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Квазипотенциальный подход к $\pi-\pi$ рассеянию в $SU_2 \times SU_2$ киральной теории

Рассматривается низкоэнергетическое пион-пионное рассеяние. Найдены в явном виде точные решения бесконечного числа квазипотенциальных уравнений с квазипотенциалом, построенным в древесном приближении нелинейной $(SU_2 \times SU_2)$-инвариантной σ-модели. Получены длины рассеяния, эффективные радиусы, фазовые сдвиги и амплитуды низкоэнергетического пион-пионного рассеяния.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Karloukovski V.I.

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Quasipotential Approach to $\pi-\pi$ Scattering in $SU_2 \times SU_2$ Chiral Theory

An infinite set of quasipotential equations is solved exactly with the quasipotential constructed in the tree approximation of the nonlinear σ-model. The scattering lengths, the s- and p-wave phase shifts, and the amplitude of the low-energy pion-pion scattering are found explicitly.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1979

1. INTRODUCTION

The σ-models /1-5/, the first Lagrangian models with chiral symmetry, were used as a convenient device to obtain unitary corrections to current algebra results. In particular, the low-energy $\pi-\pi$ scattering was studied in the framework of the linear /6,7/ and the non-linear /8-12/ σ-models by means of the superpropagator and the Padé method (excluding /7/ where quasipotential approach was used).

In the present work we study the applicability of the quasipotential approach /13/ to the $\pi-\pi$ scattering in the case when the potential is constructed in the tree approximation of the non-linear σ-model. We write the Lagrangian of the latter in Weinberg's /14/ notation

$$\mathcal{L} = \frac{1}{2} g_{ab}(\pi)\, \partial_\mu \pi^a \partial^\mu \pi^b \tag{1.1}$$

with

$$g_{ab}(\pi) = d_1(\pi^2)\delta_{ab} + d_2(\pi^2)\pi_a \pi_b \tag{1.2}$$

$$d_1(\pi^2) = F_\pi^2 [\pi^2 + f^2(\pi^2)]^{-1}; \quad d_2(\pi^2) = -F_\pi^2 (1 + 4ff' - 4\pi^2 f'^2)(\pi^2 + f^2)^{-2}. \tag{1.3}$$

Here $F_\pi \approx 94$ MeV is the pion decay constant and $f(\pi^2)$ is an arbitrary function of π^2 specifying the choice of the pion field parametrization

$$f(\pi^2) = F_\pi \sum_{n=0}^{\infty} f_n \left(\frac{\pi^2}{F_\pi^2}\right)^n . \tag{1.4}$$

Only the coefficient f_1 of this expansion appears in the $\pi-\pi$ scattering amplitude which has in tree approximation the form

$$T_{a_1 a_2 a_3 a_4}(s,t,u) = -F_\pi^{-2} [A(s,t,u)\delta_{a_1 a_2}\delta_{a_3 a_4} + A(t,s,u)\delta_{a_1 a_3}\delta_{a_2 a_4} + A(u,t,s)\delta_{a_1 a_4}\delta_{a_2 a_3}]$$

$$A(s,t,u) = (1 + 2f_1)(s+t+u) - s. \tag{1.5}$$

We solve explicitly an infinite set of quasipotential equations (labelled by a parameter α) involving the local variant proposed by Todorov [15-17]. They differ from each other off the mass shell. Any one of them should yield the same scattering matrix on the mass shell provided the complete quasipotential (constructed to accord with its relation to the underlying field theory) is used. Nevertheless, it makes sense to consider all the quasipotential approaches in their community rather than to single out any one of them. We show that this can be advantageous at least in two respects. First, one can, by means of analytic continuation with respect to the parameter α, give precise meaning to certain ambiguous or divergent expressions. Second, one can look for such values of α which ensure sufficiently fast convergence to the exact results. We have already used this principle of analytic continuation in [18] in order to remove certain divergences appearing in the quasipotential.

In the case of exact chiral symmetry the pion mass is zero, $m=0$, and the T-matrix elements (1.5) do not depend on f_1 (i.e., on the pion field parametrization) on the mass shell where $s+t+u=4m^2=0$. We shall assume, however, that the pions have their physical mass $m \approx 140$ MeV and see how does the result depend on f_1 in this case of broken chiral symmetry.

2. THE QUASIPOTENTIAL APPROACH

The Logunov-Tavkhelidze quasipotential equation [13] was written down almost at the time when chiral field theory appeared. It and all its modifications were exaustively studied and applied to different cases of two-particle interaction but there was almost no interplay between these two developments except for some isolated work like [7], for instance, in which the local version of quasipotential approach proposed by Todorov [15-17] was used to study the low-energy $\pi\pi$ scattering in the framework of the linear σ -model.

The quasipotential equation is a Lippmann-Schwinger-type equation which reads (in the centrum of mass system)

$$T_w(\vec{p},\vec{q})+V_w(\vec{p},\vec{q})+\int V_w(\vec{p},\vec{k})G_w(\vec{k})T_w(\vec{k},\vec{q})\frac{d^3k}{(2\pi)^3}=0. \qquad (2.1)$$

Here w , $\vec{q}$, and $\vec{p}$ are the c.m. energy, initial, and final momentum.

