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EXACT CLASSICAL SOLUTIONS OF THE 0(4) CHIRAL THEORY IN EUCLIDEAN SPACE-TIME

Еникова М.М., Карлуковски В.И. **E2 - 12783**

Точные классические решения 0(4) киральной теории в евклидовом пространстве-времени

Рассматривается классическая теория поля с киральной группой симметрии 0(4) в евклидовом пространстве -времени. Получены в явном виде большие семейства точных решений. Обсуждается существование решений с конечным действием.

Работа выполнена в Лаборатории теоретической физики ОИЯИ

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Enikova M., Karloukovski V.I. **E2 - 12783**

Exact Classical Solutions of the 0(4) Chiral Theory in Euclidean Space-Time

A procedure for writing explicitly a family of exact solutions of the classical 0(4) chiral theory in four-dimensional Euclidean space-time is suggested. It developes further our previous results.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The chiral theories have already had quite a long history. The chiral models served as field-theoretic realizations of current algebra and were extensively applied to particle physics. Their nice mathematical interpretation and simplicity rendered them attractive for certain theoretical investigations. The close relation to some more realistic theories increased further their significance and this aspect becomes now even more important in view of the novel relation and reduction of the gauge field theories to (generalized) σ-models[1-4].

In previous work [5-7] we have undertaken a systematic study of some chiral field-theoretic models (in Minkowski space) on a classical level. We started with the conceptionally simplest problems of finding explicitly large families of exact solutions to the corresponding field equations and studying their properties. We hope that this rigorous results could provide a sound basis and stimulate the investigation of the interesting but more difficult and involved problems of complete integrability, conservation laws and quantization.

Here we recall the following result for the O(N+1) chiral model in the realization of a unit-field lagrangian model

$$\mathcal{L} = \frac{1}{2}\partial_\mu n \partial^\mu n \qquad (1.1)$$

and field equations

$$\Box n + (\partial_\mu n \partial^\mu n) n = 0, \qquad (1.2)$$

where the real field $n(x) = (n_0(x), \dots, n_N(x))$ obeys the constraint

$$n^2(x) = 1. \qquad (1.3)$$

Denote $u_1(z) \equiv \sin z$, $u_2(z) \equiv \cos z$. Any field of the form

$$n(x) = \sum_{i_1 i_2 \ldots i_\ell = 1}^{2} a^{i_1 i_2 \ldots i_\ell} \, u_{i_1}(\tau_1) u_{i_2}(\tau_2) \ldots u_{i_\ell}(\tau_\ell) \tag{1.4}$$

is a solution to the field equations (1.2) provided the set of $O(N+1)$-vectors $a^{i_1 i_2 \ldots i_\ell}$ satisfies the algebraic system

$$\frac{1}{2^\ell} \sum_{\mathrm{sym}(i_s, j_s)} a^{i_1 i_2 \ldots i_\ell} \, a^{j_1 j_2 \ldots j_\ell} = \delta_{i_1 j_1} \, \delta_{i_2 j_2} \ldots \delta_{i_\ell j_\ell} \tag{1.5}$$

and the scalar functions $\tau_s = \tau_s(x)$ are solutions of the wave equations

$$\Box \tau_s(x) = 0 \tag{1.6}$$

obeying the constraints

$$\partial_\mu \tau_s \partial^\mu \tau_r = 0, \quad s \neq r. \tag{1.7}$$

The solutions of the algebraic system of 3^ℓ equations for the $(N+1)2^\ell$ components of the vectors $a^{i_1 i_2 \ldots i_\ell}$ can be written explicitly (for sufficiently small ℓ) and so the original problem, Eq. (1.2), is reduced to finding the solutions of the system (1.6), (1.7).

In the present work we study the last system for $\ell = 2$ in the case of the $O(4)$-invariant chiral model in four-dimensional Euclidean space-time (with metric $x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$):

$$\partial^2 \tau \equiv \sum_{n=1}^{4} \frac{\partial^2 \tau}{\partial x_n^2}, \quad \partial^2 \sigma \equiv \sum_{n=1}^{4} \frac{\partial^2 \sigma}{\partial x_n^2}, \quad \partial \tau \partial \sigma \equiv \sum_{n=1}^{4} \frac{\partial \tau}{\partial x_n} \frac{\partial \sigma}{\partial x_n} = 0 . \tag{1.8}$$

This is an overdetermined system of differential equations. In this section we formulate a procedure for writing down solutions to this system. Our final goal is to find the corresponding solutions in Minkowski space by means of analytic continuation. We have restricted the space-time dimensions to be four not only because they are four in reality but also in order to exploit some properties of the vector fields on the unit sphere S_4.

