an additional spin-spin interaction

$$\varphi(r)\left[3(\vec{\sigma}_1\vec{n}_{1\Lambda p})(\vec{\sigma}_2\vec{n}_{1\Lambda p})-(\vec{\sigma}_1\vec{\sigma}_2)\right]\to$$
$$\to\varphi(r-2i\lambda)S_{12}+\left[\varphi(r-2i\lambda)-\varphi(r)\right](\vec{\sigma}_1\vec{\sigma}_2), \qquad (2.21)$$

An angular and spin dependences of equation potential (2.20) are determined by five structures, having the same form as in the nonrelativistic theory:

$$\hat{V}(\vec{r},\vec{p})=V_S+V_T S_{12}+V_{LS}(\vec{L}\vec{S})+V_G(\vec{\sigma}_1\vec{\sigma}_2)+V_{(LS)^2}(\vec{S}\vec{L})^2 \qquad (2.22)$$

Functions $V_i(\vec{r})$ $(i=S, S_{12}, LS, G, (LS)^2)$ depend on vector modulus $\vec{r}$ and difference operators $\hat{\vec{p}}$ and $\hat{p}_0=\hat{H}$ in the following way $(\lambda=1)$:

$$V_S=(2p_0^2-1)\left[V_{YUK}(r)+\frac{16\pi}{r}\delta(r^2+1)\right]-$$
$$-4ip_0\left[\left(\frac{r-i}{r}\right)^2\exp\left(-i\frac{d}{dr}\right)-p_0\right]A(r) \qquad (2.23)$$

$$V_T=\left(\frac{r-2i}{r}\right)^2\exp\left(-2i\frac{d}{dr}\right) \qquad (2.24)$$

$$V_{LS}=4p_0\frac{r}{(r+i)^2}A(r)+\left[\frac{r(r-i)}{(r+i)^2}\exp\left(-i\frac{d}{dr}\right)-p_0\right]\frac{r}{(r+i)^2}\varphi(r) \qquad (2.25)$$

$$V_G=\left[\left(\frac{r-2i}{r}\right)^2\exp\left(-2i\frac{d}{dr}\right)-1\right]\varphi(r)-f\frac{\mu^2}{3}V_{YUK}(r)-\frac{8}{3}\frac{\pi}{r}\delta(r^2+1) \qquad (2.26)$$

$$V_{(LS)^2}=-\frac{4r}{(r+i)(r+2i)}\varphi(r) \qquad (2.27)$$

We introduce a radial function $R^j_{e's'es}(r)$

$$\Psi^S_{q,\sigma_1,\sigma_2}(\vec{r})=\sum_{jeMe'M'}R^j_{e's'es}(r)\left\{\Omega^{*S}_{jeM}(\vec{n})\right\}\left\{\Omega^{S}_{je'M'}(\vec{n})\right\}_{\sigma_1\sigma_2} \qquad (2.28)$$

The substitution of (2.28) into equation (2.27) gives us a system of four partial equations. Two of the system of equations, obeying the case of $S=0$, $e'=j$ and $S=1$, $e'=j$ are uncoupled:

$$\left[2\hat{H}_e-2E_{q,e}+V^S_{e',s}(r)\right]R^S_{e'}(r)=0\,;\; e'=j\,;\; S=0,1. \qquad (2.29)$$

$$V^{S=0}_{e'=j}(r)=-g_v^2(V_S-3V_G) \qquad (2.30)$$

$$V^{S=1}_{e'=j}(r)=-g_v^2(V_S+2V_T+V_G-V_{LS}+V_{(LS)^2}) \qquad (2.31)$$

Hamiltonian $\hat{H}'_e$ is obtained from expression (2.12) for the $\hat{H}$ through the substitution of the operator $\Delta_{\theta\phi}$ for its eigenvalue $e(e+1)$. The other two WF describing states with a spin $S=1$ are defined by the system of two coupled equations

$$\begin{cases} (2E_q - 2H_{e'=j+1}) R_{e'=j+1}(r) = V_{e'=j+1}(r)\, R_{e'=j+1}(r) + \tilde{V}_T(r) R_{e'=j-1}(r) \\ (2E_q - 2H_{e'=j-1}) R_{e'=j-1}(r) = V_{e'=j-1}(r)\, R_{e'=j-1}(r) + \tilde{V}_T(r) R_{e'=j+1}(r) \end{cases} \qquad (2.32)$$