The quasipotential equations differ in the choice of the Green function $G_w(\vec{k})$. Any quasipotential equation satisfies a few general requirements. One of them is the elastic two-particle unitarity condition

$$T_w(\vec{p},\vec{q})-T^*_w(\vec{p},\vec{q}) = \frac{i}{(4\pi)^2}\frac{b(w)}{w}\int T^*_w(\vec{p},\vec{k})T_w(\vec{k},\vec{q})d\Omega_k , \qquad (2.2)$$

where $b(w)$ is the on-mass-shell value of the momenta as a function of w.

$$p^2=q^2=\frac{1}{4}(w^2-4m^2)=b^2(w). \qquad (2.3)$$

This requirement fixes the imaginary part of the Green function

$$\mathrm{Im}\,G_w(\vec{k})=\frac{\pi}{2w}\delta(\vec{k}^2-b^2(w)). \qquad (2.4)$$

The Green functions satisfying (2.4) are of the form

$$G_w(\vec{k})=\frac{1}{2w}\frac{F(w,k^2)}{k^2-b^2(w)-i0} \qquad (2.5)$$

with

$$F(w,b^2(w))=1 \qquad (2.6)$$

but otherwise arbitrary. We shall restrict our attention here to the following simple choice of F

$$F(w,k^2)=\left[\frac{w^2}{4(k^2+m^2)}\right]^\alpha \qquad (2.7)$$

with a real parameter. For $\alpha=\frac{1}{2}$ one obtaines the Logunov-Tavkhelidze quasipotential equation and for $\alpha=0$ the Todorov quasipotential equation. We shall work with the whole infinite family of quasipotential equations corresponding to arbitrary real α.

Another requirement imposed on the quasipotential equations is that, given the perturbative expansion of the T -matrix

$$T=\sum_{n=1}^{\infty}T_n . \qquad (2.8)$$

one can write a perturbation expansion for the quasipotential

$$V = \sum_{n=1}^{\infty} V_n \tag{2.9}$$

such that Eq. (2.1) will be satisfied identically in all orders of perturbation theory. This requirement allows one to construct the perturbative expansion of the quasipotential in terms of the T-matrix

$$V_1 = -T_1\,,\quad V_2 = -T_2 + T_1 G T_1\,, \ldots \tag{2.10}$$

Suppose now that we have constructed the quasipotential up to some order. Then the quasupotential equation may be used to give us higher-order corrections to the T-matrix. The different equations will produce different corrections and it is meaningful to look for this choice of the quasipotential equation (of the parameter α) which gives the best approximation to the exact result (or to the experimental data, what is the case here).

Here we study this problem using the quasipotential constructed in the tree approximation of the nonlinear σ-model according to (1.5):

$$\begin{aligned} V^0(\vec p,\vec q) &= 2F_\pi^{-2}[(1+5f_1)w^2 - 2(2+5f_1)(p^2+q^2)] \\ V^1(\vec p,\vec q) &= -4F_\pi^{-2}(\vec p\,\vec q) \\ V^2(\vec p,\vec q) &= 2F_\pi^{-2}[(1+2f_1)w^2 - (1+4f_1)(p^2+q^2)]. \end{aligned} \tag{2.11}$$

3. THE p-WAVE

In this section we solve the isospin I=1 equation (2.1) with the quasipotential V_w^1 (cf. (2.11)) which does not depend on the parametrization of the pion field. Denoting by θ_{pq} the angle between the momenta $\vec p$, $\vec q$, we have, taking into account $\cos\theta_{pk} = \cos\theta_{pq}\cos\theta_{kq} + \sin\theta_{pq}\sin\theta_{kq}\cos(\phi_p-\phi_q)$,

$$F_\pi^2 T_w^1(p,q) - 4pq\cos\theta_{pq} - 4p\cos\theta_{pq}\int_0^\infty\int_0^\pi k\,G_w(k)\cos\theta_{kq} \times T_w^1(k,q)k^2\sin\theta_{kq}\frac{dk\,d\theta_{kq}}{(2\pi)^2} = 0. \tag{3.1}$$

The last equation implies that T_w^1 is of the form

$$T_w^1(p,q) = p\,t_1(q)\cos\theta_{pq}, \tag{3.2}$$

i.e., only a p-wave is involved in the partial-wave decomposition of the isospin I=1 T-matrix. The unknown factor $t_1(q)$ obeys the algebraic equation

$$F_\pi^2 t_1(q) - 4q - \frac{8}{3}\int_0^\infty k^4 G_w(k)\,t_1(q)\frac{dk}{(2\pi)^2} = 0. \tag{3.3}$$

We shall denote by $g_\alpha(x)$ the Green function in coordinate representation

$$g_\alpha(x) \equiv \frac{1}{2w}\left(\frac{w^2}{4}\right)^\alpha \int \frac{e^{ikx}}{(k^2+m^2)^\alpha(k^2-b^2-i0)}\frac{d^3k}{(2\pi)^3}. \tag{3.4}$$

