сообщения
объединенного
института
ядерных
исследований
дубна

E2 - 12820

M.M.Enikova, V.I.Karloukovski

APPLICATION OF THE MONODROMY OPERATOR TO QUANTUM MECHANICS PROBLEMS

1979

E2 - 12820

M.M.Enikova, V.I.Karloukovski

APPLICATION OF THE MONODROMY OPERATOR TO QUANTUM MECHANICS PROBLEMS

Еникова М.М., Карлуковски В.И. **E2 - 12820**

Применение оператора монодромии
к квантовомеханическим проблемам

Задачи рассеяния и задачи на связанные состояния рассматриваются единым образом в терминах оператора монодромии. Разработана общая вычислительная процедура для широкого класса потенциалов. В частности, можно рассматривать потенциалы, содержащие δ -функциональные члены. Этот метод можно применять как к уравнению Шредингера, так и к квазипотенциальному уравнению Тодорова.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1979

Enikova M.M., Karloukovski V.I. **E2 - 12820**

Application of the Monodromy Operator to Quantum Mechanics Problems

Bound state and scattering problems are formulated in a unified manner in terms of the matrix elements of a monodromy operator. A general computational procedure is developed for a large class of potentials involving, in particular, potentials with δ-function terms. It can be applied both to the Schrödinger equation and to the Todorov quasipotential equation.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1979

INTRODUCTION

The present day quantum mechanics is a highly developed theory with numerous computational techniques and methods. The possibility and the need of looking for others, however, is not extinguished. The purpose of our study is to reveal that the concept of monodromy operator /1/ may be useful in setting up a unified computational procedure for large variety of quantum mechanics problems. It allows one to formulate, in particular, the bound-state and scattering problems on the same level as a Cauchy problem and in this sense our procedure is supplementary to that proposed in ref. /2/ and reducing all these problems to a boundary-value problem.

A further motivation to develop the present approach (actually the one which suggested us the problem) is that it allows one to relate quantities defined at distant points. We have encountered a problem of this kind in ref. /3/ where the parameter b_{01} defined at the origin must be related to the parameter a_0 appearing in the asymptotics at infinity. Here we formulate the answer to this question.

1. THE MONODROMY OPERATOR FOR BOUNDED POTENTIALS

In this section we consider the one-dimensional Schrödinger equation

$$\phi''(x) + [k^2 - U(x)]\phi(x) = 0 \tag{1.1}$$

on the infinite interval. We assume that the potential $U(x)$ is bounded and tends to certain limits at $\pm\infty$ (we choose $U(\pm\infty) = 0$).

We shall denote by $M(\eta,\xi)$ the monodromy operator for the potential which is equal to $U(x)$ in the interval (ξ,η) and zero outside this interval. The solutions in this case for

$x<\xi$ and $\eta<x$ may be written in the form $\phi(x)=A_\xi \sin kx+B_\xi \cos kx$ and $\phi(x)=A_\eta \sin kx+B_\eta \cos kx$, respectively. The operator $M(\eta,\xi)$ maps the vector (A_ξ,B_ξ) into the vector (A_η,B_η). The monodromy operator for Eq. (1.1) is $M=\lim_{\substack{\xi\to-\infty \\ \eta\to\infty}} M(\eta,\xi)$.

A straightforward calculation gives

$$M(\eta,\xi)=R^{-1}(k_\eta\eta)D^{-1}(k_\eta)\Lambda(\eta,\xi)D(k_\xi)R(k_\xi\xi), \qquad (1.2)$$

where the matrix $\Lambda(\eta,\xi)$ can be approximated by

$$\Lambda(\eta,\xi)=\prod_{n=1}^{N} D(k_n)R(k_n(\xi_n-\xi_{n-1}))D^{-1}(k_n) \qquad (1.3)$$

and

$$R(z)\equiv\begin{bmatrix}\cos z & -\sin z\\ \sin z & \cos z\end{bmatrix}, \qquad D(z)\equiv\begin{bmatrix}1 & 0\\ 0 & z\end{bmatrix}. \qquad (1.4)$$

Here $\xi\equiv\xi_0<\xi_1<\ldots<\xi_N<\xi_{N+1}\equiv\eta$ and $k_n\equiv\sqrt{k^2-U(\xi_n)}\equiv k_{\xi_n}$.

