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Релятивистская динамика стохастических частиц

Исследуется вопрос о движении частицы в стохастическом пространстве, т.е. пространстве, координаты которого состоят из регулярных и малых стохастических частей. Показано, что в первом приближении по параметру ℓ /универсальная длина/ движение частицы во внешнем силовом поле описывается уравнениями, совпадающими по форме с уравнениями стохастической механики Д. Кершоу, Э.Нельсона, Пена-Ауэрбаха и др. Предлагается метод релятивизации данной схемы описания процессов в стохастическом пространстве, позволяющий записать уравнения движения частицы в ковариантном виде.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Relativistic Dynamics of Stochastic Particles

This paper deals with the study of the problem on the particle motion in the stochastic space, i.e., the space whose coordinates consist of regular and small stochastic parts. A free particle in this space resembles the Brownian particle the motion of which is characterized by the dispersion $\mathfrak{D}$ dependent on the universal length ℓ. It is shown that in the first approximation in parameter ℓ the particle motion in an external force field is described by equations coincident in form with equations of the stochastic mechanics by Nelson E., Kershaw D., L. de la Pena-Auerbach. A method is proposed for relativization of the given scheme of describing the processes in the stochastic space; by using this method, the equations of particle motion can be written in a covariant form.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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The interest in stochastic processes and fields has grown in the last years. This is mainly due to the fact that the close correspondence between stochastic processes and quantum mechanics /1/ , and Euclidean quantum field theory /2/ has been found. The stochastic quantization of Nelson has been generalized to the case of continuous systems /3/ and also to the cases of particles with spin /4/ and relativistic mechanics /5/.

There are also other approaches to investigation of stochastic processes and fields. Some of them start with the hypothesis on the stochastic property of the electromagnetic vacuum /6/ (this approach is called the stochastic electrodynamics). Other approaches are based, more or less, on the concept of stochastic spaces /7/ . One assumes in this case that the random behaviour is caused by the stochastic character of the Physical space (in the analogy with the Brownian movement). In other words, the stochastic character of processes is interpreted as a result of action of the space alone on the considered physical system.

Following an idea of Blokhintsev [7] we investigate in the present paper the problem of motion of a particle, the coordinates of which in a stochastic space $R_4(\hat{x})$ are defined by two terms $\hat{x}_\mu = x_\mu + b_\mu$, x_μ being the regular part of the coordinates and b_μ some random vector with a distribution $\tau(b)$ obeying the condition

$$\int d\tau(b_\mu) = 1 \quad , \qquad d\tau(b_\mu) \geq 0 .$$

In the nonrelativistic case it is sufficient to assume the stochastic property of the space component $\vec{x} \to \hat{\vec{x}} = \vec{x} + \vec{b}$, but in the relativistic case such an operation needs some explanation. The space $R_4(\hat{x})$ in a relativistic theory must be the Minkowski space. Indefiniteness of metrics of this space leads to specific problems which do not appear in the case of Euclidean space. These specific difficulties are connected with the invariance assumption and normalization condition for the probability of a value of an interval in the indefinite-metric space [8] . The invariance assumption, roughly speaking, means that the distribution $\tau(b_\mu)$ of the vector b_μ must be a function of the interval $b^2 = b^\mu b_\mu$ and the normalization condition gives the equality $\int d\tau(b^\mu b_\mu) = 1$. These two conditions cannot be, in fact, fulfilled simultaneously in the Minkowski space. However, one can get rid of the above-mentioned difficulty assuming the stochasticity of the space $R_4(\hat{x})$ to appear in the Euclidean region of the variables $\hat{x}_\mu$. Using the language of random fluctuations this means that the fluctuations appear in the Euclidean space $E_4(\hat{x})$. We make, at the same time with this assumption, a shift of the coordinates in such a way that the coordinate $x_0 = ct$ will obtain a pure-imaginary additive term, and the coordinates $\vec{x}$ will stay real. This shift procedure is deeply tight to the fundamental problem of causality and it is also directly related to the relativistic invariant description of extended objects. There exists also a correspondence of this procedure to the transformation to the pseudo-Euclidean region in the case when a construction of quantum field theory proceeds from the Euclidean picture (more details can be found in the monograph /9/ of Efimov). Thus, starting a construction in an Euclidean stochastic space, the relativistic-invariant description of the particle motion in a stochastic space may be realized. Attractivity of this approach based on the idea of stochasticity of the physical space is mainly due to the fact that it allows one to generalize the stochastic mechanics of Nelson to the relativistic case.

