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Боголюбский И.Л.

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Спектр масс в модели барионов в рамках $1/N$ разложения нерелятивистской КХД

Исследуются сферически-симметричные решения уравнения Шредингера с нелокальным потенциалом, полученного Виттеном в рамках $1/N$ разложения квантовой хромодинамики в предположении, что кварки имеют одинаковый "аромат" и попарно притягиваются по закону Кулона. С помощью ЭВМ вычислен спектр энергий связанных состояний, число которых, по-видимому, счетно, с собственными значениями, сгущающимися вблизи нуля. Получена зависимость масс соответствующих "барионов" от величины эффективной константы взаимодействия.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

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Bogolubsky I.L.

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The Mass Spectrum in the Baryon Model in $1/N$ Expansion of Nonrelativistic QCD

The spherically symmetric solutions are investigated for the Shrödinger equation with the nonlocal potential that has been obtained by Witten within the framework of $1/N$ expansion of nonrelativistic QCD assuming that all the N quarks inside the baryon are of the same flavour and the Coulomb attraction acts between each two of N quarks. The energy spectrum of the bound states has been found, their number is apparently denumerable and the eigen values ϵ_n condense in the vicinity of zero. The masses of the corresponding "baryons" have been found as a functions of the effective coupling constant value.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

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1. The great mathematical complexity of quantum chromodynamics (QCD) has stimulated the search for small parameter, which is absent in QCD in explicit form. The so-called $1/N$ expansion suggested by 't Hooft /1/ (i.e., an investigation of the limit $N \to \infty$ with N characterizing the gauge group $SU^c(N)$) proved to be rather effective, $1/N$ is considered as a small parameter when N is large. Using $1/N$ expansion, Witten siggested an approach for qualitative description of baryons, which were described here by nonanalytic in $1/N$ solutions and hence interpreted as solitons, in contrast with mesons, which can be considered within the framework of the perturbation theory in small parameter $1/N$. The main supplementary simplifications made in ref. /2/ to obtain the basic equations are the following:

1) using of nonrelativistic approximation;
2) consideration only of baryons composed of one fixed flavour quarks, e.g., Δ^{++} or Ω^-;
3) taking into consideration only of pair quark interactions with Coulomb attractive potential (the attraction is a consequence of the baryon state antisymmetry in colour $SU(N)$ indices).

In $N \to \infty$ limit the Hartree-Fock approximation has been used in ref. /2/. An equation for one-particle wave function $\phi(x)$ has been obtained (it is independent of N) assuming that N-particle wave function $\psi(x_1, \ldots, x_N) = \prod_{i=1}^{N} \phi(x_i)$:

$$-\frac{\nabla^2 \phi}{2M} - g^2 \phi(x) \int \frac{d^3 y\, \phi^* \phi(y)}{|x-y|} = \epsilon \phi(x) \qquad (1)$$

with the normalization condition $\int d^3 x \phi^* \phi(x) = 1$. Here M is the quark mass, g is an effective coupling constant.

Eq. (1) may be transformed into the differential form (in the spherically symmetric case this transformation does not lead to the irrelevant solution emergence):

$$-\frac{1}{2M}\nabla^2(\phi^{-1}\nabla^2\phi)+4\pi g^2\phi^*\phi=0. \tag{2}$$

The localized solution $\phi(\mathbf{x})$ given, the mass of corresponding N quarks bound state is determined:

$$H=N[M+\int\frac{d^3x|\nabla\phi|^2}{2M}-\frac{1}{2}g^2\iint\frac{d^3x d^3y\,\phi^*\phi(\mathbf{x})\,\phi^*\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}]. \tag{3}$$

Let us introduce the dimensionless variables ϕ_d, $\mathbf{x}_d$, ϵ_d:

$$\phi_d=\phi(8\pi g^2M)^{-3/2},\ \mathbf{x}_d=\mathbf{x}\cdot 8\pi g^2M,\ \epsilon_d=\epsilon/8\pi g^4M. \tag{4}$$