$$V_{e'=j+1}(r) = g_v^2 \left[V_S - (j+2) V_{LS} + (j+2)^2 V_{(LS)^2} - \frac{2(j+1)}{2j+1} V_T + V_\sigma \right] \qquad (2.33)$$

$$V_{e'=j-1}(r) = g_v^2 \left[V_S + (j-1) V_{LS} + (j-1)^2 V_{(LS)^2} - \frac{2(j-1)}{2j+1} V_T + V_\sigma \right] \qquad (2.34)$$

$$\tilde{V}_T(r) = -g_v^2 \, \frac{6\sqrt{j(j+1)}}{2j+1} V_T(r) \,. \qquad (2.35)$$

Writing down system (2.29), (2.32) we follow the approximation on the phenomenological use of the OBEP, accepted in paper /11/ we neglect the $\vec{P}$ quadratic terms in the spin-spin interaction $\vec{P}^2 \vec{S}^2 - (\vec{P}\vec{S})^2$ and confine ourselves to the account of the terms proportional to $m^2 (\vec{\sigma}_1 \vec{\sigma}_2)$. In expressions (2.24)-(2.27), as it has already been noticed, p_0 is understood as a difference operator $\hat{H}$. However, in obtaining numerical evaluations the reasonable approximation which is not reduced to the nonrelativistic one is the use, instead of the operator $\hat{H}$, of the eigenvalue of energy p_0 (as in the problem of a positronium /17/) of one bound partical in the part of potential (1.1), consisting of the sum $V_{conf}(r)$ and the scalar part of potential (1.4): $V = \frac{4m^2}{\mu^2 + 4\bar{x}^2}$.

3. Quasiclassical Solution of Partial Equations

We write down uncoupled equations (2.25) in such a form:

$$[\hat{H}_e - X(r)]\, R_e(r) = 0 \,, \qquad (3.1)$$

where $X(r) = (2m + E_{bound} - V(r))/2m$,

and the potential $V(r)$ is defined through expressions (2.30), (2.31).

Regular in the zero free solution $(V(r) = 0)$ of equation (3.1), according to /7/ has the form:

$$\psi^{free}_e(r,\chi) = \sqrt{\tfrac{\pi}{2} sh\chi}\, (-1)^{e+1} \frac{(-r/\chi)^{(e+1)}}{r} P^{-\frac{1}{2}-e}_{-\frac{1}{2}+i r/\chi}(ch\chi) \qquad (3.2)$$

$$ch\chi = (2m + E_{bound})/2m \qquad (3.3)$$

where $P^{\mu}_{\nu}(ch\chi)$ is the Legendre function and $(-r/\chi)^{(e+1)}$ is the generalized degree defined by the expression: $(-\frac{r}{\chi})^{(e+1)} = i^{(e+1)} \frac{\Gamma(i\frac{r}{\chi}+e+1)}{\Gamma(i r/\chi)}$.

The Legendre function $P^{-\frac{1}{2}-e}_{-\frac{1}{2}+i r/\chi}(ch\chi)$ is real. Thus in expression (3.2) only the factor $\frac{(-r/\chi)^{(e+1)}}{r}$ appears to be complex.

We separate this factor by the substitution:

$$\psi_e(r,\chi) = \frac{(-r/\chi)^{(e+1)}}{r} K_e(r,\chi). \qquad (3.4)$$

We obtain the following equation for the function $K_e(r,\chi)$:

$$\left[2\,ch\!\left(i\chi\frac{d}{dr}\right) - \frac{2(e+1)}{i r \chi}\, sh\!\left(i\chi\frac{d}{dr}\right) - X(r)\right] K_e(r,\chi) = 0. \qquad (3.5)$$

Note, that the Hamiltonian of the equation (3.5) in contrast to (3.5) appears to be a real operator. In the case of the potential is real, $K_e(r,\chi)$ is also a real function. When the interaction is absent, the Legendre function appears to be a solution of equation (3.5)

$$K_e(r,\chi) = P^{-\frac{1}{2}-e}_{-\frac{1}{2}+i r/\chi}(ch\chi) = \left(\frac{sh\chi}{2}\right)^{e+\frac{1}{2}} \frac{\exp[-\chi(i\frac{r}{\chi}+e+1)]}{\Gamma(3/2+e)} \times$$
$$\times F\left(i\frac{r}{\chi}+e+1,\, e+1;\, 2e+2;\, 1-\exp(-2\chi)\right). \qquad (3.6)$$