2. SOLUTIONS TO THE OVERDETERMINED SYSTEM

Let

$$w = w(\zeta_1, \zeta_2, \dots, \zeta_6) \tag{2.1}$$

be an arbitrary function of the six variables ζ_j, $j = 1,\dots,6$. Define the complex variables

$$z_{mn} = x_m + i x_n, \tag{2.2}$$

where x_1, x_2, x_3, x_4 are the (real) coordinates in space-time and consider the complex function

$$\tau(x_1,\dots,x_4) + i\sigma(x_1,\dots,x_4) = w(z_{12}, z_{13}, z_{14}, z_{23}, z_{24}, z_{34}). \tag{2.3}$$

The functions τ and σ are its real and imaginary parts, respectively. Differentiating (2.3) with respect to the coordinates we obtain

$$\begin{aligned}
\frac{\partial\tau}{\partial x_1} + i\frac{\partial\sigma}{\partial x_1} &= \frac{\partial w}{\partial z_{12}} + \frac{\partial w}{\partial z_{13}} + \frac{\partial w}{\partial z_{14}}, \\
\frac{\partial\tau}{\partial x_2} + i\frac{\partial\sigma}{\partial x_2} &= i\frac{\partial w}{\partial z_{12}} + \frac{\partial w}{\partial z_{23}} + \frac{\partial w}{\partial z_{24}}, \\
\frac{\partial\tau}{\partial x_3} + i\frac{\partial\sigma}{\partial x_3} &= i\frac{\partial w}{\partial z_{13}} + i\frac{\partial w}{\partial z_{23}} + \frac{\partial w}{\partial z_{34}}, \\
\frac{\partial\tau}{\partial x_4} + i\frac{\partial\sigma}{\partial x_4} &= i\frac{\partial w}{\partial z_{14}} + i\frac{\partial w}{\partial z_{24}} + \frac{\partial w}{\partial z_{34}}.
\end{aligned} \tag{2.4}$$

Differentiating once again with respect to the coordinates and summing up the results we obtain

$$\begin{aligned}
\partial^2\tau + i\partial^2\sigma = {} & 2\frac{\partial^2 w}{\partial z_{12}\partial z_{13}} + 2\frac{\partial^2 w}{\partial z_{12}\partial z_{14}} + 2\frac{\partial^2 w}{\partial z_{13}\partial z_{14}} + 2i\frac{\partial^2 w}{\partial z_{12}\partial z_{23}} + \\
& + 2i\frac{\partial^2 w}{\partial z_{12}\partial z_{24}} + 2\frac{\partial^2 w}{\partial z_{23}\partial z_{24}} - 2\frac{\partial^2 w}{\partial z_{13}\partial z_{23}} + 2i\frac{\partial^2 w}{\partial z_{13}\partial z_{34}} + 2i\frac{\partial^2 w}{\partial z_{23}\partial z_{34}} - \\
& - 2\frac{\partial^2 w}{\partial z_{14}\partial z_{24}} - 2\frac{\partial^2 w}{\partial z_{14}\partial z_{34}} - 2\frac{\partial^2 w}{\partial z_{24}\partial z_{34}}.
\end{aligned} \tag{2.5}$$

Summing the squares of (2.4) we get

$$(\partial\tau)^2 - (\partial\sigma)^2 + 2i\,\partial\tau\,\partial\sigma =$$

$$= 2\frac{\partial w}{\partial z_{12}}\frac{\partial w}{\partial z_{13}} + 2\frac{\partial w}{\partial z_{12}}\frac{\partial w}{\partial z_{14}} + 2\frac{\partial w}{\partial z_{13}}\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{12}}\frac{\partial w}{\partial z_{23}} +$$

$$+ 2i\frac{\partial w}{\partial z_{12}}\frac{\partial w}{\partial z_{24}} + 2\frac{\partial w}{\partial z_{23}}\frac{\partial w}{\partial z_{24}} - 2\frac{\partial w}{\partial z_{13}}\frac{\partial w}{\partial z_{23}} + 2i\frac{\partial w}{\partial z_{13}}\frac{\partial w}{\partial z_{34}} + \quad (2.6)$$

$$+ 2i\frac{\partial w}{\partial z_{23}}\frac{\partial w}{\partial z_{34}} - 2\frac{\partial w}{\partial z_{14}}\frac{\partial w}{\partial z_{24}} - 2\frac{\partial w}{\partial z_{14}}\frac{\partial w}{\partial z_{34}} - 2\frac{\partial w}{\partial z_{24}}\frac{\partial w}{\partial z_{34}}.$$