It obeys the equation

$$(\nabla^2 + b^2)g_\alpha(x) = \rho_\alpha(x) \tag{3.5}$$

with

$$\rho_\alpha(x) = \frac{-1}{2w}\left(\frac{w^2}{4}\right)^\alpha \int \frac{e^{ikx}}{(k^2+m^2)^\alpha}\frac{d^3k}{(2\pi)^3} = \frac{-(2m)^{3/2-\alpha}}{8\pi^{3/2}\,w\,\Gamma(\alpha)}\left(\frac{w^2}{4}\right)^\alpha r^{\alpha-\frac{3}{2}} K_{\frac{3}{2}-\alpha}(mr) \tag{3.6}$$

One can therefore write

$$g_\alpha(x) = \int \frac{e^{ib|x-x'|}}{-4\pi|x-x'|}\rho_\alpha(x')d^3x', \tag{3.7}$$

i.e., the Fourier transform (3.4) is equal to the convolution (3.7), as it should be. In particular,

$$\begin{aligned} g_\alpha(0) &= 4\pi\int_0^\infty \frac{e^{ibr}}{-4\pi r}\rho_\alpha(r)\,dr = \\ &= \frac{(2m)^{3/2-\alpha}}{8\pi^{5/2}w\,\Gamma(\alpha)}\left(\frac{w^2}{4}\right)^\alpha \int_0^\infty e^{ibr} r^{\alpha-1/2} K_{3/2-\alpha}(mr)\,dr \end{aligned} \tag{3.8}$$

which after some calculation results in

$$g_\alpha(0) = R_\alpha(w) + i\,\frac{b(w)}{8\pi w} \tag{3.9}$$

with the real part given by

$$R_\alpha(w) = \frac{m\Gamma(\alpha-1/2)}{8w\pi^{3/2}\Gamma(\alpha)}\left(\frac{w^2}{4m^2}\right)^\alpha {}_2F_1\left(1,\alpha-1/2;1/2;\frac{-b^2(w)}{m^2}\right) = \tag{3.10}$$

$$= \frac{\Gamma(\alpha-1/2)}{16\pi^{3/2}\Gamma(\alpha)}\left(1+\frac{b^2}{m^2}\right)^{\alpha-1/2} {}_2F_1\left(1,\alpha-1/2;1/2;\frac{-b^2}{m^2}\right).$$

We shall encounter in what follows the following three integrals (the first three even moments of the Green function):

$$I_{2n}(w,\alpha) \equiv \int k^{2n} G_w(k)\frac{d^3k}{(2\pi)^3}, \qquad n=0,1,2. \tag{3.11}$$

It is readily verified that

$$I_0(w,\alpha) = g_\alpha(0)$$

$$I_2(w,\alpha) = \frac{w^2}{4} g_{\alpha-1}(0) - m^2 g_\alpha(0) \tag{3.12}$$

$$I_4(w,\alpha) = \left(\frac{w^2}{4}\right)^2 g_{\alpha-2}(0) - 2m^2\frac{w^2}{4} g_{\alpha-1}(0) + m^4 g_\alpha(0).$$

All the above relations are quoted to be true [19] in certain intervals of α which we shall not write explicitly. We shall rather perform an analytic continuation in the parameter α labelling the quasipotential approaches and shall assume them true for all values of α for which the right-hand sides make sense, excluding in this way only the series of poles implied by $\Gamma(\alpha-1/2)$ in (3.10).

In this notation Eq. (3.3) takes the form

$$[F_\pi^2 - \frac{4}{3} I_2(w,\alpha)]\, t_1(q) = 4q \tag{3.13}$$

and hence

$$T^1_{w\ell}(p,q) = 4pq[F_\pi^2 - \frac{4}{3} I_2(w,\alpha)]^{-1}\delta_{\ell 1}. \tag{3.14}$$

Taking into account that the physical scattering amplitude

$$f^I(w,\theta) = \sum_{\ell=0}^{\infty}(2\ell+1) f^I_\ell(w) P_\ell(\cos\theta) \tag{3.15}$$

is related to the T-matrix by [7]:

$$T^I_w(p,q) = 8\pi w f^I(w,\theta),$$

$$T^I_{w\ell}(p,q) = 8\pi w(2\ell+1) f^I_\ell(w), \quad p=q=b(w) \tag{3.16}$$

one finally obtains:

$$\frac{1}{f^1_1(w)} = \frac{6\pi w}{b^2(w)} F_\pi^2 - \frac{8\pi w}{b^2(w)} I_2(w,\alpha) = \tag{3.17}$$

$$= \frac{2\pi w}{b^2(w)}[3F_\pi^2 - w^2 R_{\alpha-1}(w) + 4m^2 R_\alpha(w)] - ib(w).$$

The relation between the scattering amplitude and the phase shifts

$$\frac{1}{f^I_\ell(w)} = b(w)[\operatorname{cotg}\delta^I_\ell(w) - i] \tag{3.18}$$

yields

$$\operatorname{cotg}\delta^1_1(w) = \frac{2\pi w}{b^3(w)}[3F_\pi^2 - w^2 R_{\alpha-1}(w) + 4m^2 R_\alpha(w)]. \tag{3.19}$$