The conditions of the orthogonality for these functions will be written down analogously to (1.6b):

$$\frac{1}{\lambda}\int_0^\infty sh^2x\,dx\,\overset{*}{P}{}^{-\frac12+\ell}_{-\frac12+i\frac{r}{\lambda}}(chx)\;P^{-\frac12-\ell}_{-\frac12+i\frac{r'}{\lambda}}(chx) = \frac{\delta(r-r')}{\left(-\frac{r}{\lambda}\right)^{*(\ell+1)}\left(-\frac{r'}{\lambda}\right)^{(\ell+1)}} \tag{3.7}$$

$$\frac{1}{\lambda}\int_0^\infty dr\,\overset{*}{P}{}^{-\frac12-\ell}_{-\frac12+i\frac{r}{\lambda}}(chx)\;P^{-\frac12-\ell}_{-\frac12+i\frac{r}{\lambda}}(chx')\left[\left(-\frac{r}{\lambda}\right)^{(\ell+1)}\right]^{*}\left(-\frac{r}{\lambda}\right)^{(\ell+1)} = \frac{\delta(x-x')}{\sqrt{shx\,shx'}}.$$

We solve expressions (3.5) by the WKB method. Suppose:

$$K_\ell(r)=\exp\frac{i}{\hbar}g(r);\quad g(r)=g_0(r)+\frac{\hbar}{i}g_1(r)\dots \tag{3.8}$$

Inserting (3.8) into (3.5) and keeping terms of the zero order in $\hbar$, we derive a differential equation for $g_0(r)$:

$$\frac{\lambda}{\hbar}g_0'(r)=\mathrm{arcch}\,X_\Lambda(r)-i\,\mathrm{arctg}\,\Lambda\frac{\lambda}{r}$$

$$X_\Lambda(r)=X(r)\cdot\left[1+\left(\Lambda\frac{\lambda}{r}\right)^2\right]^{-1/2},\quad \Lambda=\ell+1. \tag{3.9}$$

The imaginary part in (3.9) contributes into the preexponential factor. With the account of the term of the order $\hbar$, we arrive at the equation for $g_1'(r)$:

$$g_1'(r)=-\frac{X_\Lambda'(r)+i\frac{\lambda\Lambda}{r^2+\lambda^2\Lambda^2}}{2\left(X_\Lambda^2(r)-1\right)}X_\Lambda(r). \tag{3.10}$$

Thus we obtain for the function $K_\ell(r)$:

$$K_{q\ell}(r)=\left[X_\Lambda^2(r)-1\right]^{-1/4}\left(r^2/\lambda^2+\Lambda^2\right)^{-\frac{\Lambda}{2}}\times$$
$$\times\exp\frac{i}{\lambda}\int_{r_-}^{r}dr'\left[\mathrm{arcch}\,X_\Lambda(r')-\frac{X_\Lambda(r')}{2[X_\Lambda(r')-1]}\cdot\frac{\lambda\Lambda}{r'^2+\lambda^2\Lambda^2}\right]. \tag{3.11}$$

Here r_- is a turning point, defined from the condition $X_\Lambda(r_\pm)=1$. In the case $r\gtrsim r_-\gg\lambda$ that is characteristic of energies

$E\ll 2mc^2$ [1] formula (3.11) gets simplified. We write it down in the form of a standing wave

$$K_{q\ell}(r)_{r\gg\lambda}=\left[X_\Lambda^2(r)-1\right]^{-1/4}\left(\frac{r}{\lambda}\right)^{-\ell-1}\sin\frac{1}{\lambda}\int_{r_-}^{r}dr'\,\mathrm{arcch}\,X_\Lambda(r'). \tag{3.12}$$

The factor $r^{-\ell-1}$ compensates the separated expression $\left(\frac{r}{\lambda}\right)^{(\ell+1)}_{r\to\infty}\to(r/\lambda)^{\ell+1}$. With the help of (3.12) we obtain a quasi-classical condition of quantization

$$\int_{r_-}^{r_+}dr'\,\mathrm{arcch}\,X_\Lambda(r')=\lambda\pi\left(n+\frac12\right). \tag{3.13}$$

For the majority of potentials a spectrum is formed at the distances $r\gg\lambda$. This allows us to neglect imaginary terms in $V_i(r)$ (2.23-2.27). In such an approach, potentials (2.30, 2.31) become real and we can find energy levels for a singlet $(S=0)$ and triplet $(S=1,\ \ell=j)$ states.