At this way we come at the following lemma:

If the complex function w of the complex variables (2.2) satisfies the following two differential equations:

$$\frac{\partial}{\partial z_{12}}\left[\frac{\partial w}{\partial z_{13}} + 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} + i\frac{\partial w}{\partial z_{24}}\right] +$$

$$+ \frac{\partial}{\partial z_{13}}\left[\frac{\partial w}{\partial z_{12}} + 2\frac{\partial w}{\partial z_{14}} - 2\frac{\partial w}{\partial z_{23}} + i\frac{\partial w}{\partial z_{34}}\right] +$$

$$+ \frac{\partial}{\partial z_{24}}\left[i\frac{\partial w}{\partial z_{12}} - 2\frac{\partial w}{\partial z_{14}} + 2\frac{\partial w}{\partial z_{23}} - \frac{\partial w}{\partial z_{34}}\right] + \quad (2.7)$$

$$+ \frac{\partial}{\partial z_{34}}\left[i\frac{\partial w}{\partial z_{13}} - 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} - \frac{\partial w}{\partial z_{24}}\right]$$

and

$$\frac{\partial w}{\partial z_{12}}\left[\frac{\partial w}{\partial z_{13}} + 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} + i\frac{\partial w}{\partial z_{24}}\right] +$$

$$+\frac{\partial w}{\partial z_{13}}\left[\frac{\partial w}{\partial z_{12}}+2\frac{\partial w}{\partial z_{14}}-2\frac{\partial w}{\partial z_{23}}+i\frac{\partial w}{\partial z_{34}}\right]+ \tag{2.8}$$

$$+\frac{\partial w}{\partial z_{24}}\left[i\frac{\partial w}{\partial z_{12}}-2\frac{\partial w}{\partial z_{14}}+2\frac{\partial w}{\partial z_{23}}-\frac{\partial w}{\partial z_{34}}\right]+$$

$$+\frac{\partial w}{\partial z_{34}}\left[i\frac{\partial w}{\partial z_{13}}-2\frac{\partial w}{\partial z_{14}}+2i\frac{\partial w}{\partial z_{23}}-\frac{\partial w}{\partial z_{24}}\right]$$

then its real and imaginary parts, $\tau(x)$ and $\sigma(x)$, are solutions of the system (1.8). Moreover, it follows also from Eq.(2.6), that

$$(\partial\tau)^2=(\partial\sigma)^2. \tag{2.9}$$

The meaning of Eq.(2.8) is that the vector whose components are the quantities in the brackets must be orthogonal to the vector

$$\left(\frac{\partial w}{\partial z_{12}},\frac{\partial w}{\partial z_{13}},\frac{\partial w}{\partial z_{24}},\frac{\partial w}{\partial z_{34}}\right), \tag{2.10}$$

i.e., it is a combination of the three orthogonal vector fields

$$\left(\frac{\partial w}{\partial z_{13}},-\frac{\partial w}{\partial z_{12}},\frac{\partial w}{\partial z_{34}},-\frac{\partial w}{\partial z_{24}}\right),$$

$$\left(\frac{\partial w}{\partial z_{24}},-\frac{\partial w}{\partial z_{34}},-\frac{\partial w}{\partial z_{12}},\frac{\partial w}{\partial z_{13}}\right), \tag{2.11}$$

$$\left(\frac{\partial w}{\partial z_{34}},\frac{\partial w}{\partial z_{24}},-\frac{\partial w}{\partial z_{13}},-\frac{\partial w}{\partial z_{12}}\right),$$