Comparing the last result with the (relativistic) effective-range expansion

$$b^{2\ell+1}(w)\operatorname{cotg}\delta^I_\ell(w) = \frac{1}{a^I_\ell} + \frac{1}{2} r^I_\ell b^2(w) + \dots \tag{3.20}$$

one immediately obtains the corresponding scattering lengths and effective ranges

$$\frac{1}{a^1_1} = 12\pi m F_\pi^2 - \frac{\Gamma(\alpha-3/2)}{2\sqrt{\pi}\,\Gamma(\alpha)} m^3, \tag{3.21}$$

$$r^1_1 = \frac{12\pi}{m} F_\pi^2 + \frac{3\Gamma(\alpha-3/2)}{2\sqrt{\pi}\,\Gamma(\alpha-1)} m. \tag{3.22}$$

For $\alpha=n+1, n=0,1,2,\dots$, for the series of (pseudo-) local quasipotential equations the hypergeometric function in (3.10) reduces to a polynomial

$$
{}_2F_1\left(1,n+1/2;1/2;\frac{-b^2}{m^2}\right)=\left(1+\frac{b^2}{m^2}\right)^{-1}{}_2F_1\left(1,-n;1/2;\frac{b^2/m^2}{b^2/m^2+1}\right)=
$$

$$
=\frac{4m^2}{w^2}\,{}_2F_1\left(1,-n;1/2;\frac{4b^2}{w^2}\right)= \qquad (3.23)
$$

$$
=\frac{4m^2}{w^2}\left[1-\frac{n}{1/2}\frac{4b^2}{w^2}+\frac{n(n-1)}{(1/2)(3/2)}\left(\frac{4b^2}{w^2}\right)^2+\dots+(-1)^n\frac{n(n-1)\dots1}{1/2\cdot3/2\dots(n-1/2)}\left(\frac{4b^2}{w^2}\right)^n\right].
$$

In particular, the phase shifts and the amplitude in the case $n=0$ (an equation proposed by Filippov) are given by

$$
\mathrm{tg}\,\delta_1^1(w)=\frac{b^3(w)}{6\pi w F_\pi^2-m^3}
$$

$$
\frac{1}{f_1^1(w)}=\frac{6\pi w}{b^2(w)}F_\pi^2-\frac{m^3}{b^2(w)}-ib(w) \qquad (3.24)
$$

while for the Todorov equation ($n=-1$) they are

$$
\mathrm{tg}\,\delta_1^1(w)=\frac{b^3(w)}{6\pi w F_\pi^2}
$$

$$
\frac{1}{f_1^1(w)}=\frac{6\pi w}{b^2(w)}F_\pi^2-ib(w). \qquad (3.25)
$$

In both cases

$$
\lim_{w\to\infty}\mathrm{tg}\,\delta_1^1(w)=\infty \qquad (3.26)
$$

so that it is impossible to impose the conventional in potential scattering theory requirement $\lim_{w\to\infty}\delta_1^1(w)=0$. It could be rather replaced by $\lim_{w\to\infty}\delta_1^1(w)=\pi/2$. The phase shifts for different α are pictured in Fig. 1. For $n\geq 2$ one already has $\delta_1^1=0$ at infinity (see Fig. 1). The effective-range parameters are given in Table 1 for some of the local equations.

4. THE S-WAVE

The isospin $I=0$ and 2 quasipotentials corresponding to the tree approximation of T-matrix elements (1.5) are