If we consider the problem of random flights within the framework of space stochasticity, then it can be shown easily that the probability that a particle will be displaced by $\Delta\vec{r}$ during an interval Δt on account of space stochasticity is given by

$$\psi(\Delta\vec{r},\Delta t)=(4\pi\mathcal{D}\Delta t)^{-3/2}\exp\left\{-|\Delta\vec{r}|^2/4\mathcal{D}\Delta t\right\},$$

where
$$\mathcal{D}=\frac{1}{2}\ell^2\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N}\left(a_j\sqrt{n}\right)^2.$$

n is the number of displacements per unit time, a_j an arbitrary sequence of real numbers and ℓ the universal length. The fluctuations of the particle position resulting from this stochasticity of space are assumed to be describable as the Markoff process. Accordingly, the position probability density $\rho(\vec{x},t)$ must obey the Smoluchowski equations

$$\rho(\vec{x}, t \pm \Delta t) = \int \rho(\vec{x} \mp \delta\vec{x}_{\pm}, t)\, \Psi_{\pm}(\vec{x} \mp \delta\vec{x}_{\pm}, t; \delta\vec{x}_{\pm}, \Delta t)\, d^3(\delta x_{\pm}) \quad (2)$$

where

$$\Psi_{\pm}(\vec{x}, t; \delta\vec{x}_{\pm}, \Delta t) = (4\pi\mathcal{D}_{\pm}\Delta t)^{-3/2} \exp\left\{-(\delta\vec{x}_{\pm} - \vec{v}_{\pm}(\vec{x}, t)\Delta t)^2 / 4\mathcal{D}_{\pm}\Delta t\right\},$$

$\delta\vec{x}_{\pm} = \vec{v}_{\pm}(\vec{x}, t)\Delta t + \Delta\vec{x}_{\pm}$ is the total dispacement during the time interval Δt. The quantities $\vec{v}_{+}$, $\vec{v}_{-}$, $\vec{v} = (\vec{v}_{+} + \vec{v}_{-})/2$ and $\vec{u} = (\vec{v}_{+} - \vec{v}_{-})/2$ are called the forward and backward total drift and stochastic velocity, respectively.

Due to Kershaw /10/ we can constitute equations of the type (2) for the mean velocity $v_j^{\pm}(\vec{x}, t \pm \Delta t)$ in the potential field $F_j = -\partial U / \partial x_j$ by the following formula

$$v_j^{\pm}(\vec{x}, t \pm \Delta t) = \frac{1}{N^{\pm}} \int \left[v_j^{\pm}(\vec{x} \mp \delta\vec{x}_{\pm}, t) \pm \frac{\Delta t}{m} F_j^{\pm}(\vec{x} \mp \delta\vec{x}_{\pm}, t) \right] \cdot \quad (3)$$
$$\rho(\vec{x} \mp \delta\vec{x}_{\pm}, t)\, \Psi_{\pm}(\vec{x} \mp \delta\vec{x}_{\pm}, t; \delta\vec{x}_{\pm}, \Delta t)\, d^3(\delta x_{\pm}),$$

where $N^{\pm}$ are the normalization constants

$$N^{\pm} = \int \rho(\vec{x} \mp \delta\vec{x}_{\pm}, t)\, \Psi_{\pm}(\vec{x} \mp \delta\vec{x}_{\pm}, t; \delta\vec{x}_{\pm}, \Delta t)\, d^3(\delta x_{\pm}).$$

Upper (lower) sign corresponds to v_j^{+} (v_j^{-}). We assume $\mathcal{D}_{+} = \mathcal{D}_{-} = \mathcal{D}$ expand $v_j^{\pm}, \rho, \Psi_{\pm}$ and $F_j^{\pm}$ in the Taylor series, integrate and retain only the terms of the first order in Δt, then we get (4)

$$m\left(\frac{\partial v_j^{\pm}}{\partial t} + v_i^{\pm} \nabla_i v_j^{\pm}\right) = F_j^{\pm} \pm m\mathcal{D}\frac{1}{\rho}\left(\nabla^2(\rho v_j^{\pm}) - v_j^{\pm}\nabla^2\rho\right),$$

and

$$\left.\begin{aligned} \frac{\partial\rho}{\partial t} &= -\nabla_j(\rho v_j^{+}) + \mathcal{D}\nabla^2\rho \\ \frac{\partial\rho}{\partial t} &= -\nabla_j(\rho v_j^{-}) - \mathcal{D}\nabla^2\rho \end{aligned}\right\} \quad (5)$$

