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E2 - 12742

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ON THE EVOLUTION EQUATIIONS, SOLVABLE THROUGH THE INVERSE SCATTERING METHOD.

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Submitted to "Bulgarian Journal of Physics"

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E2 - 12742

Об эволюционных уравнениях, решаемых методом обратной задачи рассеяния. II. Гамильтонова структура и преобразования Беклунда

Единым образом выведены основные результаты теории нелинейных эволюционных уравнений, исходя из спектральной теории, развитой в первой части работы.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

Gerdjikov V.S., Khristov E.Kh.

E2 - 12742

On the Evolution Equations, Solvable Through the Inverse Scattering Method. II. Hamiltonian Structure and Bäcklund Transformations

In a uniform manner we derive uniquely the main results of the nonlinear evolution equations' theory, starting from the spectral theory developed in the first part of this paper.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

§1. INTRODUCTION

In the present paper we consider the nonlinear evolution equations (NLEE), related to the one-parameter family of Dirac operators (see refs. [1,2] and the review paper [3]):

$$\ell[w]y \equiv \left[\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\frac{d}{dx} + \begin{pmatrix}0 & q\\ -r & 0\end{pmatrix}\right]\begin{pmatrix}y_1\\ y_2\end{pmatrix} = i\zeta y(x,\zeta,t),\quad w = \begin{pmatrix}q\\ -r\end{pmatrix}, \qquad (1.1)$$

where $q(x,t)$ and $r(x,t)$ are the functions of Schwartz type with respect to x for all values of the parameter t. It is well known, that the NLEE, solvable through the inverse scattering method (ISM) for the operator $\ell[w]$ are generated by the integro-differential operator L_+ (or L_-) [2]:

$$L_{\pm} \equiv \frac{i}{2}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + 2\begin{pmatrix}q(x,t)\\ -r(x,t)\end{pmatrix}\int_x^{\pm\infty} dy\,(r(y,t),\, q(y,t))\right\}, \qquad (1.2)$$

i.e., each NLEE is constructed with the help of a certain function of the operator $L_+(L_-)$.

In ref. [4] we investigated the properties of the eigenfunctions, and in ref. [5] we constructed the spectral theory of the operators $\Lambda_{\pm}$, which are natural generalizations of the operators $L_{\pm}$. In the present paper, starting from the results of ref. [4,5], we derive in uniform manner the main results of the NLEE theory, namely:

1. The description of the class of NLEE related to $\ell[w]$ and their conservation laws [2,3] (§3).

2. The description of the hierarchy of Hamiltonian structures [6] and the proof of complete integrability of the NLEE (explicit calculation of the action-angle variables) [7,8,9] (§4).

3. The description of the class of Backlund transformations [10] for these NLEE (§5).

Besides technical simplifications in the proofs of the theorems and in the calculations, the spectral theory of the operators $L_{\pm}$ has led us to some complementary results. For example, the symplectic expansion introduced in ref. /4/ enabled us to describe the Lagrangian manifold of the NLEE (§4). We also write down the conserved quantities of these equations in a compact form (§3), which enabled us to write down any of the above-mentioned NLEE in the explicitly Hamiltonian form; we briefly discuss the properties of the NLEE, generated by singular functions of the operators $L_{\pm}$ (§4).

The general formulae are illustrated by two important examples: i) the nonlinear Schrödinger equation /1,11,12/ and ii) the sine-Gordon equation /13,14/.

The authors have the pleasure to thank sincerely Dr. P.P.Kulish for his constant attention and fruitful discussions.

§2. PRELIMINARIES

Consider the pair of operators

$$\ell_n[w_n]y_n = \left\|\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\frac{d}{dx} + \begin{pmatrix}0 & q_n\\ r_n & 0\end{pmatrix}\right\|\begin{pmatrix}y_{n,1}\\ y_{n,2}\end{pmatrix} = i\zeta y_n(x,\zeta),\quad w_n = \begin{pmatrix}q_n\\ -r_n\end{pmatrix},\quad n = 1,2, \tag{2.1}$$

where the coefficients $q_n(x)$, $r_n(x)$ are complex-valued functions of Schwartz type. We call the set $T_n = T_n^+ \cup T_n^-$