Eqs. (1), (2) in these variables have the following form, correspondingly:

$$-4\pi\nabla^2\phi(\mathbf{x})-\phi(\mathbf{x})\int\frac{d^3y\,\phi^*\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|}=\epsilon\phi(\mathbf{x}), \tag{5}$$

$$\nabla^2(\phi^{-1}\nabla^2\phi)=\phi^*\phi \tag{6}$$

(here and further "d" is omitted), and the energy of the bound state with the wave function $\phi(\mathbf{x})$ is:

$$H=NM[1+8\pi g^4(4\pi\int d^3x|\nabla\phi|^2-\frac{1}{2}\iint\frac{d^3xd^3y\,\phi^*\phi(\mathbf{x})\phi^*\phi(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|})]=NM(1+8\pi g^4\epsilon). \tag{7}$$

To obtain ϵ we use the relationship following from equation (5):

$$\epsilon=-4\pi(\phi^{-1}\nabla^2\phi)_{\mathbf{x}=0}-\int\frac{d^3y\,\phi^*\phi(\mathbf{y})}{|\mathbf{y}|}. \tag{8}$$

2. To investigate an asymptotic form of possible spherically symmetric solutions of Eq. (6), we assume that at $r\to\infty$ $\phi\simeq A\exp(-kr)$. Substituting this expression into (6) we obtain that $\phi\simeq 2k^2\exp(-kr)\exp(i\psi)$ at $r\to\infty$. Further we shall deal only with real solutions $\phi(r)$, so $\psi=0$.

We shall essentially use further the following important property of the localized solutions of Eq. (6) (without the normalization condition), namely: if $\phi(r)$ is the solution of (6), then $\phi_B(r)=B\phi(\sqrt{B}r)$ is the solution of (6), too. Using this fact, one may obtain the normalized solution of (6) $\eta(r)$, $\int\eta^2\cdot 4\pi r^2dr=1$ if some solution $\phi(r)$ is at hand. Indeed, represent $\phi(r)$ in the form $\phi(r)=B\eta(\sqrt{B}r)$; it is easy to find that $||\phi||=\int\phi^2\cdot 4\pi r^2dr=\sqrt{B}$ and we obtain finally

$$\eta(r)=\phi(r||\phi||^{-1})||\phi||^{-2}. \tag{9}$$

Now we can easily determine ϵ in terms of arbitrary solution $\phi(r)$ of (6), which may be not normalized:

$$\begin{aligned}\epsilon &= -4\pi\frac{\nabla^2\eta}{\eta}\Big|_{\mathbf{x}=0}-\int\frac{d^3y\,\eta^2(y)}{|\mathbf{y}|}=\\ &=-4\pi||\phi||^{-2}[\frac{\nabla^2\phi}{\phi}\Big|_{r=0}+\int dr\cdot r\phi^2(r)]=\\ &=-4\pi||\phi||^{-2}[3(\phi^{-1}\phi''_{r^2})_{r=0}+\int dr\cdot r\phi^2(r)].\end{aligned} \tag{10}$$

3. To find spherically symmetric solution of the fourth order Eq. (6) one should put four supplementary conditions; two of them are evident: A) $\phi(r)\to 0$ at $r\to\infty$ and B) $\int\phi^2d^3r=1$. Two more conditions are: C) $\phi'_r(0)=0$ and D) $\phi'''_{r^3}(0)=0$; they appear as a result of $\phi(r)$ expansion in Taylor series in r at $r\to 0$ and substituting this series into (6). It is convenient to search at first for the solutions of (6) without normalization condition B). Fix some negative value $\phi''(0)$ and, choosing the parameter $a_0=\phi(0)$, try to satisfy condition A) (the "shooting method"). To solve Eq. (6) numerically, previously expand $\phi(r)$ at $r\to 0$:

$$\phi(r)\simeq a_0+a_2r^2+a_4r^4, \tag{11}$$

Substituting (11) into (6), we find:

$$a_4=(a_0^3+36a_2^2/a_0^{-1})/120. \tag{12}$$

Using (11) and (12) we begin numerical solution at the region $r\simeq 0$, and further solve equation (6), being transformed to the form convenient for the numerical investigation, using the finite-difference scheme of the second order accuracy.