We pass to the variables v_j, u_j and sum the equations in (4) and (5), so we obtain

$$\left.\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla_j(\rho v_j) \\ d_c v_j - d_s u_j &= \frac{1}{m} F_j \end{aligned}\right\} \tag{6}$$

where $u_i = \mathcal{D}\nabla_i \ln\rho$, $F_j = (F_j^+ + F_j^-)/2$, $d_c = \frac{\partial}{\partial t} + v_i \nabla_i$ and $d_s = u_i \nabla_i + \mathcal{D}\nabla^2$.

We see that starting with the hyperthesis on the space stochasticity and the assumption $\mathcal{D} = \hbar/2m$, we obtain the same fundamental equations (6), which were derived by Nelson /1/, Pena-Auerbach /5/ and Skagerstam /11/ by different methods.

It is very interesting to investigate the relativistic motion of a particle in the space which is assumed to be stochastic. Since the problem of the construction of Markoff diffusion processes in a relativistic-covariant way is a long-standing open problem (see /5/ and the references therein). Now we pass to this question. As shown above, in the relativistic case we consider formally the motion of a particle suffering the random flights on account of stochasticity of four-dimensional Euclidean space $E_4(\hat{x})$. Let the particle make N displacements, then its position is given by $B_\mu = \sum_{j=1}^{N} \beta_j b_\mu$, where the β_j's form an arbitrary set of N real numbers and the vector $b_\mu = (b_4, \vec{b})$ is distributed with the probability density $\tau(b_\nu b^\nu) d^4 b$, $b_\nu b^\nu = b_4^2 + \vec{b}^2$. The probability $\rho(B_\mu) d^4 B$ that B_μ belongs to the region (B_μ, $B_\mu + dB_\mu$) is given by

$$\rho(B_\mu) d^4 B = d^4 B (4\pi\mathcal{D}s)^{-2} \exp\left\{-B^2/4\mathcal{D}s\right\},$$

where s is some invariant parameter (proper time) . Further in the analogy with the 3-dimensional case we determine the transition probability

$$\psi(\Delta y_E, \Delta s) = (4\pi \mathcal{D} \Delta s)^{-2} \exp\left\{-|\Delta y_E|^2 / 4\mathcal{D}\Delta s\right\} .$$

The Smoluchowski equation acquires the following form

$$\rho_E(x_\mu^E, s+\Delta s) = \int d^4 y_E\; \psi(y_E, \Delta s) \rho_E(x_\mu^E - y_\mu^E, s). \tag{7}$$

and the diffusion equation is given by

$$\frac{\partial \rho_E}{\partial s} = \mathcal{D} \Box_E \rho_E , \quad \Box_E = \frac{\partial^2}{\partial x_4^2} + \frac{\vec{\partial}^2}{\partial \vec{x}^2} . \tag{8}$$

If $\psi(y_E, \Delta s)$ depends on a point x_μ^E , for example,

$$\psi(x_\mu^E, y_E, \Delta s) = (4\pi \mathcal{D} \Delta s)^{-2} \exp\left\{-(y_E - y_E^o(x_\mu^E))^2 / 4\mathcal{D}\Delta s\right\} ,$$

where $(y_E^o)^\mu = u_E^\mu \Delta s$, $u_E^\mu = (u_4, \vec{u})$ is the 4-dimensional Euclidean velocity of a particle, then instead of (8) one obtains the Fokker-Planck-type equation in the Euclidean space

$$\frac{\partial \rho_E}{\partial s} = -\partial_\mu^E (u_E^\mu \rho_E) + \mathcal{D} \Box_E \rho_E , \quad \partial_\mu^E = \left(\frac{\partial}{\partial x_4}, \vec{\nabla}\right) .$$