$$T_n^{\pm} = \{\rho_n^{\pm}(\zeta), \zeta \in R;\ \zeta_{n,k}^{\pm}, C_{n,k}^{\pm},\ k = 1,\dots,N_n\},$$

$$\rho_n^{\pm}(\zeta) = \frac{b_n^{\pm}(\zeta)}{a_n^{\pm}(\zeta)},\quad C_{n,k}^{\pm} = \frac{b_{n,k}^{\pm}}{\dot a_{n,k}^{\pm}},\quad \zeta_{n,k}^{\pm}:\ a_n^{\pm}(\zeta_{n,k}^{\pm}) = 0,$$

the scattering data for the problems (2.1), $\dot a_{n,k}^{\pm} = \frac{da^{\pm}}{d\zeta}\Big|_{\zeta=\zeta_{n,k}^{\pm}}$.

Here, for simplicity, we suppose, that the discrete spectrum of the operators $\ell_n : \sigma_n = \sigma_n^+ \cup \sigma_n^-$

$$\sigma_n^{\pm} = \{\zeta_{n,k}^{\pm},\ \mathrm{Im}\,\zeta_{n,k}^{\pm} \gtrless 0,\ k = 1,\dots,N_n^{\pm}\} \tag{2.2}$$

consists of a finite number of eigenvalues, and that $N_n^+ = N_n^- = N_n$. The functions $a_n^{\pm}(\zeta)$ are analytic for $\mathrm{Im}\,\zeta \gtrless 0$, respectively and are uniquely recovered from T_n by using the dispersion relation /2/

$$A_n(\zeta) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\xi}{\xi-\zeta}\ln(1+\rho_n^+(\xi)\rho_n^-(\xi)) + \sum_{k=1}^{N}\ln\frac{\zeta-\zeta_{n,k}^+}{\zeta-\zeta_{n,k}^-}, \tag{2.3}$$

where $A_n(\zeta) = \pm\ln a_n^{\pm}(\zeta)$, $\mathrm{Im}\,\zeta \gtrless 0$ and $A_n(\zeta) = \frac{1}{2}\ln a_n^+/a_n^-(\zeta)$ for $\mathrm{Im}\,\zeta = 0$. When $\zeta \in R$ the integral in (2.3) should be understood in the sense of principal value.

Without discussing the solution of the inverse scattering problem for the systems (2.1), (see /1-3/), we go to the expansions over the products $\Psi^{\pm}(x,\zeta) \equiv \psi_1^{\pm} \circ \psi_2^{\pm} = \begin{pmatrix}\psi_{1,2}^{\pm}\psi_{2,1}^{\pm}\\ \psi_{1,2}^{\pm}\psi_{2,2}^{\pm}\end{pmatrix}(x,\zeta)$ of the Jost solutions of (2.1); $\psi_n^{\pm}(x,\zeta)$ are uniquely determined by their asymptotics at $x \to \infty$ (for more details, see refs. /4,5/).

<u>Lemma 1</u>. For the vector-functions $w_+ = w_1 + w_2$, and $w_- = \tau_3(w_2 - w_1)$, $\tau_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ the following expansion formulae

$$w_+(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta\,[(\rho_2^+ + \rho_1^+)\Psi^+(x,\zeta) + (\rho_2^- + \rho_1^-)\Psi^-(x,\zeta)] + 2i\sum_{\substack{k=1\\ n=1,2}}^{N}[C_{n,k}^+\Psi_{n,k}^+(x) - C_{n,k}^-\Psi_{n,k}^-(x)], \tag{2.4}$$

$$w_-(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta\,[(\rho_2^+ - \rho_1^+)\Psi^+(x,\zeta) - (\rho_2^- - \rho_1^-)\Psi^-(x,\zeta)] - 2i\sum_{\substack{k=1\\ n=1,2}}^{N}[C_{n,k}^+\Psi_{n,k}^+(x) + C_{n,k}^-\Psi_{n,k}^-(x)], \tag{2.5}$$

hold, where $\Psi_{n,k}^{\pm}(x) = \Psi^{\pm}(x,\zeta_{n,k}^{\pm})$, and the coefficients $\rho_n^{\pm}$, $C_{n,k}^{\pm}$ enter into the sets T_n (2.2) of scattering data.