We note that (1.2) and (1.3) imply

$$\det\Lambda(\eta,\xi)=1, \qquad \det M(\eta,\xi)=k_\eta/k_\xi, \qquad (1.5)$$

so that for $k_\xi=k_\eta$ the monodromy operator belongs to $SL(2,R)$. For an elegant discussion of this property of the monodromy matrix (whose physical content is the conservation of the probability current) we refer the reader to /1/. We note also that

$$M(\zeta,\eta)M(\eta,\xi)=M(\zeta,\xi), \qquad \Lambda(\zeta,\eta)\Lambda(\eta,\xi)=\Lambda(\zeta,\xi) \qquad (1.6)$$

and

$$M(\xi,\xi)=1, \qquad \Lambda(\xi,\xi)=1. \qquad (1.7)$$

It follows from (1.6) that

$$\Lambda(\eta+h,\xi)=\Lambda(\eta+h,\eta)\Lambda(\eta,\xi) \qquad (1.8)$$

and representation (1.3) implies that for h infinitesimal

$$\Lambda(\eta+h,\eta)=D(k_\eta)R(k_\eta h)D^{-1}(k_\eta)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\ -k_\eta^2 & 0\end{bmatrix}h+O(h^2). \qquad (1.9)$$

On the other hand

$$\Lambda(\eta + h, \xi) = \Lambda(\eta, \xi) + h\frac{d}{d\eta}\Lambda(\eta, \xi) + O(h^2). \tag{1.10}$$

In this way we obtain the following differential equation for the Λ-matrix

$$\frac{d}{d\eta}\Lambda(\eta, \xi) = Q(\eta)\Lambda(\eta, \xi), \quad Q(\eta) \equiv \begin{pmatrix} 0 & 1 \\ -k^2(\eta) & 0 \end{pmatrix} \tag{1.11}$$

with the initial conditions (1.7).

Equation (1.3) can be written in the form of an ordered exponent

$$\Lambda(\eta, \xi) = T \exp[\int_{\xi}^{\eta} Q(\tau)d\tau]. \tag{1.12}$$

2. GENERALIZATIONS. SINGULAR POTENTIALS

In this section we consider the Schrödinger equation

$$[\Delta + k^2 - U(r)]\phi(x) = 0 \tag{2.1}$$

with a spherically symmetric potential $U(r)$ which can be expanded in a power series

$$U(r) = \sum_{n=-1}^{\infty} U_n r^n \tag{2.2}$$

in vicinity of the origin and approaches zero at infinity. Upon a partial-wave expansion

$$\phi(x) = \sum_{\ell=0}^{\infty} \phi_\ell(r) P_\ell(\cos\theta), \tag{2.3}$$

we write the radial equation in the form

$$[\frac{d^2}{dr^2} + k^2 - \frac{\ell(\ell+1)}{r^2} - U(r)]\, r\phi_\ell(r) = 0. \tag{2.4}$$

Its general solution is a combination of a regular and a singular solutions

$$r\phi_\ell(r) = A_\ell u_\ell(r) + B_\ell v_\ell(r) \tag{2.5}$$

whose behaviour at the origin is defined by their power series expansion

$$u_\ell(r) = r^{\ell+1} \sum_{s=0}^{\infty} a_{\ell s} r^s$$

$$v_\ell(r) = r^{-\ell} \sum_{s=0}^{\infty} b_{\ell s} r^s + \gamma_\ell u_\ell(r) \ln r. \tag{2.6}$$

We normalize them by choosing conventionally

$$a_{\ell 0} = [(2\ell+1)!!]^{-1}, \quad b_{\ell 0} = -(2\ell-1)!! \tag{2.7}$$

This determines all the coefficients $a_{\ell s}$ of the regular solution

$$a_{\ell 1} = \frac{U_{-1}}{2(\ell+1)} a_{\ell 0}, \; 2s(\ell+1)a_{\ell s} = k^2 a_{\ell, s-2} - \sum_{j=0}^{s-1} U_{s-2-j}\, a_{\ell j}, \tag{2.8}$$

the first coefficients of the singular solution

$$b_{\ell 1} = -\frac{U_{-1}}{2\ell} b_{\ell 0}, \; s(2\ell+1-s) b_{\ell s} = k^2 b_{\ell, s-2} - \sum_{j=0}^{s-1} U_{s-2-j}\, b_{\ell j}, \tag{2.9}$$

and γ_ℓ

$$3(2\ell+1) a_{\ell, 2\ell+1} \gamma_\ell = -k^2 b_{\ell, 2\ell-1} + \sum_{j=0}^{2\ell} U_{2\ell-1-j}\, b_{\ell j}. \tag{2.10}$$