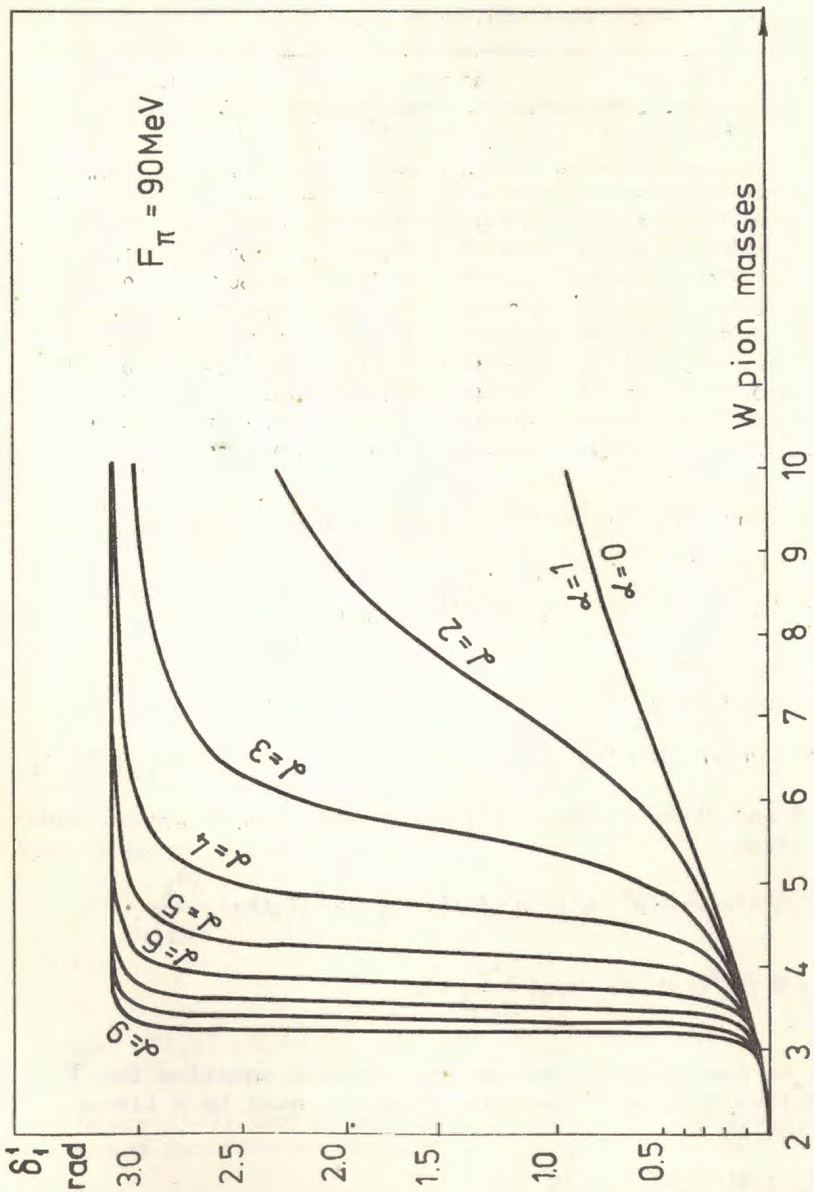

Fig. 1. Dependence of the p-wave phase shifts on the c.m. energy for different α.

Table 1

Isospin-one scattering lengths and effective ranges for different values of F_π and a

F_π	80 MeV		90 MeV		100 MeV	
α	a_1^1 (pion mass)$^{-3}$	r_1^1 pion mass	a_1^1 (pion mass)$^{-3}$	r_1^1 pion mass	a_1^1 (pion mass)$^{-3}$	r_1^1 pion mass
0	0.07893	12.669	0.06237	16.035	0.05052	19. 796
1	0.08569	12.669	0.06651	16.035	0.05320	19. 796
3	-0.07946	13.306	-0.06270	16.671	-0.05073	20. 432
4	-0.07923	13.179	-0.06255	16.544	-0.05064	20. 305
5	-0.07913	13.106	-0.06249	16.471	-0.05060	20. 232
6	-0.07908	13.057	-0.06246	16.423	-0.05058	20. 184
7	-0.07904	13.022	-0.06244	16.387	-0.05056	20. 149
8	-0.07902	12.995	-0.06242	16.360	-0.05055	20. 121
9	-0.07900	12.973	-0.06241	16.339	-0.05055	20. 100

$$V^\circ(p,q) = 2F_\pi^{-2}\,[(1+5f_1)w^2 - 2(2+5f_1)(p^2+q^2)]$$
$$V^2(p,q) = 2F_\pi^{-2}[(1+2f_1)w^2 - (1+4f_1)(p^2+q^2)]. \tag{4.1}$$

For a quasipotential of the form

$$V(p,q) = A - B(p^2+q^2) \tag{4.2}$$

with A and B independent of p and q, the quasipotential equation reads

$$T_w(p,q) + A - B(p^2+q^2) + (A - Bp^2)\int G_w(k^2)T_w(k,q)\frac{d^3k}{(2\pi)^3} -$$
$$-B\int k^2 G_w(k^2)T_w(k,q)\frac{d^3k}{(2\pi)^3} = 0, \tag{4.3}$$

i.e., we have again a degenerate integral equation for T. It follows from here that the T-matrix must be a linear function in p^2

$$T_w(p,q) = t_0(q^2) + t_2(q^2)p^2. \tag{4.4}$$

It does not depent on the angle between p and q, i.e., the zeroth-order term of the quasipotential contributes only to the s-wave for $I=0$ or 2. Inserting (4.4) into (4.3) we obtain a system of two linear equations

$$[1 + AI_0(w,a) - BI_2(w,a)]t_0(q) + [AI_2(w,a) - BI_4(w,a)]t_2(q) = Bq^2 - A \tag{4.5}$$
$$-BI_0(w,a)t_0(q) + [1 - BI_2(w,a)]t_2(q) = B$$

to determine the two functions $t_0(q)$ and $t_2(q)$. It follows from (4.4) and (4.5) that

$$T_w(p,q) = \frac{-A + B(q^2+p^2) + B^2[q^2p^2I_0 - I_2(q^2+p^2) + I_4]}{1 + AI_0 - 2BI_2 + B^2(I_2^2 - I_0I_4)}, \tag{4.6}$$

where $I_{2n} = I_{2n}(w,a)$ are given by (3.12). On the mass shell we have

$$T^I(w) = \frac{-A_I + 2B_Ib^2(w) + B_I^2[I_0b^4(w) - 2I_2b^2(w) + I_4]}{1 + A_II_0 - 2B_II_2 + B_I^2(I_2^2 - I_0I_4)} \tag{4.7}$$

with A_I and B_I given by

$$A_0 = 2F_\pi^{-2}(1+5f_1)w^2, \quad B_0 = 4F_\pi^{-2}(2+5f_1)$$
$$A_2 = 2F_\pi^{-2}(1+2f_1)w^2, \quad B_2 = 2F_\pi^{-2}(1+4f_1). \tag{4.8}$$

