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Иванов Е.А.

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Теория Янга-Миллса в сигма-модельном представлении

Показано, что предложенная ранее В.И.Огиевецким и автором интерпретация теории Янга-Миллса как теории спонтанного нарушения естественно приводит к ее новому представлению на языке билокальной нелинейной σ-модели. Обсуждаются возникающие возможности.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Ivanov E.A.

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Yang-Mills Theory in Sigma-Model Representation

The suggested earlier by V.I.Ogievetsky and the author interpretation of the Yang-Mills theory as the theory of spontaneous breakdown is shown to naturally lead to the new representation of this theory in terms of bilocal nonlinear σ-model. The possibilities arising are discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Recently, it has been suggested [1,2] that the Yang-Mills theory is completely integrable (both on classical and on quantum levels) and this property could become manifest upon passing to suitable unconventional variables (some considerations along a similar line are contained in ref. [3,3a]). The search for new convenient variables for this theory is important also in connection with the problem of adequate description of its symmetric, gauge-invariant phase which is expected to realize the colour confinement [4,5].

Here we present a new formulation of gauge theories which explicitly demonstrates their common nature with nonlinear σ-models and seems promising from both points of view mentioned above.

2. The formulation we are going to describe is based upon the observation made earlier by V.I.Ogievetsky and the author. We have shown in ref. [6] that any gauge theory is the nonlinear realization of certain infinite-parameter group $K = G \ltimes \mathcal{P}$ with $G^0 \times \mathcal{P}$ as the vacuum stability subgroup (see also [7]). Here G^0 is the relevant global symmetry subgroup, G is isomorphic to the connected component of the corresponding local group spanned by all gauge functions decomposable in the Taylor series around $x_\mu = 0$ and $\mathcal{P}$ is usual Poincaré group.

As a starting point, we introduce an extra coordinate-Lorentz 4-vector y_μ and represent the generators of K as follows:

$$P_\mu = \partial^y_\mu \, , \; L_{\mu\nu} = i(y_\mu \partial^y_\nu - y_\nu \partial^y_\mu) ,$$
$$Q^i_\mu = y_\mu Q^i \, , \ldots , Q^i_{\mu_1 \ldots \mu_n} = y_{\mu_1} \ldots y_{\mu_n} Q^i \, , \ldots . \qquad (1)$$

Here P_μ, $L_{\mu\nu}$, Q^i are, resp., the 4-translation generator, the generators of the Lorentz group L, and those of G^0. The tensor generators Q^i_μ, $Q^i_{\mu_1\mu_2}$, ... $Q^i_{\mu_1...\mu_n}$ .... together with Q^i form the algebra of the infinite-parameter group G /6,7/.

The representation (1) allows one to convert the infinite set of parameters of the coset space $K/G^0 \times L$ ($b^i_\mu(x)$, $b^i_{\mu_1\mu_2}(x)$,.... $b^i_{\mu_1...\mu_n}(x)$...) into the single object: the bilocal Goldstone field $b(x,y) \equiv b^i(x,y)Q^i = \sum_{n\geq 1} \frac{1}{n!} b^i_{\mu_1...\mu_n}(x) y^{\mu_1} ... y^{\mu_n} Q^i$.

It is completely specified by the requirement of decomposability in powers of y_μ around $y_\mu = 0$ and by the condition

$$b(x,0) = 0. \tag{2}$$

In the basis (1) an element of coset $K/G^0 \times L$ takes the form:

$$G(x,b) = \exp\{i x^\mu P_\mu\} \exp\{i \sum_{n\geq 1} \frac{1}{n!} b^i_{\mu_1...\mu_n}(x) Q^{i\mu_1...\mu_n}\} =$$

$$= \exp\{-x^\mu \partial^y_\mu\} \exp\{ib(x,y)\}.$$

When the group K acts on $G(x,b)$ from the left, the bilocal field $b(x,y)$ undergoes the following transformations:

$$\mathcal{P}: \; b'(x,y) = b(\Lambda^{-1}(x-a), \Lambda^{-1}y) \tag{3}$$

$$G: \; \exp\{ib'(x,y)\} = \exp\{i\lambda(x+y)\} \exp\{ib(x,y)\} \exp[-i\lambda(x)\}, \tag{4}$$

where $\lambda(y) = \lambda(0) + \sum_{n\geq 1} \frac{1}{n!} \lambda_{\mu_1...\mu_n}(0) y^{\mu_1} ... y^{\mu_n}$ is the generating function for constant parameters of G.