The coefficient $b_{\ell, 2\ell+1}$ remains arbitrary and all the next coefficients $b_{\ell s}$ depend on its choice

$$s(2\ell+1-s) b_{\ell s} = \gamma_\ell (2s+2\ell+1) a_{\ell s} + k^2 b_{\ell, s-2} - \sum_{j=0}^{s-1} U_{s-2-j}\, b_{\ell j}, \quad s > 2\ell+1, \tag{2.11}$$

i.e., the singular solutions are one-parameter family. We adopt the convention that $v_\ell(r)$ is the particular (singular) solution corresponding to

$$b_{\ell, 2\ell+1} = 0. \tag{2.12}$$

Let us recall also that the regular and the singular solutions have the following asymptotic behaviour at infinity

$$u_\ell(r) \underset{r\to\infty}{\sim} \alpha_\ell \sin(kr - \tfrac{1}{2}\pi\ell + \eta_\ell)$$

$$v_\ell(r) \underset{r\to\infty}{\sim} -\beta_\ell \cos(kr - \tfrac{1}{2}\pi\ell + \tilde{\eta}_\ell) \tag{2.13}$$

with amplitudes related by

$$k\, \alpha_\ell \beta_\ell = 1. \tag{2.14}$$

The phase shift $\tilde{\eta}_\ell$ of the singular solution depends on the free parameter $b_{\ell,2\ell+1}$. This allows one to define another distinguished singular solution $w_\ell(r)$ by the requirement $\tilde{\eta}_\ell = \eta_\ell$ so that

$$w_\ell(r) \underset{r\to\infty}{\sim} -\beta_\ell \cos(kr - \frac{1}{2}\pi\ell + \eta_\ell). \tag{2.15}$$

The problem of finding that particular value of $b_{\ell,2\ell+1}$ for which

$$\tilde{\eta}_\ell(b_{\ell,2\ell+1}) = \eta_\ell \tag{2.16}$$

will be solved in the next section.

In the case when $\ell = 0$ and $U(r)$ is bounded the method discussed in the previous section can be applied to define and calculate the monodromy operator. In particular, it follows from Eqs. (1.23)-(1.26) that

$$\Lambda_0(\rho, 0) = \begin{bmatrix} v_0(\rho) & u_0(\rho) \\ v_0'(\rho) & u_0'(\rho) \end{bmatrix}. \tag{2.17}$$

We generalize this result and define $\Lambda_\ell(\rho,0)$ for any partial wave (in the case when the potential may have singulariry of the form (2.2)) by

$$\Lambda_\ell(\rho,0) = \begin{bmatrix} \Lambda_{11}^\ell(\rho,0) & \Lambda_{12}^\ell(\rho,0) \\ \Lambda_{21}^\ell(\rho,0) & \Lambda_{22}^\ell(\rho,0) \end{bmatrix} = \begin{bmatrix} v_\ell(\rho) & u_\ell(\rho) \\ v_\ell'(\rho) & u_\ell'(\rho) \end{bmatrix}. \tag{2.18}$$

The matrix elements of (2.18) obey the following system of differential equations and initial asymptotics:

$$\frac{d}{d\rho}\Lambda_{12}^\ell(\rho,0) = \Lambda_{22}^\ell(\rho,0), \tag{2.19}$$

$$\frac{d}{d\rho}\Lambda_{22}^\ell(\rho,0) = -[k^2 - \frac{\ell(\ell+1)}{\rho^2} - U(\rho)]\Lambda_{12}^\ell(\rho,0),$$

$$\Lambda_{12}^\ell(\rho,0) \underset{\rho\to 0}{\sim} a_{\ell 0}\rho^{\ell+1}, \quad \Lambda_{22}^\ell(\rho,0) \underset{\rho\to 0}{\sim} (\ell+1)a_{\ell 0}\rho^\ell \tag{2.20}$$

and

$$\frac{d}{d\rho}\Lambda_{11}^\ell(\rho,0) = \Lambda_{21}^\ell(\rho,0),$$

$$\frac{d}{d\rho}\Lambda_{21}^{\ell}(\rho, 0) = -[k^2 - \frac{\ell(\ell+1)}{\rho^2} - U(\rho)]\Lambda_{11}^{\ell}(\rho, 0), \quad (2.21)$$

$$\Lambda_{11}^{\ell}(\rho, 0) \underset{\rho\to 0}{\sim} b_{\ell 0}\,\rho^{-\ell}, \quad \Lambda_{21}^{\ell}(\rho,0) \underset{\rho\to 0}{\sim} \ell\, b_{\ell 0}\,\rho^{-\ell-1} + \gamma_\ell(\ell+1)a_{\ell 0}\rho^{\ell}\ln\rho \quad (2.22)$$

with $a_{\ell s}, b_{\ell s}$ given by (2.7)-(2.11).