According to (3.12) we have

$$I_2^2 - I_0I_4 = \left(\frac{w^2}{4}\right)^2[g_{a-1}^2(0) - g_{a-2}(0)g_a(0)] \tag{4.9}$$

and

$$I_0b^4 - 2I_2b^2 + I_4 = \left(\frac{w^2}{4}\right)^2[g_{a-2}(0) - 2g_{a-1}(0) + g_a(0)] \tag{4.10}$$

and hence (4.7) can be expressed in terms of the values of the Green function at the origin $g_a = g_a(0)$ in the following way

$$\frac{1}{T_0^I(w)} = \frac{1+A_I g_a - 2B_I(\frac{w^2}{4} g_{a-1} - m^2 g_a) + B_I^2(\frac{w^2}{4})^2 (g_{a-1}^2 - g_{a-2}\, g_a)}{-A_I + 2B_I b^2 + B_I^2(\frac{w^2}{4})^2 \, (g_{a-2} - 2g_{a-1} + g_a)} \tag{4.11}$$

Separating the real and imaginary parts of g_a according to (3.12), one obtains

$$\frac{1}{f_0^I(w)} = \frac{8\pi w}{T_0^I(w)} = b(w)(\mathrm{cotg}\,\delta_0^I - i)$$

$$b(w)\,\mathrm{cotg}\,\delta_0^I(w) = \tag{4.12}$$

$$= 8\pi w \, \frac{1 + A_I R_a - 2B_I[(b^2+m^2)R_{a-1} - m^2 R_a] + B_I^2(b^2+m^2)^2(R_{a-1}^2 - R_{a-2}R_a)}{-A_I + 2B_I b^2 + B_I^2(b^2+m^2)^2(R_{a-2} - 2R_{a-1} + R_a)}.$$

We expand the last equation in powers of b^2 using the relation

$$R_a(w) = c_a + d_a b^2 + \ldots, \quad d_a = (\frac{1}{2} - a)\frac{c_a}{m^2}, \quad c_a = \frac{\Gamma(a - \frac{1}{2})}{16\pi^{3/2}\,\Gamma(a)} \tag{4.13}$$

and obtain in this way the scattering lengths

$$a_0^I = D_0^I(8\pi N_0^I)^{-1} \tag{4.14}$$

and the effective ranges

$$r_0^I = 8\pi \frac{N_0^I}{D_0^I} + 16\pi \, \frac{N_2^I D_0^I - N_0^I D_2^I}{(D_0^I)^2}, \tag{4.15}$$

where (we denote further $A_I = \tilde{A}_I \frac{w^2}{4}$)

$$N_0^I = 1 + m^2(\tilde{A}_I + 2B_I)c_a - 2m^2 B_I c_{a-1} + m^4 B_I^2 (c_{a-1}^2 - c_{a-2}c_a)$$

$$D_0^I = -m^2 \tilde{A}_I + m^4 B_I^2 \, (c_{a-2} - 2c_{a-1} + c_a)$$

$$N_2^I = [(\frac{3}{2} - a)\tilde{A}_I + 2(\frac{1}{2} - a)B_I]c_a - 2(\frac{5}{2} - a)B_I c_{a-1} + 2(\frac{5}{2} - a)m^2 B_I^2(c_{a-1}^2 - c_{a-2}c_a) \tag{4.16}$$

$$D_2^I = -\tilde{A}_I + 2B_I + m^2 B_I^2[(\frac{9}{2} - a)c_{a-2} - 2(\frac{7}{2} - a)c_{a-1} + (\frac{5}{2} - a)c_a].$$

In particular, in the case of the Todorov equation ($a = 0$, $R_0(w) = R_{-1}(w) = R_{-2}(w = 0)$ we have

$$b(w)\,\mathrm{cotg}\,\delta_0^I(w) = \frac{8\pi w}{-A_I + 2B_I b^2} = \frac{16\pi\sqrt{1+b^2}}{-\tilde{A}_I\sqrt{1+b^2} + 2B_I b^2} \tag{4.17}$$

which implies

$$a_0^I = -\frac{1}{16\pi}\tilde{A}_I, \quad r_0^I = \frac{16\pi}{\tilde{A}_I}(1 - \frac{4B_I}{\tilde{A}_I}) \tag{4.18}$$

and in the case $a = 1$, Filippov equation,

$$b(w)\,\mathrm{cotg}\,\delta_0^I(w) = 8\pi w \, \frac{1 + A_I R_1(w) + 2m^2 B_I R_1(w)}{-A_I + 2B_I b^2 + B_I^2(b^2+m^2)^2 R_1(w)} \tag{4.19}$$

$$R_1(w) = \frac{m}{8\pi w}$$

so that

$$a_0^I = (\frac{B_I^2}{16\pi} - \tilde{A}_I)(16\pi + \tilde{A}_I + 2B_I)^{-1} \tag{4.20}$$

and

$$r_0^I = -2\,\frac{16\pi + \tilde{A}_I + 3B_I}{B_I^2/16\pi - \tilde{A}_I} - \frac{\tilde{A}_I + 4B_I}{(B_I^2/16\pi - \tilde{A}_I)^2} \tag{4.21}$$

The values of the s-wave scattering lengths and effective ranges for some of the local quasipotential equations and different pion parametrizations are given in Table 2. The scattering phase shifts $\delta_0^I(w)$ for different f_1 are shown in Fig. 3.