4. Eq. (6) is nonlinear. Spatially non-one-dimensional nonlinear equations often have more than one solution even in the class of functions independent of the angular variables. For example, D-dimensional equations of the type

$$\Delta_{rr}\phi+F(\phi)=0,\quad \Delta_{rr}=\frac{\partial^2}{\partial r^2}+\frac{D-1}{r}\frac{\partial}{\partial r}, \tag{13}$$

where $F(\phi)$ is nonlinear function, have as a rule denumerable set of solutions, $\phi_n(r)$ being numbered by the number of the function $\phi_n(r)$ nodes plus one ($n = k + 1$, k is the number of nodes), and the value $\phi_n(0)$ increases with the n increasing.

By this reason the investigation of solutions of (6) was not restricted by the search for the nodeless solution. The whole family of localized solutions of (6) has been found, it is convenient to number them by $n=k+1$, where k is the number of the solution nodes. Some of these solutions (not yet normalized) are represented in Fig. 1, and their characteristic parameters are brought together in Table 1. The "size" of the solution $\phi_n(r)$ is given by $R^{(n)}_{char}[\phi]$; $R^{(n)}_{char}[\phi]$ has been defined so that $\phi(R^{(n)}_{char}) = 10^{-2}|\phi|^{(n)}_{max}$, where $|\phi|^{(n)}_{max}$, is the magnitude of n-th (counting from the center $r = 0$) maximum of the solution $\phi_n(r)$ absolute value.

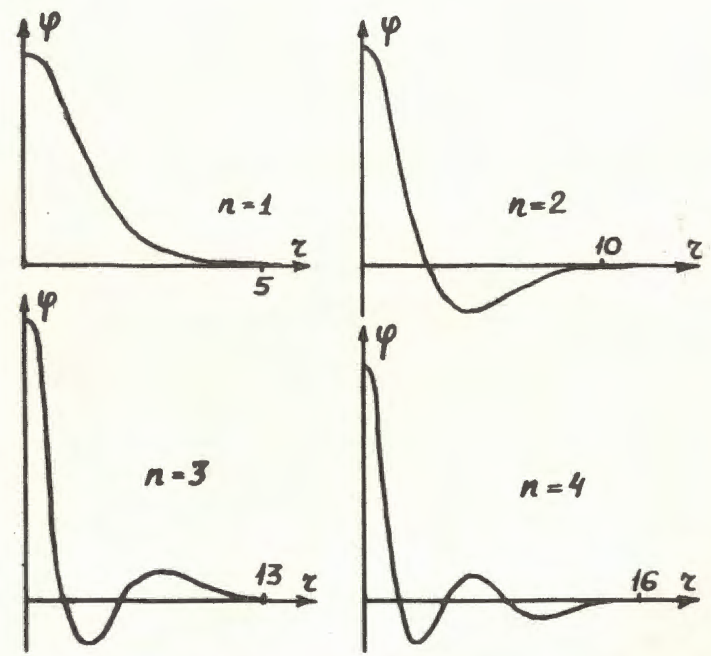

Fig. 1. The first four solutions to Eq. (6) ($||\phi_n|| \neq 1$).

Table 1

The characteristic parameters of non-normalized solutions of Eq. (6)

n	a_2	a_0	$\|\varphi\|$	$R_{char}[\varphi]$
I	-I.5	3.I300	77.I	4
2	-.5	I.574406	I2I.6	I2
3	-I.	2.II30I7	2I8.0	I4
4	-I.	2.049462	290.2	I9
5	-I.	2.00683	36I.5	23
6	-I.	I.97540	432.6	27
7	-I.	I.95080	503.I	3I
8	-I.	I.93035	572.I	36
9	-I.	I.9I3200	64I.2	40