Suppose that $f_\ell(r)$, $g_\ell(r)$ is another set of fundamental solutions of (2.4), or, even more generally, a set of functions with different from zero Wronskian

$$W(f_\ell, g_\ell) = f_\ell' g_\ell - g_\ell' f_\ell \equiv \Delta_\ell \neq 0 \quad (2.23)$$

(not necessary solutions of Eq. (2.4)). One can then write

$$Y_\ell(\rho,0) = \begin{bmatrix} g_\ell(\rho) & f_\ell(\rho) \\ g_\ell'(\rho) & f_\ell'(\rho) \end{bmatrix}, \quad Y_\ell^{-1}(\rho,0) = \frac{1}{\Delta_\ell}\begin{bmatrix} f_\ell'(\rho) & -f_\ell(\rho) \\ -g_\ell'(\rho) & g_\ell(\rho) \end{bmatrix} \quad (2.24)$$

and define a monodromy matrix

$$M_\ell(\rho, 0) = Y_\ell^{-1}(\rho, 0)\Lambda_\ell(\rho, 0). \quad (2.25)$$

This monodromy operator transforms the vector with components A_ℓ and B_ℓ, the integration constants in (2.5), into another vector

$$\begin{bmatrix} B_\ell(\rho) \\ A_\ell(\rho) \end{bmatrix} = M_\ell(\rho, 0)\begin{bmatrix} B_\ell \\ A_\ell \end{bmatrix}. \quad (2.26)$$

Let us consider the auxiliary function

$$\Phi_\ell(r;\rho) = B_\ell(\rho)\, g_\ell(r) + A_\ell(\rho) f_\ell(r). \quad (2.27)$$

In the case when $f_\ell(r)$ and $g_\ell(r)$ are solutions of the radial equation (2.4) the function $\Phi_\ell(r;\rho)$ also is a solution of this equation. It can be written in detail

$$\Phi_\ell(r;\rho) = \frac{B_\ell}{\Delta_\ell}[(f_\ell'(\rho)g_\ell(r) - g_\ell'(\rho)f_\ell(r))v_\ell(r) + g_\ell(\rho)f_\ell(r) - f_\ell(\rho)g_\ell(r))v_\ell'(r)] +$$
$$+ \frac{A_\ell}{\Delta_\ell}[(f_\ell'(\rho)g_\ell(r) - g_\ell'(\rho)f_\ell(r))u_\ell(r) + (g_\ell(\rho)f_\ell(r) - f_\ell(\rho)g_\ell(r))u_\ell'(r)]. \quad (2.28)$$

It follows from here that

$$\Phi_\ell(r;r) = B_\ell v_\ell(r) + A_\ell u_\ell(r) = r\phi_\ell(r) \quad (2.29)$$

i.e., $\Phi_\ell(r;r)$ is a solution of the radial equation (2.4) independent of whether $f_\ell(r)$ and $g_\ell(r)$ are solutions or not. So we have two ways to represent a solution $r\phi_\ell(r)$ of the radial equation

$$B_\ell(r)g_\ell(r) + A_\ell(r)f_\ell(r) = r\phi_\ell(r) = B_\ell v_\ell(r) + A_\ell u_\ell(r) \tag{2.30}$$

and the link between the two sets of coefficients A_ℓ, B_ℓ and $A_\ell(r)$, $B_\ell(r)$ is provided by the monodromy operator $M_\ell(r,0)$ (cf. Eq. (2.26)). The functions $A_\ell(r)$ and $B_\ell(r)$ reduce to constants provided $f_\ell(r)$ and $g_\ell(r)$ are solutions of (2.4) or they approach certain constants for $r \to \infty$ in the case when $f_\ell(r)$ and $g_\ell(r)$ tend to solutions asymptotically at infinity.

3. APPLICATION TO BOUND STATE AND SCATTERING PROBLEMS

In the preceding Section we have defined a monodromy operator

$$M_\ell(r,0) = Y_\ell^{-1}(r,0)\Lambda_\ell(r,0) = \begin{bmatrix} g_\ell(r) & f_\ell(r) \\ g'_\ell(r) & f'_\ell(r) \end{bmatrix}^{-1} \begin{bmatrix} v_\ell(r) & u_\ell(r) \\ v'_\ell(r) & u'_\ell(r) \end{bmatrix}, \tag{3.1}$$

where the elements of the Λ-matrix $v_\ell(r)$ and $u_\ell(r)$ are the singular and the regular solutions of Eq. (2.4), defined by (2.6), (2.7) and (2.12). To find the phase shifts η_ℓ in a scattering problem we choose

$$f_\ell(r) = \sin(kr - \frac{1}{2}\pi\ell), \quad g_\ell(r) = \cos(kr - \frac{1}{2}\pi\ell), \tag{3.2}$$

which are asymptotic solutions of (2.4). In particular, the asymptotics of the regular solution is (cf. (2.13))

$$u_\ell(r) \underset{r\to\infty}{\sim} a_\ell \sin\eta_\ell\, g_\ell(r) + a_\ell \cos\eta_\ell\, f_\ell(r). \tag{3.3}$$