We see that after the choice of the best a ensuring steepest descent to the exact solution (to the experimen-

Table 2

Isospin zero and two s -wave scattering lengths and effective ranges for $F_\pi = 90$ MeV and different values of f_1 and α

f_1	α	a_0^0 (pion mass)$^{-1}$	a_0^2 (pion mass)$^{-1}$	r_0^0 (pion mass)$^{-1}$	r_0^2 (pion mass)$^{-1}$
-0.5	0				
	1				
	3	0.69544	0.0032101	35.953	2.495×10^4
	4	0.67193	0.0005407	37.366	6.747×10^5
	5	0.65765	0.0002036	38.307	4.590×10^6
	6	0.64753	0.0001020	39.022	1.810×10^7
	7	0.63992	0.0000596	39.592	5.284×10^7
	8	0.63395	0.0000383	40.064	1.274×10^8
	9	0.62912	0.0000264	40.464	2.691×10^8
-0.4	0	0.37419	-0.07484	-2.6724	53.449
	1	0.59793	-0.07447	-3.2917	25.868
	3	0.43527	-0.07071	5.6006	-320.48
	4	0.42374	-0.07244	5.7733	-312.44
	5	0.41684	-0.07295	5.8851	-309.96
	6	0.41214	-0.07323	5.9661	-308.56
	7	0.40869	-0.07341	6.0290	-307.61
	8	0.40602	-0.07354	6.0799	-306.90
	9	0.40387	-0.07364	6.1227	-306.34
-1/2	0	0.24946	-0.12473	-8.0173	16.035
	1	0.28945	-0.11649	-7.4019	16.665
	3	0.28181	-0.11794	84.061	-200.17
	4	0.27314	-0.11955	87.032	-197.18
	5	0.26923	-0.12031	88.508	-195.70
	6	0.26680	-0.12081	89.487	-194.72
	7	0.26508	-0.12116	90.212	-193.99
	8	0.26379	-0.12143	90.784	-193.42
	9	0.26278	-0.12165	91.254	-192.95
-0.25	0	0.09355	-0.18710	-74.828	5.3449
	1	0.09539	-0.15761	-24.170	12.583
	3	0.10884	-0.17483	193.99	-137.47
	4	0.09946	-0,17676	211.86	-135.75
	5	0.09758	-0.17799	216.22	-134.63
	6	0.09675	-0.17886	218.36	-133.82
	7	0.09627	-0.17952	219.68	-133.19
	8	0.09595	-0.18004	220.62	-132.68
	9	0.09572	-0.18046	221.33	-132.25

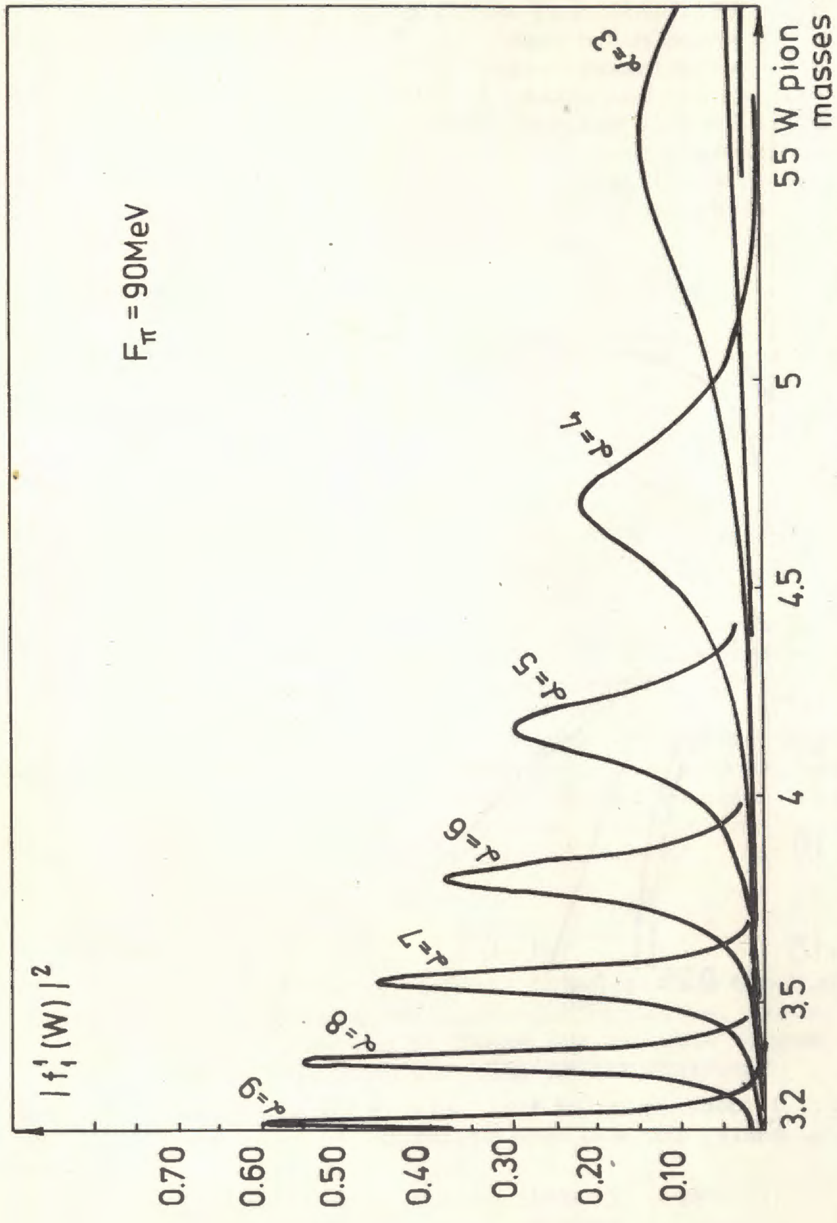

Fig. 2. Resonant behaviour of the p -wave scattering amplitudes for different α.