<u>Proof.</u> Lemma 1 follows directly from theorem 1 in /1/, from the relations $(\tilde\Phi = (\Phi_2, -\Phi_1))$

$$[w_{\pm}, \Phi] \equiv \int_{-\infty}^{\infty} dx\,\tilde\Phi(x,\zeta)w_{\pm}(x) = (\phi_{1,2}\phi_{2,1} \pm \phi_{1,1}\phi_{2,2})(x,\zeta)\Big|_{x=-\infty}^{\infty} \tag{2.6}$$

and from the known asymptotics of the Jost solutions for $x \to \pm\infty$ (see /1-3/). The lemma is proved.

In the particular case * for

$$q_1 \equiv q_2 = q(x,t), \quad r_1 \equiv r_2 = r(x,t) \qquad (2.7)$$

we obtain the expansions over the "squares" $\Psi^{\pm} \equiv \psi^{\pm} \circ \psi^{\pm}(x,\zeta)$, $\Phi^{\pm} \equiv \phi^{\pm} \circ \phi^{\pm}(x,\zeta)$ of Jost solutions $\psi^{\pm}, \phi^{\pm}$ of the system (1.1)($\phi^{\pm}$ are defined by their asymptotics for $x \to -\infty$)

Lemma 2. For the vector-functions $w = \begin{pmatrix} q \\ -r \end{pmatrix}$ and $r_3 w_t = \begin{pmatrix} q_t \\ r_t \end{pmatrix}$ $r_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ the following expansion formulae:

$$w(x,t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta (\rho^+ \Psi^+ + \rho^- \Psi^-)(x,\zeta,t) + 2i \sum_{k=1}^{N} (C_k^+ \Psi_k^+ - C_k^- \Psi_k^-)(x,t), \quad (2.8a)$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta (\sigma^+ \Phi^+ + \sigma^- \Phi^-)(x,\zeta,t) + 2i \sum_{k=1}^{N} (M_k^+ \Phi_k^+ - M_k^- \Phi_k^-)(x,t), \quad (2.8b)$$

$$r_3 w_t = \frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta (\rho_t^+ \Psi^+ - \rho_t^- \Psi^-)(x,\zeta,t) - 2i \sum_{k=1}^{N} (X_k^+ + X_k^-)(x,t)$$

$$= -\frac{1}{\pi} \int_{-\infty}^{\infty} d\zeta (\sigma_t^+ \Phi^+ - \sigma_t^- \Phi^-)(x,\zeta,t) + 2i \sum_{k=1}^{N} (Y_k^+ + Y_k^-)(x,t), \qquad (2.9a)$$

$$X_k^{\pm} = (C_{k,t}^{\pm} \Psi_k^{\pm} + C_k^{\pm} \zeta_{k,t}^{\pm} \dot{\Psi}_k^{\pm})(x,t), \qquad (2.9b)$$

$$Y_k^{\pm} = (M_{k,t}^{\pm} \Phi_k^{\pm} + M_k^{\pm} \zeta_{k,t}^{\pm} \dot{\Phi}_k^{\pm})(x,t)$$

hold, where by $S = S^+ \cup S^-$

$$S^{\pm} = \{ \sigma^{\pm} = \frac{b^{\mp}(\zeta)}{a^{\pm}(\zeta)}, \ \zeta \in R, \ \zeta_k^{\pm}, \ M_k^{\pm} = (a_k^{\pm} b_k^{\pm})^{-1}, \ k = 1,\dots,N \}$$

we have denoted an equivalent to T set of scattering data of the operator $\ell[w]$. In what follows by $\sum_{k=1}^{N} X_k^{\pm}$ we mean $\sum_{k=1}^{N} (X_k^+ + X_k^-)$.

Proof. The expansion (2.8a) follows directly from the condition (2.7) and the expansion (2.4); (2.8b) is derived

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*Since from the context it will be clear whether the condition (2.7) holds or not, we will denote the "products" $\Psi^{\pm}$ and $\Phi^{\pm}$ by the same symbols as the "squares". The set of scattering data of the system (1.1) will be denoted by T. omitting everywhere in (2.2) the index n.

from (2.7) and an analogic to (2.4) expansion for w_+ over the system $\{\Phi\}$ (see theorem 1 in ref. /5/).