On the other hand, the regular solution can be represented, according to (2.30) and (2.26) by

$$u_\ell(r) = B_\ell^{reg}(r)g_\ell(r) + A_\ell^{reg}(r)f_\ell(r) \tag{3.4}$$

with

$$\begin{bmatrix} B_\ell^{reg}(r) \\ A_\ell^{reg}(r) \end{bmatrix} = M_\ell(r,0)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} M_{12}^\ell(r,0) \\ M_{22}^\ell(r,0) \end{bmatrix}, \tag{3.5}$$

which implies

$$\frac{M^{\ell}_{12}(r,0)}{M^{\ell}_{22}(r,0)} \underset{r\to\infty}{\sim} \operatorname{tg}\eta_\ell \tag{3.6}$$

and allows one to compute the phase shifts η_ℓ by calculating the matrix elements of the monodromy operator for sufficiently large values of r.

Consider now the set of singular solutions normalized by (2.7). Any one of them can be represented in accord with (2.30) and (2.26) in the form

$$\chi_\ell^{sing}(r) = B_\ell^{sing}(r)\, g\ (r) + A_\ell^{sing}(r) f_\ell(r) \tag{3.7}$$

with

$$\begin{bmatrix} B_\ell^{sing}(r) \\ A_\ell^{sing}(r) \end{bmatrix} = M_\ell(r,0) \begin{bmatrix} 1 \\ b_{\ell,2\ell+1} \end{bmatrix} = \begin{bmatrix} M^{\ell}_{11}(r,0)+b_{\ell,2\ell+1}M^{\ell}_{12}(r,0) \\ M^{\ell}_{21}(r,0)+b_{\ell,2\ell+1}M^{\ell}_{22}(r,0) \end{bmatrix} \tag{3.8}$$

and the asymptotics (2.13)

$$\chi_\ell^{sing}(r) \underset{r\to\infty}{\sim} -\beta_\ell \cos\tilde\eta_\ell\, g_\ell(r) + \beta_\ell \sin\tilde\eta_\ell\, f_\ell(r) \tag{3.9}$$

gives

$$\frac{M^{\ell}_{21}(r,0) + b_{\ell,2\ell+1} M^{\ell}_{22}(r,0)}{M^{\ell}_{11}(r,0) + b_{\ell,2\ell+1} M^{\ell}_{12}(r,0)} \underset{r\to\infty}{\sim} -\operatorname{tg}\tilde\eta_\ell . \tag{3.10}$$

Now we can easily find the value of $b_{\ell,2\ell+1}$ for which $\tilde\eta_\ell = \eta_\ell$. Denoting $M^{\ell}_{ij} = \lim_{r\to\infty} M^{\ell}_{ij}(r,\ 0)$ we have, comparing (3.6) and (3.10),

$$\frac{M^{\ell}_{21} + b_{\ell,2\ell+1}M^{\ell}_{22}}{M^{\ell}_{11} + b_{\ell,2\ell+1}M^{\ell}_{12}} = -\frac{M^{\ell}_{12}}{M^{\ell}_{22}} .$$

Solving it with respect to $b_{\ell,2\ell+1}$ we obtain

$$b_{\ell,2\ell+1} = -\frac{M^{\ell}_{11}M^{\ell}_{12} + M^{\ell}_{21}M^{\ell}_{22}}{(M^{\ell}_{12})^2 + (M^{\ell}_{22})^2} . \tag{3.11}$$

In the case of a bound-state problem the radial equation (2.4) has the form

$$[\frac{d^2}{dr^2} - \kappa^2 - \frac{\ell(\ell+1)}{r^2} - U(r)] r\phi_\ell(r) = 0 \tag{3.12}$$

and the asymptotics of the regular solution is

$$u_\ell(r) \underset{r\to\infty}{\sim} C_\ell e^{-\kappa r} + D_\ell e^{\kappa r} \tag{3.13}$$

and we choose accordingly

$$f_\ell(r) = e^{-\kappa r}, \quad g_\ell(r) = e^{\kappa r} \tag{3.14}$$

so that (3.13) becomes

$$u_\ell(r) \underset{r\to\infty}{\sim} C_\ell f_\ell(r) + D_\ell g_\ell(r). \tag{3.15}$$