The covariant derivatives of Goldstonions $\nabla_\mu b_\rho(x)$,.... $\nabla_\mu b_{\rho_1...\rho_n}(x)$,... defined in /6/ combine now into the bilocal Gartan form

$$\omega_\mu(x,y) = -b_\mu(x) + \sum_{n\geq 1} \frac{1}{n!} \nabla_\mu b_{\rho_1...\rho_n}(x) y^{\rho_1} y^{\rho_2} .... y^{\rho_n}. \tag{5}$$

It is introduced by the relation

$$\exp\{-ib(x,y)\}(\partial^x_\mu - \partial^y_\mu) \exp\{ib(x,y)\} = i\omega_\mu(x,y) \tag{6}$$

and as a consequence satisfies the generalized Maurer-Cartan equation:

$$(\partial^x_\mu - \partial^y_\mu)\omega_\rho(x,y) - (\partial^x_\rho - \partial^y_\rho)\omega_\mu(x,y) + i[\omega_\mu(x,y), \omega_\rho(x,y)] = 0, \tag{7}$$

(which is equivalent to the statement that the generalized Yang-Mills connection defined by $\omega_\rho(x,y)$ is trivial on the certain subspace of 8-dimensional space (x,y)). It follows from the definition (6) that under transformations (4) $\omega_\rho(x,y)$ behaves like the Yang-Mills field:

$$\omega'_\mu(x,y) = \exp\{i\lambda(x)\}[\omega_\mu(x,y) - i\partial^x_\mu] \exp\{-i\lambda(x)\}. \tag{8}$$

The infinite sequence of differential conditions of the inverse Higgs phenomenon /8/ by which unessential Goldstonions $b^i_{\mu_1...\mu_n}(x)$ $(n\geq 2)$ have been eliminated in ref. /8/ at the expense of $b^i_\mu(x)$ and its derivatives is represented, in the bilocal notation, by the one manifestly covariant equation:

$$y^\mu[\omega_\mu(x,y) + b_\mu(x)] = 0 \tag{9}$$

or, with making use of the definition (6):

$$y^\mu(\partial^x_\mu - \partial^y_\mu) \exp\{-ib(x,y)\} = iy^\mu b_\mu(x) \exp\{-ib(x,y)\}. \tag{10}$$

The solution of eq. (10) is given by

$$\exp\{-i\bar{b}(x,y)\} = T \exp\{-i \int_0^1 y^\mu b_\mu [x + (1-\lambda)y] d\lambda\}, \qquad (11)$$

where the symbol T means the ordering in matrices Q^i within the interval $0 \leq \lambda \leq 1$. That is easily recognized as the path integral of the Yang-Mills field along the straight line going from the point $x+y$ to x. The corresponding Cartan form $\bar{\omega}_\rho(x,y)$ in its power expansion in y_μ reads as

$$\bar{\omega}_\rho(x,y) = -b_\rho(x) + \frac{1}{2} G_{\rho\mu}(x) y^\mu + \sum_{n\geq 2} \frac{1}{(n+1)!} \nabla_{\rho_1} \nabla_{\rho_2} \cdots \nabla_{\rho_{n-1}} G_{\rho\rho_n}(x) y^{\rho_1} \cdots y^{\rho_n}, \qquad (12)$$

where $G_{\mu\rho} = \partial_\mu b_\rho - \partial_\rho b_\mu - i[b_\mu, b_\rho]$ is the standard Yang-Mills strength, $\nabla_\rho = \partial_\rho - i[b_\rho,$ is the Yang-Mills covariant derivative.

Thus, the "string functional" of gauge fields which is now under intensive study [1,2,9-11] naturally arises in our approach as the most economical representation for cosets G/G^0. In refs.[1,2,9-11] such functionals are introduced "by hand", as a certain ansatz, while in the present scheme their appearance is the result of consistent application of methods of the general theory of nonlinear realizations. Just this theory prescribes the definition of covariants according to the formula (6), i.e., through ordinary differentiation of the end points of the path (which can be conceived as an infinitesimal rotation of the path as a whole around the point $(x+y)$). In the standard approach to the path integrals, covariants are defined in the essentially nonlocal fashion, through infinitesimal deformations of separate sections of the path.