The expansion (2.9) follows directly from theorem 1 in /5/, from the condition (2.7) and the relations

$$[r_3 w_t, U^{\pm}](\zeta) = -\rho_t^{\pm}(\zeta,t), \qquad [r_3 w_t, V^{\pm}](\zeta) = -\sigma_t^{\pm}(\zeta,t),$$

$$[r_3 w_t, U_k^{(1)\pm}] = -C_k^{\pm} \zeta_{k,t}^{\pm}, \qquad [r_3 w_t, V_k^{(1)\pm}] = -M_k^{\pm} \zeta_{k,t}^{\pm}, \qquad (2.10)$$

$$[r_3 w_t, U_k^{(2)\pm}] = -C_{k,t}^{\pm}, \qquad [r_3 w_t, V_k^{(2)\pm}] = -M_{k,t}^{\pm}.$$

where the skew-scalar product $[\,,\,]$ is defined in (2.6), and the systems of functions $\{U\}$ and $\{V\}$ are introduced in /5/. A detailed derivation of the relations (2.10) is given in /15/.

At the end let us give the expansions of w and $r_3 w_t$ over the symplectic basis $\{P,Q\}$ introduced in /4,5/.

Lemma 3. For the vector-functions w and $r_3 w_t$ the following expansion formulae

$$w(x,t) = -\int_{-\infty}^{\infty} d\zeta P(x,\zeta,t) - \sum_{k=1}^{N} (P_k^+ + P_k^-)(x,t), \qquad (2.11)$$

$$r_3 w_t = \int_{-\infty}^{\infty} d\zeta [\tilde{p}_t(\zeta,t) Q(x,\zeta,t) - \tilde{q}_t(\zeta,t) P(x,\zeta,t)] +$$

$$+ \sum_{k=1}^{N} [\tilde{p}_{k,t}^{\pm} Q_k^{\pm}(x,t) - \tilde{q}_{k,t}^{\pm} P_k^{\pm}(x,t)] \qquad (2.12)$$

hold, where

$$\tilde{p}(\zeta,t) = -\frac{1}{\pi} \ln(1 + \rho^+ \rho^-), \qquad \tilde{p}_k^{\pm} = \pm 2i \zeta_k^{\pm},$$

$$\tilde{q}(\zeta,t) = -\frac{1}{2} \ln b^+/b^-, \qquad \tilde{q}_k^{\pm} = \mp \ln b_k^{\pm} \qquad (2.13)$$

Proof. The expansion (2.11) is obtained from (2.8a) or (2.8b) using the definition of the system $\{P,Q\}$ (see formulae (6.1)-(6.3) in ref. /5/). The expansion (2.12) follows from theorem 4 in /5/ , from the relations (2.10) and from the definition of the systems $\{P,Q\}$ and $\{U\}$, $\{V\}$ (see /5/). The lemma is proved.

§3. DESCRIPTION OF THE CLASS OF NLEE RELATED TO THE DIRAC SYSTEM AND THEIR CONSERVATION LAWS

It is well known, that the main idea of the ISM consists in the change of variables $w \to T$ (or S), i.e., in the transition from w to its expansion coefficients over the system $\{\Psi\}$ (or $\{\Phi\}$) (see formulae (2.8)); at the same time the expansion coefficients of w_t over $\{\Psi\}$ (or $\{\Phi\}$) are determined by the t-derivatives of the scattering data T (or S). The uniqueness and invertibility of this change of variables follows directly from the uniqueness and invertibility of the expansions over $\{\Psi\}$ (or $\{\Phi\}$) (see /4/). This allows one to consider the ISM as a generalized Fourier transform /2/, in which as a generalization of the exponents $e^{i\zeta x}$ there enter $\Psi^{\pm}(x,\zeta)$ (or $\Phi^{\pm}(x,\zeta)$) and instead of the operator $\frac{1}{i}\frac{d}{dx}$ naturally appears the integro-differential operators L_+ (or L_-).