4. Calculation of Electromagnetic Decay Width of Meson

To illustrate the use of quasi-classical formulas, obtained in §3, we calculate the $|\Psi(0)|^2$ value of WF quadratic at $r=0$ [2] that is present in formulas for electromagnetic width of mesons (1.9), (1.10). In the case of the massless-particle (gluon) exchange to the first term of the quasi-potential (1.3) $V=-g_V^2\frac{4m^2}{\vec{\Delta}^2}$ there corresponds the potential $V(r)=\frac{cth\,(rm\pi)}{r}$ in the r-space. There have been obtained in (19) exact solutions and a spectrum in the field of such a potential. The function $cth(rm\pi)$ changes essentially only at a distance of the order of λ from the origin of coordinate and, as is seen in /7/, its presence does not influence the form of the spectrum.

[1] For the free motion, for example: $\frac{r_-}{\lambda}\approx\frac{2mc^2\Lambda}{E}\gg 1$.

[2] Note $r=0$ means that particles are at the Compton wave length distance /15/. A possibility of substitution of nonrelativistic WF in (1.9) for $|\Psi^{nonrel}(\lambda)|^2$ has been discussed in /15/.

Consider a potential, being a combination of the OBEP transform in RCR, taking into account only the Coulomb part and a confining potential. The value of energy levels is in general influenced by the potential behaviour at large distances $r \gg \lambda$. In this region we may neglect a hypergoemetrical factor in the Coulomb potential

$$V(r) = -\frac{\alpha}{r} + V_{conf}(r), \qquad (4.1)$$

at small r a Coulomb term dominates (region I at the Figure). In this region a WF $\Psi(r)$ coincides with the exact regular solution at $r=0$ in the potential $-\alpha/r$:

$$\Psi_\ell(r) = C_\ell \frac{\Gamma(i\frac{r}{\lambda}+\ell+1)}{r\,\Gamma(i\frac{r}{\lambda})} K_\ell(r) \qquad (4.2)$$

$$K_\ell(r) = \exp(-i\tfrac{r}{\lambda} - i\chi\varkappa)\, F(\ell+1+\tfrac{ir}{\lambda};\, \ell+1+i\varkappa,\, 2\ell+2;\, 1-e^{-2\chi}) \qquad (4.3)$$

$$\varkappa = \alpha/2\,\mathrm{sh}\chi,$$

where C_ℓ is an unknown normalizing factor. This factor enters into the definition $\Psi(0)$.

To obtain this we consider an asymptotics of the exact solution (4.2) at large r, written in the form of a standing wave

$$\Psi_\ell(r) \xrightarrow[r\to\infty]{} C_\ell \frac{2\Gamma(2\ell+2)\exp[-\frac{\pi}{2}\varkappa+(\ell+1)\chi]}{(2\,\mathrm{sh}\chi)^{\ell+1}\, Re[\Gamma(\ell+1-i\varkappa)]} \times \qquad (4.4)$$

$$\times \frac{1}{r}\sin[\chi r + \varkappa\ln(2r\,\mathrm{sh}\chi) + \eta_\ell - \frac{\pi}{2}\ell],$$

where η_ℓ is the relativistic Coulomb phase:

$$\eta_\ell = \arg\Gamma(\ell+1-i\varkappa). \qquad (4.5)$$

The Coulomb phase can be also calculated through the WKB method:

$$\eta_\ell^{WKB} = \frac{1}{\lambda}\lim_{r\to\infty}\Big[\int_{r_-}^{r}\mathrm{arcch}\, X_\Lambda^{coul}(r')dr' - \int_{r_-}^{r}\mathrm{arcch}\, X_\Lambda^{free}(r')dr'\Big]. \qquad (4.6)$$