Now the question of transition to the pseudo-Euclidean space arises. According to the above deduction, we may write the equation (7) in the following form

$$\rho(x_\mu, s \pm \Delta s) = \int d^4 y_E\; \psi_\pm(y_E, \Delta s) \rho(\vec{x} \mp \vec{y}, x_0 + i y_4, s) , \tag{9}$$

where the variables $x_\mu = (x_0, \vec{x})$ are pseudo-Euclidean and $\psi_\pm$ can be chosen in the form

$$\psi_\pm = (4\pi \mathcal{D} \Delta s)^{-2} \exp\left\{-(y_E - y_\pm)^2 / 4\mathcal{D}\Delta s\right\} , \quad y_\pm = \left(\pm i u_\pm^o \Delta s,\ \vec{u}_\pm \Delta s\right) ,$$

$u_\pm^\mu$ are four-dimensional velocity vectors. We obtain easily from this the two Fokker-Plack equations

$$\left.\begin{aligned} \frac{\partial \rho}{\partial s} &= -\partial_\mu (\rho u_+^\mu) + \mathcal{D} \Box \rho \\ \frac{\partial \rho}{\partial s} &= -\partial_\mu (\rho u_-^\mu) - \mathcal{D} \Box \rho \end{aligned}\right\} \tag{10}$$

$$\partial_\mu = \left(\frac{\partial}{\partial x_0}, \vec{\nabla}\right), \qquad \Box = -\frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial \vec{x}^2} .$$

In what follows we shall assume that the derivative with respect to S used here could be interpreted as a direvative with respect to the direction of some (arbitrarily chosen) vector V_μ. Especially, if V_μ is the particle velocity, then S may be interpreted as the proper time of this particle. This problem is discussed in more detail in the extended version of this paper .

It is known that the conservation of mass (interpreted as the probability density times volume) means that the mass cannot disappear through any hyperplane characterized by the vector V_μ. This implies, especially, that the probability density of the particle along its world line is constant, i.e.,

$$\frac{\partial \rho}{\partial s} = (\vec{d}\rho \cdot \vec{u}) = \frac{\partial \rho}{\partial x_0} u_0 + \frac{\vec{v}}{\sqrt{1-\beta^2}} \frac{\partial \rho}{\partial \vec{x}} = 0 . \tag{11}$$

The notation used is borrowed from the textbook /12/ . The last equation expressed the conservation of the particle mass (the probability density of the current). We see from (10) that the condition (11) leads, in fact, to the following two equations:

$$\partial_\mu (\rho u_+^\mu) = \mathcal{D} \cdot \Box \rho \qquad \text{and} \qquad \partial_\mu (\rho u_-^\mu) = -\mathcal{D}\, \Box \rho .$$

Further we shall define vectors of the four-dimensional current:

$$j_\pm^\mu = \rho u_\pm^\mu , \qquad j^\mu = \frac{1}{2}(j_+^\mu + j_-^\mu) , \qquad v^\mu = j^\mu / \rho .$$

Then $j^\mu(x)$ obeys the continuity equation

$$\partial_\mu j^\mu(x) = 0 . \tag{12}$$

In the framework elaborated here the relativistic equations for $u_\pm^\mu$ are obtained from the following integral equations:

$$u_\pm^\mu(x, S_\pm + \varepsilon \Delta S_\pm) = \frac{1}{N^\pm} \int \Big[u_\pm^\mu(\vec{x} - \varepsilon\vec{y}, x_0 + iy_4, S_\pm) + \frac{\varepsilon \Delta S_\pm}{m} F_\pm^\mu(\vec{x}-\varepsilon\vec{y}, x_0 + iy_4) \Big] \cdot$$
$$\cdot \rho(\vec{x} - \varepsilon \vec{y}, x_0 + iy_4)\, \psi_\pm(y, \Delta S_\pm)\, d^4 y_E , \tag{13}$$

where

$$N^{\pm}=\int d^4y_E\, \Psi_{\pm}\, \rho\,, \qquad \varepsilon=\begin{cases} 1 & for\ \ u^{\mu}_{+} \\ -1 & for\ \ u^{\mu}_{-}\,. \end{cases}$$

s_+ (s_-) is the proper time of the particle u^{μ}_{+} (u^{μ}_{-}) (or they may be interpreted as some parameters the derivatives with respect to which are equal to $\partial/\partial s_{\pm}=u^{\nu}_{\pm}\partial_{\nu}$). The equations (13) imply in this case