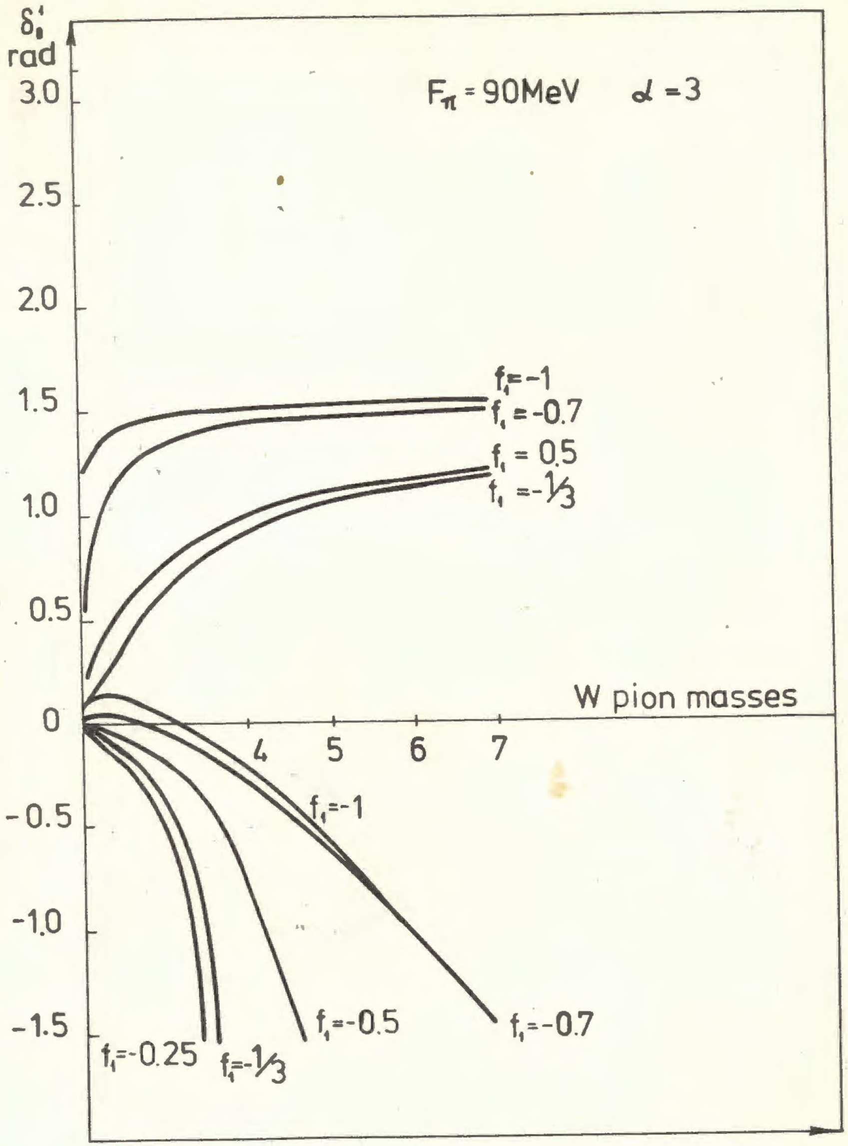

Fig. 3. Dependence of the s-wave phase shifts on the c.m. energy for $\alpha=3$ and different f_1.

tal p-wave data in our case) was done there remains a strong dependence in the s-wave on the parameter f_1 related to the paramatrization of the pion field. In the case we are considering the pion field has a mass, $m = 140$ MeV, and consequently the chiral symmetry is broken. For massless pions the symmetry is exact and the results do not depend on f_1, as mentioned already in Sec. 1. For massive pions $m^2 f_1$ plays the role of a symmetry breaking parameter and we can say alternatively that the s-wave scattering lengths, effective ranges and phase shifts strongly depend on the symmetry breaking.

Table 3

The position (in MeV) of the p-wave resonance as a function of α

F_π \ α	0	1	3	4	5	6	7	8	9
90			774.2	651.4	575.4	525.9	490.0	463.8	443.1
95			792.1	662.4	583.7	531.4	495.5	467.9	447.2

In conclusion we note that the quasipotential approach supplied by the principle of analyticity in the parameter α (labelling the different quasipotential approaches) can be successfully applied to extend the chiral current algebra results to higher energies for pion-pion scattering. Even only the zeroth order term of the non-linear σ-model used as an input to construct the quasipotential ensures a fairly good approximation to the experimental data. It is interesting to study the effect of the higher orders.

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