On the other hand, we have the representations (3.4), (3.5), and the condition for a bound state solution, $D_\ell = 0$, reads

$$M^\ell_{12}(r, 0) \underset{r\to\infty}{\sim} 0. \tag{3.16}$$

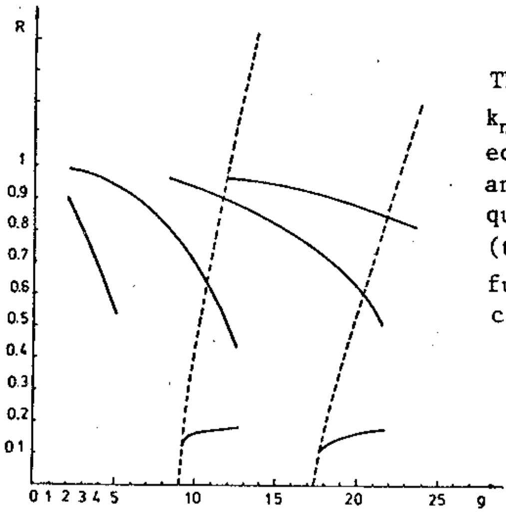
In this way Eqs. (3.6), (3.10), (3.11), and (3.16) may be used to formulate some bound state and scattering problems entirely in terms of the monodromy operator. The computation of the last one can be reduced, according to (3.1), to the computation of the Λ-matrix for which we have the system of differential equations (2.19), (2.21) with the asymptotics (2.20), (2.22) at the origin. This asymptotics are not quite convenient to be accounted for in computer calculations. One can, however, reformulate the problem in terms of others more convenient sets of functions. We have used the functions $Z^\ell_{ij}(r,0)$:

$$\begin{aligned}
\Lambda^\ell_{12}(r, 0) &= r^{\ell+1} Z^\ell_{12}(r, 0), \\
\Lambda^\ell_{22}(r, 0) &= r^{\ell+1} Z^\ell_{22}(r, 0) + \frac{\ell+1}{r} Z^\ell_{12}(r, 0), \\
\Lambda^\ell_{11}(r, 0) &= r^\ell Z^\ell_{11}(r,0) + \sum_{s=0}^{2\ell-1} b_{\ell s} r^{-\ell+s} + \gamma_\ell a_{\ell 0} r^{\ell+1} \ln r, \\
\Lambda^\ell_{21}(r, 0) &= r^\ell Z^\ell_{21}(r, 0) + \ell r^{\ell-1} Z^\ell_{11}(r, 0) + \sum_{s=0}^{2\ell-1} (-\ell+s) b_{\ell s} r^{-\ell+s-1} + \\
&\quad + \gamma_\ell a_{\ell 0} [r^\ell + (\ell+1) r^\ell \ln r].
\end{aligned} \tag{3.17}$$

As an illustration we have calculated the $\ell = 1$ bound states in the case of a Yukawa potential

$$U(r) = -g\frac{e^{-r}}{r}, \quad g > 0. \tag{3.18}$$

The dependence of the bound state momenta $k_n = \sqrt{2mE_n}$ on the coupling constant g is pictured in the Figure (the dashed lines).



The $\ell = 1$ bound-state momenta $k_n = \sqrt{2mE_n}$ for the Schrödinger equation (the dashed lines) and $k_n = b(w_n)$ for the Todorov quasipotential equation (the solid lines) as a function of the coupling constant g.

4. POTENTIALS CONTAINING δ-FUNCTION TERMS

We briefly recall here (for details, see /3/) that if one adds a δ-function term to the potential

$$v(r) = U(r) + V_0\,\delta(x) \tag{4.1}$$

the corresponding quantum mechanical problems become ambiguous. The correct mathematical formulation of these problems involves some free renormalization parameters. In particular, the scattering amplitude for the potential (4.1) has the form

$$f(w,\theta;V_0) = \sum_{\ell=0}^{\infty}(2\ell+1)f_\ell(w;V_0)P_\ell(\cos\theta),\quad f_\ell(w;V_0) = \frac{1}{k}\sin\eta_\ell\, e^{i\eta_\ell},\quad \ell \geq 1 \tag{4.2}$$

$$f_0(w;V_0) = \frac{1}{k}\;\frac{Q_0\alpha_0\sin\eta_0 - \beta_0\cos\eta_0}{Q_0\alpha_0 + i\beta_0}\,e^{i\eta_0},$$

where

$$Q_0 = \frac{4\pi}{V_0} - b_{01} + \nu_1 + \nu_2 U_{-1}. \tag{4.3}$$

Here $\nu_1 = \nu_1(k)$ and $\nu_2 = \nu_2(k)$ are renormalization parameters. All other quantities in (4.2) and (4.3) have already been defined. The δ-function term only affects the s-wave. The corresponding s-wave phase shift is increased by a quantity τ_0

$$\delta_0 = \eta_0 + \tau_0, \quad \mathrm{tg}\,\tau_0 = \frac{-\beta_0}{Q_0 \alpha_0}. \tag{4.4}$$