In this way we come to

$$\frac{\partial w}{\partial z_{13}}+2\frac{\partial w}{\partial z_{14}}+2i\frac{\partial w}{\partial z_{23}}+i\frac{\partial w}{\partial z_{24}}=\lambda\frac{\partial w}{\partial z_{13}}+\mu\frac{\partial w}{\partial z_{24}}+\nu\frac{\partial w}{\partial z_{34}},$$

$$\frac{\partial w}{\partial z_{12}} + 2\frac{\partial w}{\partial z_{14}} - 2\frac{\partial w}{\partial z_{23}} + i\frac{\partial w}{\partial z_{34}} = -\lambda\frac{\partial w}{\partial z_{12}} - \mu\frac{\partial w}{\partial z_{34}} + \nu\frac{\partial w}{\partial z_{24}},$$

$$i\frac{\partial w}{\partial z_{12}} - 2\frac{\partial w}{\partial z_{14}} + 2\frac{\partial w}{\partial z_{23}} - \frac{\partial w}{\partial z_{34}} = \lambda\frac{\partial w}{\partial z_{34}} - \mu\frac{\partial w}{\partial z_{12}} - \nu\frac{\partial w}{\partial z_{13}}, \tag{2.12}$$

$$i\frac{\partial w}{\partial z_{13}} - 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} - \frac{\partial w}{\partial z_{24}} = -\lambda\frac{\partial w}{\partial z_{24}} + \mu\frac{\partial w}{\partial z_{13}} - \frac{\partial w}{\partial z_{12}}.$$

Then Eq. (2.7) becomes

$$\{\lambda, w\}_{z_{12}z_{13}} + \{\lambda, w\}_{z_{24}z_{34}} + \{\mu, w\}_{z_{12}z_{24}} + \{\mu, w\}_{z_{34}z_{13}} + \{\nu, w\}_{z_{12}z_{34}} + \{\nu, w\}_{z_{13}z_{24}} = 0, \tag{2.13}$$

where

$$\{A, B\}_{ab} \equiv \frac{\partial A}{\partial a}\frac{\partial B}{\partial b} - \frac{\partial A}{\partial b}\frac{\partial B}{\partial a} \tag{2.14}$$

and its simplest solution which we shall study in detail in what follows is

$$\lambda = \text{const}, \quad \mu = \text{const}, \quad \nu = \text{const}. \tag{2.15}$$

In this case the problem is completely linearized. It remains to solve the linear system (2.12) with constant coefficients which may be written in the form

$$(1-\lambda)\frac{\partial w}{\partial z_{13}} + 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} + (i-\mu)\frac{\partial w}{\partial z_{24}} - \nu\frac{\partial w}{\partial z_{34}} = 0,$$

$$(1+\lambda)\frac{\partial w}{\partial z_{12}} + 2\frac{\partial w}{\partial z_{14}} - 2\frac{\partial w}{\partial z_{23}} - \nu\frac{\partial w}{\partial z_{24}} + (i+\mu)\frac{\partial w}{\partial z_{34}} = 0,$$

$$(i+\mu)\frac{\partial w}{\partial z_{12}} + \nu\frac{\partial w}{\partial z_{13}} - 2\frac{\partial w}{\partial z_{14}} + 2\frac{\partial w}{\partial z_{23}} - (1+\lambda)\frac{\partial w}{\partial z_{34}} = 0, \qquad (2.16)$$

$$\nu\frac{\partial w}{\partial z_{12}} + (i-\mu)\frac{\partial w}{\partial z_{13}} - 2\frac{\partial w}{\partial z_{14}} + 2i\frac{\partial w}{\partial z_{23}} + (\lambda-1)\frac{\partial w}{\partial z_{24}} = 0,$$

Any function w of the variable

$$\xi = A_{12}z_{12} + A_{13}z_{13} + A_{24}z_{24} + A_{34}z_{34} + A_{14}z_{14} + A_{23}z_{23} \qquad (2.17)$$

satisfies the system (2.16) provided the constant coefficients A_{ij} obey the linear system

$$M\begin{bmatrix} A_{12} \\ A_{13} \\ A_{24} \\ A_{34} \end{bmatrix} = \begin{bmatrix} -2A_{14} - 2iA_{23} \\ -2A_{14} + 2A_{23} \\ 2A_{14} - 2A_{23} \\ 2A_{14} - 2iA_{23} \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 1-\lambda & i-\mu & -\nu \\ 1+\lambda & 0 & -\nu & i+\mu \\ i+\mu & \nu & 0 & -1-\lambda \\ \nu & i-\mu & -1+\lambda & 0 \end{bmatrix} \qquad (2.18)$$

with

$$\det M = (\lambda^2 + \mu^2 + \nu^2)^2 - 4(\lambda + i\mu)^2. \qquad (2.19)$$

We shall denote by A'_{ij} the solutions of (2.18) with $A'_{14} = 1$, $A'_{23} = 0$, and by A''_{ij} the solutions of (2.18) with $A''_{14} = 0$, $A''_{23} = 1$. The set of solutions to the system (2.16) consists of the complex functions $w = w(\xi', \xi'')$ of the two complex variables