$$\left.\begin{aligned} \frac{\partial u^{\mu}_{+}}{\partial s_{+}} &= 2u^{\nu}\partial_{\nu}u^{\mu}_{+}+\mathcal{D}\Box u^{\mu}_{+}+F^{\mu}_{+}/m\,, \\ \frac{\partial u^{\mu}_{-}}{\partial s_{-}} &= -2u^{\nu}\partial_{\nu}u^{\mu}_{-}-\mathcal{D}\Box u^{\mu}_{-}+F^{\mu}_{-}/m \end{aligned}\right\} \qquad (14)$$

where

$$u^{\mu}=\frac{1}{2}(u^{\mu}_{+}-u^{\mu}_{-})=-\mathcal{D}\,\partial^{\mu}\ln\rho=\left(-\mathcal{D}\frac{\partial}{\partial x_0}\ln\rho,\ \mathcal{D}\vec{\nabla}\ln\rho\right).$$

We sum the equations (14) having in mind the definition of the derivative with respect to a given direction; thus we obtain

$$\frac{1}{m}F^{\mu}=\frac{1}{2m}\left(F^{\mu}_{+}+F^{\mu}_{-}\right)=\mathcal{D}_c v^{\mu}-\mathcal{D}_s u^{\mu}\,, \qquad (15)$$

$$\mathcal{D}_c=v^{\nu}\partial_{\nu}\,, \qquad \mathcal{D}_s=u^{\nu}\partial_{\nu}+\mathcal{D}\Box\,.$$

The equation (15) together with the equation of continuity (12) represent themselves the covariant analogy of (6) in the relativistic case. Let us notice that the right-hand side of the equation (15) coincides exactly with the expression for acceleration obtained on the basis of some assumptions in the framework of the mathematical approach of Nelson (Lehr-Park /5/, Guerra-Ruggiero /5/). Our equations (12) and (15) coincide with the Klein-Gordon equation if $\mathcal{D}=\hbar/2m$,
$F^{\mu}=\frac{e}{c}(\partial^{\mu}A^{\lambda}-\partial^{\lambda}A^{\mu})v_{\lambda}$, $v^{\mu}+\frac{e}{mc}A^{\mu}=\frac{1}{m}\partial^{\mu}S$ where A^{μ} is the electromagnetic potential and S is the world scalar.

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References:

1. E.Nelson: Phys.Rev., 150, 1079 (1966); Dynamical Theories of Brownian Motion (Princeton, N.J., 1967).
2. E.Nelson: Journ.Funct.Anal., 12, 97, 211 (1973).
3. F.Guerra and P.Ruggiero: Phys.Rev.Lett., 31, 1022 (1973).
4. T.G.Dankel: Arch.Ration.Mech.Anal., 37, 192 (1970); Journ.Math.Phys., 18, 253 (1977).
5. J.P.Caubet: Le mouvement Brownien relativiste, in Lecture Notes in Mathematical, No. 559 (Heidelberg, 1976); L.de la Peña-Auerbach: Journ.Math.Phys., 12, 453 (1971); W.J.Lehr and J.L.Park: Journ. Math.Phys., 18, 1235 (1977); F.Guerra and P.Ruggiero: Lett. Nuovo Cimento 23, 529 (1978).
6. P.Braffort and C.Tzara: Compt.Rend.Acad.Sci.Paris, 239, 1779 (1954); T.W.Marshall: Proc.Roy.Soc. A276, 475 (1963); M.Surdin: Int.Journ.Theor.Phys., 4(2), 117 (1971); E.Santos: Nuovo Cimento 19, 57 (1974); T.H.Boyer. Phys.Rev., D11, 790, 809 (1975).
7. D.I.Blokhintsev. Fiz.elem.chastits at.yad. 5, 606 (1975). (Sov.Journ.Part. Nucl. 5, 242 (1975)) and references quoted therein; M.Dineikhan and Kh.Namsrai. Theor.and Math.Phys., 33, 32 (1977) (in Russian). C.Frederick: Phys.Rev. D13, 3183 (1976).
8. D.I.Blokhintsev. Space and Time in Microword (Nauka, Moscow 1970)(In Russian).
9. G.V.Efimov. Nonlocal Interactions of quantized fields (Nauka, Moscow, 1977) (in Russian).

10. D.Kershaw. Phys.Rev. 136, B1850 (1964).

11. B.Skagerstam: Inst.Theor.Phys. report 75-21 Gothenburg, Sweden (1975).

12. Ch.W.Misner, K.S.Thorne and J.A.Wheeler. Gravitation (W.H.Freeman and Company, San Francisco 1970).

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