One can express the scattering amplitude (4.2) and the phase shift (4.4) in terms of the matrix elements $M_{ij} \equiv M^{\circ}_{ij}$ of the s-wave monodromy operator. To do this, we note that according to (3.3)-(3.5)

$$\alpha_0 = \sqrt{M_{12}^2 + M_{22}^2} \tag{4.5}$$

and according to (3.7)-(3.9)

$$\beta_0^2 = (M_{11} + b_{01} M_{12})^2 + (M_{21} + b_{01} M_{22})^2. \tag{4.6}$$

Taking into account (3.11) one obtains

$$\beta_0 = \frac{|M_{11}M_{22} - M_{12}M_{21}|}{\sqrt{M_{12}^2 + M_{22}^2}}. \tag{4.7}$$

Equation (2.14) implies

$$1 = k\alpha_0\beta_0 = k\,|\det M| \tag{4.8}$$

to be compared with (1.15) and $\det M = \det Y^{-1} \det \Lambda = -1/k$. One can write, finally

$$\beta_0 = \frac{1}{k}(M_{12}^2 + M_{22}^2)^{-1/2} \tag{4.9}$$

we note also that

$$\mathrm{tg}\,\eta_0 = \frac{M_{12}}{M_{22}}, \quad \cos\eta_0 = \frac{M_{22}}{\sqrt{M_{12}^2 + M_{22}^2}}, \quad \sin\eta_0 = \frac{M_{12}}{\sqrt{M_{12}^2 + M_{22}^2}} \tag{4.10}$$

so that the $V_0=0$ scattering matrix can be written in the form

$$f_0(w;V_0)=(4\pi+\nu V_0+V_0\frac{M_{11}}{M_{12}})[4\pi+\nu V_0+V_0\frac{(M_{22}+iM_{12})(M_{21}-iM_{11})}{M_{12}^2+M_{22}^2}]^{-1} f_0(w;0) \quad (4.11)$$

while in general, for V_0 arbitrary, Eq. (4.2) becomes

$$f_0(w;0)=\frac{1}{k}\sin\eta_0 e^{i\eta_0}=\frac{1}{k}\frac{M_{12}}{M_{12}^2+M_{22}^2}(M_{22}+iM_{12}), \quad (4.12)$$

where we have denoted

$$\nu \equiv \nu_1+\nu_2 U_{-1}. \quad (4.13)$$

The quantity added to the $V_0=0$ phase shift is given by

$$\mathrm{tg}\,\tau_0=-\frac{V_0}{k}[(4\pi+\nu V_0)(M_{12}^2+M_{22}^2)+V_0(M_{11}M_{12}+M_{21}M_{22})]^{-1}. \quad (4.14)$$

The bound state condition (3.16) is changed also in the presence of a δ-function term. It was demonstrated in /3/ that the correct form of the Schrödinger equation in such a case is

$$(\nabla^2-\kappa^2)\psi(\mathbf{x})=[U(r)+V_0\delta_{\nu_1\nu_2}]\psi(\mathbf{x}), \quad (4.15)$$

where $\delta_{\nu_1\nu_2}$ is a suitable extension of the operator "multiplication by $\delta(\mathbf{x})$ ". The s-wave solution contains an admixture of the singular solution $v_0(r)$

$$r\psi_0(r)=A_0u_0(r)+B_0v_0(r) \quad (4.16)$$

and it follows from Eq. (4.15) that the integration constants A_0, B_0 are related by

$$4\pi B_0=V_0[A_0+(b_{01}-\nu_1-\nu_2U_{-1})B_0] \quad (4.17)$$

so that (4.16) becomes

$$r\psi_0(r)=A_0[u_0(r)+Q_0^{-1}v_0(r)]. \quad (4.18)$$

We recall that $b_{01}=0$ for the solution $v_0(r)$ appearing in (4.16) and (4.18).

Using the asymptotics (3.15) of the regular solution as well as the analogous asymptotics of the singular solution

$$v_0(r)\underset{r\to\infty}{\sim}\tilde{C}_0 f_0(r)+\tilde{D}_0 g_0(r) \quad (4.19)$$

we can write (omitting the normalization factor A_0)

$$r\psi_0(r) \underset{r\to\infty}{\sim} (C_0 + Q_0^{-1}\tilde{C}_0) f_0(r) + (D_0 + Q_0^{-1}\tilde{D}_0) g_0(r) \tag{4.20}$$

and the condition for bound states reads

$$D_0 + Q_0^{-1}\tilde{D}_0 = 0 . \tag{4.21}$$

On the other hand we have for (4.18) the representation

$$r\psi_0(r) = A_0(r) f_0(r) + B_0(r) g_0(r) \tag{4.22}$$

with

$$\begin{bmatrix} B_0(r) \\ A_0(r) \end{bmatrix} = M_0(r,0) \begin{bmatrix} Q_0^{-1} \\ 1 \end{bmatrix}, \quad M(r,0) = \begin{bmatrix} g_0(r) & f_0(r) \\ g_0'(r) & f_0'(r) \end{bmatrix}^{-1} \begin{bmatrix} v_0(r) & u_0(r) \\ v_0'(r) & u_0'(r) \end{bmatrix} . \tag{4.23}$$