Now to obtain solutions $\eta_n(r)$ satisfying the normalization condition B) one should make use of (9); in particular, we find

$$\text{a)}\quad \eta_n(0) = \phi_n(0)||\phi||^{-2},$$

$$\text{b)}\quad \eta''_n(0) = \phi''_n(0)||\phi||^{-4}, \qquad (14)$$

$$\text{c)}\quad r_{char}[\eta] = R_{char}[\phi]||\phi||^{-2},$$

where $r_{char}[\eta]$ is defined for $\eta(r)$ analogously to the definition of $R_{char}[\phi]$ for $\phi(r)$. Table 2 contains the parameters of the normalized solutions of $\eta_n(r)$. One can see that the value $\eta_n(0)$ decreases with the increase of solution number n but not increases as $\phi_n(0)$ usually does for the equations of type (13). Using Table 1 and formula (14b) one can see the validity of the relationship

$$\eta''_n(0) = -(nA)^{-4}, \quad A = 72.3. \qquad (15)$$

Table 2

The characteristic parameters of normalized solutions of Eq. (6)

n	$\eta(0)$	r_{char}	ϵ
I	$5.37\cdot IO^{-4}$	$2.3\cdot IO^{4}$	$-.648\cdot IO^{-2}$
2	$I.075\cdot IO^{-4}$	$I.75\cdot IO^{5}$	$-.I225\cdot IO^{-2}$
3	$4.44\cdot IO^{-5}$	$6.75\cdot IO^{5}$	$-.498\cdot IO^{-3}$
4	$2.44\cdot IO^{-5}$	$I.6\cdot IO^{6}$	$-.268\cdot IO^{-3}$
5	$I.53\cdot IO^{-5}$	$3.I\cdot IO^{6}$	$-.I67\cdot IO^{-3}$
6	$I.06\cdot IO^{-5}$	$5.0\cdot IO^{6}$	$-.II4\cdot IO^{-3}$
7	$7.8\cdot IO^{-6}$	$7.75\cdot IO^{6}$	$-.827\cdot IO^{-4}$
8	$5.9\cdot IO^{-6}$	$I.I8\cdot IO^{7}$	$-.628\cdot IO^{-4}$
9	$4.66\cdot IO^{-6}$	$I.65\cdot IO^{7}$	$-.492\cdot IO^{-4}$

The obtained solutions $(\eta_n(r), \epsilon_n)$ are the normalized solutions of Eq. (5), this fact has been confirmed (in addition to the analytic consideration) by the direct substitution $(\eta_n(r), \epsilon_n)$ into (5)*.

5. The sequence ϵ_n is of the most interest; the dependence ϵ_n on n, plotted in Fig. 2 (see also Table 2), allows one to assume that the eigenvalue problem under investigation has a denumerable set of the solutions, corresponding to the bound states. The statement of the oscillation theorem is apparently valid for the nonlinear equation (6) with the conditions A)-D): the n-th eigenvalue (in the order of the increase) corresponds to eigenfunction having (n-1) nodes, $n = 1,2,3,\dots$.

*The author is grateful to E.P.Zhidkov and V.G.Makhankov for the detailed discussion of this problem.

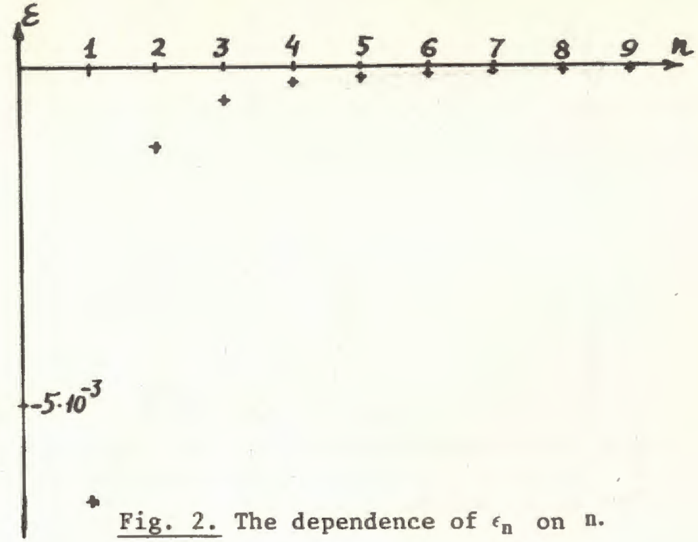

Fig. 2. The dependence of ϵ_n on n.