Let us go now to the derivation in somewhat more general form of the main result in /2/ and prove

Theorem 1. In order that the functions $q(x,t)$ and $r(x,t)$ be the solutions of the NLEE:

$$\sigma_3 w_t + \Omega(L_+)\, w(x,t) = 0, \quad 0 < t < \infty, \tag{3.1}$$

where the rational function $\Omega(L_+)$ of the operator L_+ is defined according to theorem 3 in /5/, it is necessary and sufficient that the scattering data $\dot{T}$ (2.2) of the operator $\ell[w]$ (1.1) satisfy the linear equations:

$$\rho^{\pm}_t \mp \Omega(\zeta)\rho^{\pm}(\zeta,t) = 0, \quad C^{\pm}_{k,t} \mp \Omega(\zeta^{\pm}_k) C^{\pm}_k(t) = 0, \quad \zeta^{\pm}_{k,t} = 0\,. \tag{3.2}$$

Furthermore, if q and r satisfy (3.1), then they satisfy also the NLEE

$$\sigma_3 w_t + \Omega(L_-)\, w(x,t) = 0 \tag{3.3}$$

and the scattering data S - the linear equations

$$\sigma^{\pm}_t \pm \Omega(\zeta)\sigma^{\pm}(\zeta,t) = 0, \quad M^{\pm}_{k,t} \pm \Omega(\zeta^{\pm}_k) M^{\pm}_k(t) = 0, \quad \zeta^{\pm}_{k,t} = 0. \tag{3.4}$$

Proof. Let us insert the expansions (2.8a) and (2.9a) into the l.h.s. of (3.1) and use the theorem 3 in /5/. This gives us

$$\frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta\, [(\rho^{+}_t - \Omega(\zeta)\rho^{+})\Psi^{+} - (\rho^{-}_t + \Omega(\zeta)\rho^{-})\Psi^{-}](x,\zeta,t) - $$
$$- 2i \sum_{k=1}^{N} [(C^{\pm}_{k,t} \mp \Omega(\zeta^{\pm}_k) C^{\pm}_k)\Psi^{\pm}_k + C^{\pm}_k \zeta^{\pm}_{k,t}\dot{\Psi}^{\pm}_k](x,t) = \sigma_3 w_t + \Omega(L_+) w. \tag{3.5}$$

From the uniqueness and invertibility of the expansions over the system $\{\Psi\}$ it follows, that (3.1) holds if and only if (3.2) holds.

From the expansion (2.11) and from lemma 5 in ref. /5/ (i.e., from $L_{\pm}P(x,\zeta) = \zeta P(x,\zeta)$) we see, that $\Omega(L_+)w = \Omega(L_-)w$; from here the equivalence of the NLEE (3.1) and (3.3) follows directly.

Quite analogously, using the expansions (2.8b) and (2.9b) we can prove, that the NLEE (3.3) hold if and only if (3.4) holds. The theorem is proved.

From (3.2) and (3.4) there follows directly the relations:

$$\frac{d\tilde{p}(\zeta,t)}{dt} = 0, \qquad \frac{d\tilde{q}(\zeta,t)}{dt} = -\Omega(\zeta), \qquad \zeta \in \mathbb{R} \cup \sigma, \tag{3.6}$$

where $\tilde{p}$ and $\tilde{q}$ are introduced in (2.13). The answer (3.6) is obtained also by inserting the expansions (2.11) and (2.12) into (3.1) or (3.3).

It is well known, that the NLEE (3.1) have an infinite number of conservation laws. This is related to the fact, that the regularized functional determinant of the operator $\ell[w]$ (1.1)

$$\mathrm{Det}(\ell - \zeta)(\ell_0 - \zeta)^{-1} \equiv \exp \mathrm{Tr} \ln (\ell - \zeta)(\ell_0 - \zeta)^{-1} = e^{A(\zeta)}\,, \tag{3.7}$$

where $\ell_0 = \ell|_{q=r=0}$ and $A(\zeta)$ is given by (2.3), is independent of t. Really, from (2.3) and (3.6) it follows immediately, that $dA(\zeta,t)/dt = 0$. As conserved quantities we may choose, for example, the expansion coefficients of $A(\zeta)$ over the powers of $1/\zeta$ or ζ:

$$A(\zeta) = \sum_{m=1}^{\infty} \frac{C_m}{\zeta^m} = \sum_{m=0}^{\infty} C_{-m}\zeta^{m} \tag{3.8}$$