We would like also to stress the following. As is seen from the above consideration, the inverse Higgs phenomenon in its usual, minimal formulation [6,8] picks out the straight line in a lot of paths between $x+y$ and x. However, without contradiction with the transformation laws (3), (4) it is equally possible to take as a representative of cosets G/G^0 the string functional along any other path (this path should of course be such that the related $b(x,y)$ admit the power expansion about $y_\mu = 0$). The choice of curvelinear path corresponds to a certain modification of differential constraints of the inverse Higgs phenomenon. Namely, in this case the "straight line" condition (9) is replaced by the more general one

$$y^\mu [\omega_\mu(x,y) + b_\mu(x)] = \Delta(x,y), \qquad (13)$$

where $\Delta(x,y) = a\, y^2 y^\lambda \nabla^\rho G_{\rho\lambda} + O(y^4)$ is a covariant functional of the strength $G_{\rho\lambda}(x)$ and degrees of covariant derivatives of $G_{\rho\lambda}(x)$ (a can be an arbitrary number). Knowing its structure completely specifies the configuration of path in the corresponding string functional. Indeed, the latter can always be represented by the formula (11) in which $y^\rho b_\rho(x)$ is changed to $y^\rho b_\rho(x) - \frac{1}{\lambda}\Delta(x,\lambda y)$. Note that any such string functional can be decomposed as

$$\exp\{i\tilde{b}(x,y)\} = \exp\{i\bar{b}(x,y)\} \exp\{ih(x,y)\}$$
$$(h' = e^{i\lambda(x)} h e^{-i\lambda(x)}), \qquad (14)$$

where a nonminimal factor $\exp\{ih(x,y)\}$ describes a deviation from the straight path and is expressed, in its y-expansion, through powers of covariant derivatives of $G_{\rho\lambda}(x)$. The functionals $\Delta(x,y)$ and $h(x,y)$ are related as

$$\Delta(x,y) = \frac{1}{i} \exp\{-ih(x,y)\} y^\mu (\nabla^x_\mu - \partial^y_\mu) \exp\{ih(x,y)\}.$$

To conclude this Section we emphasize that the simple group meaning indicated above can be attributed only to the "open string" functionals of gauge fields. It is as yet not clear how to accomodate within the present scheme the closed contours which are of primary interest in papers [1,2,9-11].

3. The basic relation (6) has the form typical for decompositions by which the covariant derivatives are defined in nonlinear σ-models for principal chiral fields. Therefore, the Yang-Mills theory can be interpreted as a sector of the nonlinear σ-model for the bilocal principal chiral field $b(x,y)$ on the group G^0. This sector is ext-

racted by the conditions (9) or (13) with $b_\mu(x) = \partial^y_\mu b(x,y)|_{y=0}$, $b(x,0) = 0$ by definition*.

In σ-models of the above type the equations of motion (with no sources) can be written as the condition of vanishing the divergence of the corresponding Cartan form. It is natural to ask whether it is possible to represent the standard sourceless Yang-Mills equations

$$\nabla^\rho G_{\rho\lambda}(x) = 0 \tag{15}$$

as an analogous closed differential condition on the bilocal Cartan form $\omega_\mu(x,y)$ (supplementary with respect to the "kinematical" conditions (7), (9) or (7), (13)). From the point of view of the hypothesis of complete integrability of the Yang-Mills theory it is desirable that this condition be of the first order in derivatives.

In the trivial, Abelian case the equation (15) (i.e., the free Maxwell equation) is equivalent to the manifestly covariant condition that the "straight-line" Cartan form $\bar{\omega}_\mu(x,y)$ (12) be divergenceless with respect to y-differentiation:

$$\partial^y_\mu \bar{\omega}^\mu(x,y) = 0. \tag{16}$$

In the non-Abelian case such an equivalency (for the straight path) holds only up to the third order in y_μ and it cannot be restored without adding to the l.h.s. of (16) terms with higher derivatives of $\bar{\omega}_\mu(x,y)$**. It can be shown, however, that for self-dual fields $(G_{\mu\rho} = \pm\frac{1}{2}\epsilon_{\mu\rho\lambda\nu}G^{\lambda\nu})$ and for light-like paths $(y^2=0)$ in (11) (the Minkowski space-time is considered) the equation (16) is satisfied to each order in y_μ. It is not clear if the inverse statement is valid, i.e., whether the self-duality follows from the condition (16) with $y^2=0$.