$$\xi' = \xi_1 + i\xi_2 = A'_{12}z_{12} + A'_{13}z_{13} + A'_{24}z_{24} + A'_{34}z_{34} + z_{14},$$
$$\xi'' = \xi_3 + i\xi_4 = A''_{12}z_{12} + A''_{13}z_{13} + A''_{24}z_{24} + A''_{34}z_{34} + z_{23}, \qquad (2.20)$$

where A_{ij} are algebraic functions of the complex parameters $\lambda = \lambda_1 + i\lambda_2$, $\mu = \mu_1 + i\mu_2$, and $\nu = \nu_1 + i\nu_2$. Equation (2.20) can be rewritten in the form

$$\xi_j = \sum_{n=1}^{4} A_j^n x_n, \quad j = 1, \ldots, 4, \qquad (2.21)$$

where $A_j^n = A_j^n(\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1, \nu_2)$ are real algebraic functions of the six real parameters λ_k, μ_k, ν_k). So we arrive at the following results:

Given any anlytic function of the two complex variables (2.20), its real and imaginary parts, $\tau(x)$ and $\sigma(x)$, provide us with a solution of the overdetermined system of differential equations (1.8).

3. SOLUTIONS WITH FINITE ACTION

The $\ell=2$ family of solutions to the field equations (1.2) with O(4) chiral symmetry we have written explicitly has the form

$$n(x) = a^{11}\cos\tau\cos\sigma + a^{12}\cos\tau\sin\sigma + a^{21}\sin\tau\cos\sigma + a^{22}\sin\tau\sin\sigma \quad (3.1)$$

in which the vectors $a^{i_1 i_2}$ (satisfying Eq.(1.5)) can be chosen to be

$$\begin{aligned} &a^{11} = (1, 0, 0, 0), \qquad a^{22} = (\cos\alpha,\ \sin\alpha,\ 0, 0), \\ &a^{12} = (0, 0, 0, 1), \qquad a^{21} = (0, 0,\ \sin\alpha,\ -\cos\alpha), \end{aligned} \quad (3.2)$$

and the functions $\tau(x)$ and $\sigma(x)$ obey (1.8) in Euclidean four-dimensional space-time. These functions $\tau(x)$ and $\sigma(x)$ can be constructed according to the preceding section starting with any analytic function w of the two complex variables (2.20).

The action

$$S = \int \mathcal{L}\, dx = \frac{1}{2} \int \partial_\mu n \partial^\mu n\, dx \quad (3.3)$$

takes the form

$$S = \frac{1}{2} \int [(\partial\tau)^2 + (\partial\sigma)^2]\, d^4x \quad (3.4)$$

on the manifold of solutions (3.1) in the Euclidean space-time. In terms of the function $w(\xi', \xi'')$ one obtains

$$(\partial\tau)^2 + (\partial\sigma)^2 = \frac{\partial w}{\partial\xi'} \frac{\partial\bar{w}}{\partial\bar{\xi}'} \sum_{j=1}^{2} \sum_{n=1}^{4} (A_j^n)^2 + \frac{\partial w}{\partial\xi''} \frac{\partial\bar{w}}{\partial\bar{\xi}''} \sum_{j=3}^{4} \sum_{n=1}^{4} (A_j^n)^2 +$$

$$+ 2\mathrm{Re}\left\{\frac{\partial w}{\partial\xi'} \frac{\partial\bar{w}}{\partial\bar{\xi}''}\right\} \sum_{j=1}^{2} A_j A_{j+2} + 2\mathrm{Im}\left\{\frac{\partial w}{\partial\xi'} \frac{\partial\bar{w}}{\partial\bar{\xi}''}\right\} \sum_{n=1}^{4} (A_1^n A_4^n - A_2^n A_3^n).$$
$$(3.5)$$

This implies that the action (3.4) is bounded by

$$S \leq C(\lambda, \mu, \nu) \int \left| \frac{\partial w}{\partial \xi'} + \frac{\partial w}{\partial \xi''} \right|^2 d\xi_1 \ldots d\xi_4 \qquad (3.6)$$

and in this way the problem of constructing solutions with finite action is reduced to the problem of finding complex functions $w(\xi', \xi'')$ for which the integral in the right-hand side of (3.6) is finite.

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