Assume the Coulomb constant to be small $\alpha \ll 1$. This is characteristic for the description of the J/Ψ and Υ -particles in the potential model /20/ and agrees with the hypothesis of asymptotic freedom. Further with the neglect of the terms of the first order in α and by integration in (4.6) we arrive at the following result:

$$\eta_\ell^{WKB} = \varkappa - \varkappa\ln\sqrt{\varkappa^2+\Lambda^2} - \Lambda\arcsin\frac{\varkappa}{\sqrt{\varkappa^2+\Lambda^2}}. \qquad (4.7)$$

The forms of expressions (4.5) and (4.7) coincide with their nonrelativistic analogs. The difference is that $\varkappa$ is determined by relation (4.3), that in the nonrelativistic limit turns into a dimensionless quantity, used in the Schrödinger equation $\varkappa \xrightarrow[c\to\infty]{} \frac{e^2}{\hbar v}$. Phases η_ℓ and η_ℓ^{WKB} for $\varkappa \gg 1$ differ only by terms of the order $\varkappa^{-1}$ /21/. Hence in the Coulomb field the exact solution $\Psi^{coul}_{(r)}$ and the quasiclassical wave function $\Psi^{WKB}_{(r)}$ coincide up to a factor in the region of large r (range II, figure 1) under the condition $\alpha \ll 1$, $\varkappa \gg 1$. If we can neglect a confining part of the potential (4.1) at such a large r, then in region II it is possible to equate an exact Coulomb solution (it is true in regions II and III /22/) and quasi-classical solution obtained for regions II and III /22/. Thus, we obtain the following normalizing constants:

$$C_\ell\, 2\Gamma(2\ell+2)\frac{\exp[-\frac{\pi}{2}\varkappa+(\ell+1)\chi]}{(2\,\mathrm{sh}\chi)^{\ell+1}} = \frac{C^{WKB}}{[X_\Lambda^2(r)-1]^{1/4}}. \qquad (4.8)$$

The constant C^{WKB} is derived in the same way as in nonrelativistic case

$$\int |R_{n\ell}(r')|^2 r'^2 dr' = \frac{|C^{WKB}|^2}{2}\int_{r_-}^{r_+}\frac{dr'}{\sqrt{X_\Lambda^2(r')-1}} = 1. \qquad (4.9)$$

Integral in (4.10) is easily calculated by differentiation of quantization conditions (3.13) over n :

$$|C^{WKB}|^2 \approx \frac{2}{\pi\lambda} \frac{dX_{n,\ell}}{dn} \; ; \; X_{n,\ell} = (2m+E_{n\ell})/2m \, . \tag{4.10}$$

Calculating to exact Coulomb WF in the limit of $r \to 0$ and obtaining C_ℓ^+ value from (4.8) and (4.10), we finally obtain:

$$|\Psi_{n\ell}(0)|^2 = \lambda^{-3} \prod_{\ell'=0}^{\ell} (\ell'^2 + \varkappa^2) \times \frac{\exp \pi\varkappa}{4\varkappa \, sh\, \pi\varkappa} \left| P^{-\frac{1}{2}-\ell}_{-\frac{1}{2}+i\varkappa}(ch\chi) \right|^2 \frac{dX_{n,\ell}}{dn} \, . \tag{4.11}$$

This expression differs from zero even for states with $\ell \neq 0$. So, using (4.11) we can apply formulae (1.9), (1.10) to the description of electromagnetic width of states with any values of ℓ .

In this paper a system has been obtained of partial equations describing two particles with spin 1/2, interacting through the one-vector-boson exchange. There have been solved uncoupled equations of the system by WKB method for a wide class of confining potentials and a condition of quantization and the expression for $|\Psi_{n\ell}(0)|^2$ have been found. These results can be applied to the calculation of mass spectrum and width of electromagnetic decays of systems of $e^+e^-, \mu^+\mu^-, c\bar{c}, b\bar{b}, N\bar{N}$ type.

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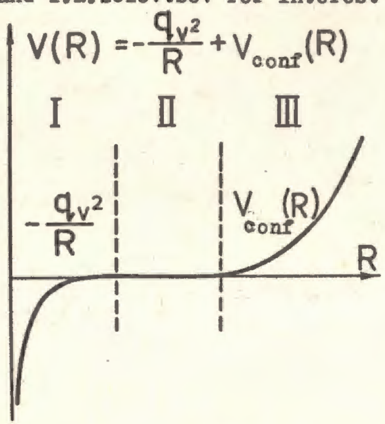

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