Hence the bound state condition (4.21) can be expressed entirely in terms of the monodromy operator

$$Q_0^{-1} M_{11}(r,0) + M_{12}(r,0) \underset{r\to\infty}{\sim} 0 \tag{4.24}$$

where

$$Q_0 = \frac{4\pi}{V_0} + \nu_1 + \nu_2 U_{-1} \tag{4.25}$$

or

$$\frac{V_0}{4\pi + \nu V_0} M_{11}(r,0) + M_{12}(r,0) \underset{r\to\infty}{\sim} 0 . \tag{4.26}$$

It reduces to (3.16) for $V_0=0$, i.e., when the δ-function term disappears.

5. THE TODOROV QUASIPOTENTIAL EQUATION

A very convenient (pseudo-) local variant of the quasi-potential approach /4/ was proposed by Todorov /5-7/. For a scalar quasipotential the Todorov equation has in coordinate representation the form

$$[\Delta + b^2(w) - \frac{1}{2w} v_w(x)] \Phi_w(x) = 0, \tag{5.1}$$

where $v_w(x)$ is the pseudo-local quasipotential depending, in general, on the c.m. energy w and

$$b^2(w) = \frac{1}{4w^2}\,[w^4 - 2(m_1^2 + m_2^2)w^2 + (m_1^2 - m_2^2)^2] \qquad (5.2)$$

is the on-mass-shell value of the c.m. relative momentum of the two particles of masses m_1 and m_2. Its close relation to the Schrödinger equation allows one to apply to it, without essential modifications, the monodromy operator method described in the preceding sections and to compute the phase shifts, the bound state energies, the resonance positions, and widths, etc. This can be done for a very large class of quasipotentials. In particular, one can treat in this way the quasipotential containing δ -function terms which usually appear in the framework of the quantum field theory /6-8/.

As an example we have calculated in the case of two particles of equal mass $m_1 = m_2 = m = 1$ the nearest to the threshold $w = 2m$ p-wave bound-state energies for the Yukawa potential

$$V_w(x) = 4g\,\frac{e^{-r}}{r} \qquad (5.3)$$

for various coupling constants g. The figure illustrates dependence of the corresponding momenta $b = b(w)$ on g in comparison with the Schrödinger case. In the case of the quasipotential equation the picture is drastically changed and the number of the bound states becomes infinite. The series of bound states accumulates at $w = 0$ in vicinity of which the known analytic properties are violated. This change in the behaviour of the bound-state energies (momenta) can be explained and visualized by the method discussed in section 4 of ref. /9/

REFERENCES

1. Arnold V.I. Additional Chapters to the Theory of Ordinary Differential Equations. "Nauka", M., 1978 (in Russian).
2. Ponomarev L.I., Puzynin I.V., Puzynina T.P. J.Comp.Phys., 1973, 13, p.1; Ponomarev L.I., Puzynin I.V., Puzynina T.P. J.Comp.Phys., 1976, 21. Bang J. et al. Nucl.Phys., 1976, A261, 1, p.59. Ponomarev L.I. et al. JINR, P2-9915, Dubna, 1976.
3. Karloukovski V.I. Rep.Math.Phys., 1976, 10, p.87.
4. Logunov A.A., Tavkhelidze A.N. Nuovo Cimento, 1963, 29, p.380.
5. Todorov I.T. Phys.Rev., 1971, D3, p.2351.

6. Todorov I.T. Quasipotential Approach to the Two-Body Problem in Quantum Field Theory. In: Properties of Fundamental Interactions, vol. 9C, ed. A.Zichichi (Editrice Compositori, Bologna, 1973).
7. Rizov V.A., Todorov I.T. Elem.chast. i atom. yad., 1975, 6, p.669.
8. Karloukovski V.I., Todorov I.T. Quasipotential Approach to the ρ-Resonance in $\pi\pi$-Scattering. Preprint DPh.T/74/47, Saclay, 1974.
9. Karloukovski V.I. Bulg.J.Phys., 1977, 4, p.595.

Received by Publishing Department
on October 1 1979.