Given the value ϵ_n, one can determine the mass of n-th baryon state H_n as a function of g, using formula (7). The dependences $h_n(g) = \frac{H_n(g)}{MN} = 1+8\pi g^4 \epsilon_n$ are plotted in Fig. 3.

The mechanism of excited state spectrum generation that has been found in this paper completes the possibilities that had been pointed out in ref./2/. In particular, the mechanism of the excitation spectrum generation is discussed in ref./2/; such excitation spectrum may arise when one quantizes localized solutions of the non-stationary analogue of Eq. (1) that depend on time non-trivially (they have $|\phi|^2(x,t)$ periodically changing in time). To really have such a possibility, one should certainly have such pulsating solutions. But as is known such pulsating non-radiating solutions exist in the case of nonlinear evolution equations only for some special models (one-dimensional Shrödinger equations with the cubic /3/ and logarithmic /4/ nonlinearities, the relativistic-invariant sine-Gordon equation /5/ ; in physically interesting three-dimensional case there is known today the unique model with such "pulson" solutions - Klein-Gordon equation with the logarithmic nonlinearity; the quantization of the oscillating solution of this equation has been made in /6/). It is an open question now whether such pulsating solutions exist in this model. Note that the nonstationary analogue of (5) can be trans-

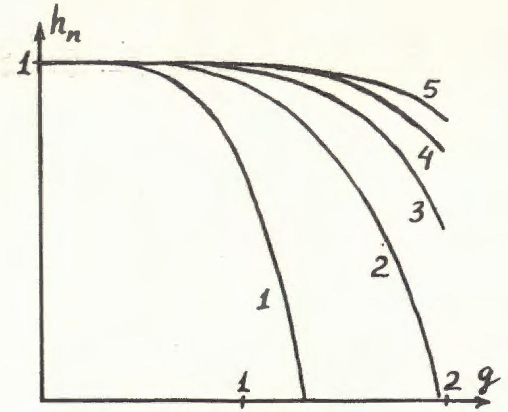

Fig. 3. The dependence of "baryon" mass H_n on the effective coupling constant g for $n = 1,2,3,4,5$; $h_n = H_n/MN$.

formed by dividing it by ϕ and acting with the operator ∇^2 (see /2/) to the form (dimensionless variables!)

$$i\nabla^2 (\phi^{-1} \frac{\partial \phi}{\partial t}) = \phi^* \phi - \nabla^2 (\phi^{-1} \nabla^2 \phi). \tag{16}$$

A numerical investigation of Eq. (16) may be easier than that for the nonstationary analogue of (5).

6. The mechanism of the energy spectrum generation in the nonrelativistic model that has been found in this paper is to the author's mind more natural from the point of view of the nonrelativistic quantum mechanics. The n number may be called the "main quantum number" on the analogy with the hydrogen atom theory. The main conclusion of this paper about the emergence of radial excitation spectrum will apparently survive in the $N \to \infty$ limit if one considers instead of Coulomb potential some other potentials that seem more natural today, say, $V(r) = -\frac{a^2}{r} + \beta^2 r$ (the number of the bound state may of course not be infinite). It is very probably that there are some analogues of the $N = \infty$ bound states in the most interesting case $N = 3$. At last to the author's mind it is possible that in the $N \to \infty$ limit there exists the set of solutions to Eqs. (5), (6) characterized by orbital quantum number ℓ together with the "main quantum number" n.

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REFERENCES

1. 't Hooft G. Nucl.Phys., 1974, B72, p.461; ibid., 1974, B75, p.461.
2. Witten E. Preprint HUTP-79/A007, Cambridge, 1979.
3. a) Zakharov V.E., Shabat A.B. Zh.E.T.F., 1971, 61, p.118; ibid, 1973, 64, p.1627; b) Abdulloev Kh.O., Bogolubsky I.L., Makhankov V.G. Phys.Lett., 1974, A48, p.161.
4. Bialynicki-Birula I., Mycielski J. Ann.Phys., 1976, 100, p.62.
5. Zakharov V.E., Takhtajan L.A., Faddeev L.D. DAN SSSR, 1974, 219, p.1334.
6. Bogolubsky I.L. Zh.E.T.F., 1979, 76, p.422.

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