Perhaps, more important is the following observation. Even in the general case there exists a string functional $\exp\{i\tilde{b}^0(x,y)\}$ such that the associated Cartan form $\tilde{\omega}^0_\mu(x,y)$ is divergenceless with respect to y-differentiation

$$\partial^y_\mu \tilde{\omega}^{0\mu}(x,y) = 0 \tag{17}$$

on solutions of the Yang-Mills equation (15). Conversely, the necessary condition for (17) to be fulfilled is that $b_\mu(x)$ obey (15). The corresponding functional $h^0(x,y)$ (by which $\tilde{b}^0(x,y)$ is completely specified in virtue of (14)) is determined from the equation:

$$\partial^\mu_y \{\exp\{-ih^0(x,y)\}(\partial^x_\mu - \partial^y_\mu + i\bar{\omega}_\mu)\exp\{ih^0(x,y)\}\}|_{\nabla^\mu G_{\mu\rho}=0} = 0 \tag{18}$$

uniquely, up to possible terms vanishing on solutions of eq. (15). The path in $\exp\{i\tilde{b}^0(x,y)\}$ is essentially curvelinear (it becomes straight for an arbitrary y_μ only in the Abelian case) and is presumably formed by the Yang-Mills field $b_\mu(x)$ itself. In other words, one may expect one-to-one correspondence between different classes of solutions of the Yang-Mills equation and permissible configurations of paths in the string functional $\exp\{i\tilde{b}^0(x,y)\}$. We hope to explore this interesting possibility in further publications.

For the time being, we do not know to what extent the above considerations may be useful in proving the hypothetical complete integrability of the Yang-Mills theory. But the fact that the Yang-Mills equations can be represented as local first-order differential conditions on the certain (path-dependent) vector form in the extended space (the condition of trivial connection (7) and the conservation condition (17)) seems highly nontrivial and deserves

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*More analogy with nonlinear σ-models for principal fields can be achieved by giving up the condition (2). In this case the law (4) is modified so that the last factor is absent and the group G^0 turns out to be completely broken, $b(x,0)$ being the Goldstone field by which the breakdown of G^0 is accompanied. The resulting theory is the massive Yang-Mills theory with the mass generated in the invariant manner (through the Higgs mechanism) /12/. We shall consider this interesting extension of the present scheme elsewhere.

**Note a possible parallel at this point with recent results of Witten /3/ and Isenberg et al. /3a/ .

further examination*. In particular, it would be interesting to learn whether this property implies the existence of infinite series of the conservation laws for the Yang-Mills theory which is the standard signal of complete integrability.

4. We have shown that the Yang-Mills theory in its standard, non-symmetric phase, treated as the theory of spontaneous breakdown [8], admits the natural embedding into the bilocal nonlinear σ-model on the group G^0. Based on this, and exploiting the analogy with usual σ-models it seems reasonable to assume that the symmetric phase of the Yang-Mills theory associated with the gauge-invariant vacuum should be described within the corresponding bilocal linear σ-model. In the simplest case $G^0 = SU(2)$, the minimal way to linearize the basic transformation law (4) is to consider a bilocal matrix $U(x, y) = U_0(x, y) + \frac{1}{2} i \tau^k U_k(x,y)$ which transforms according to (4) but does not satisfy the exponentization condition $UU^+ = I$. The infinite set of ordinary fields in the decomposition of $U(x, y)$ in powers of y_μ transforms under the action of the group K linearly and homogeneously. More detailed treatment of this possibility and also of the questions discussed in the previous Sections will be given elsewhere.

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* In papers [1,2] the sourceless Yang-Mills equations have been reformulated as first-order differential conditions on a vector form in the nonlocal space of closed contours. The main difference of our approach is its essentially greater locality: everywhere the ordinary differentiation figures instead of the employed in [1,2] procedure of varying a running point of path.

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