The dependence of C_m on the scattering data is easily obtained by comparing (3.8) and (2.3):

$$C_m = -\frac{-i\,\mathrm{sign}\, m}{2\pi} \int_{-\infty}^{\infty} d\zeta\, \zeta^{m-1} \ln(1 + \rho^{+}\rho^{-}(\zeta)) - \frac{1}{|m|}\sum_{k=1}^{N} [(\zeta^{+}_k)^m - (\zeta^{-}_k)^m], \tag{3.9}$$

where $m = \pm 1, \pm 2, \ldots$ Besides, recurrent relations are known, which allow one to calculate C_m as functionals of q and r. In the appendix we will obtain also a compact expression for $A(\zeta)$ through L_+ in the form:

$$\frac{dA}{d\zeta} = \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy \; \tilde{w}(y)(L_+ - \zeta)^{-1} w(y) \tag{3.10}$$

which immediately gives:

$$C_m = \frac{1}{|m|} \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy \; \tilde{w}(y) L_+^{m} w(y) . \tag{3.11}$$

The relations (3.9) and (3.11), and also (see [9] and the appendix)

$$\delta C_m = \frac{i \operatorname{sign} m}{2} \int_{-\infty}^{\infty} dx \; \widetilde{r_3 \delta w}(x) L_+^{m-1} w(x) \tag{3.12}$$

will be used in the next paragraph in obtaining the Hamiltonian structure of the NLEE (3.1); in (3.12) by $r_3 \delta w$ we have denoted the variations $\binom{\delta q}{\delta r}$ of the coefficients of the linear problem (1.1).

§4. HAMILTONIAN STRUCTURE OF THE NLEE AND ACTION-ANGLE VARIABLES

The Hamiltonian structures of a number of concrete NLEE, related to the problem (1.1) are well known (see refs. [7,9,11,12,14]). It is also known, that with each NLEE we can relate a whole hierarchy of Hamiltonian structures [6]. In this paragraph, using the expressions (3.11) and (3.12) for C_m and δC_m we will explicitly write down the Hamiltonian structure (i.e., the 2-form and the Hamiltonian) corresponding to the NLEE:

$$f(L_+) w_t + g(L_+) w = 0 , \tag{4.1}$$

where f and g are rational functions. Obviously eq. (4.1) is more general than (3.1) since it contains also the cases, when the functions $\Omega(\zeta)$ have poles on the spectrum of the operator $\ell[w]$; we will comment on that below. Besides the record (4.1) is not unique, and this naturally leads to the notion of hierarchy of Hamiltonian structures.

Quite analogously to theorem 1 we prove

Theorem 2. Let f and g be rational functions with no poles in a certain neighbourhood of the spectrum of the operator $\ell[w]$. Then, in order that $q(x,t)$ and $r(x,t)$ satisfy the NLEE (4.1) it is necessary and sufficient for the scattering data T to satisfy the linear system:

$$\begin{gathered} f(\zeta)\rho_t^{\pm} \mp g(\zeta)\rho^{\pm}(\zeta, t) = 0, \quad C_k^{\pm} f(\zeta_k^{\pm})\zeta_{k,t}^{\pm} = 0 , \\ f(\zeta_k^{\pm}) C_{k,t}^{\pm} \mp g(\zeta_k^{\pm}) C_k^{\pm}(t) + C_k^{\pm} f'(\zeta_k^{\pm})\zeta_{k,t}^{\pm} = 0 . \end{gathered} \tag{4.2}$$

Let us show that the NLEE (4.1) is a Hamiltonian one. Let us introduce on the manifold of vector-functions $\{q, r\}$ the Hamiltonian functional:

$$H_g = \Sigma h_m C_m[q, r] = - \int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy \; \tilde{w}(y) g_1(L_+) w(y) , \tag{4.3}$$

where $g(z) = -\Sigma h_m \frac{\operatorname{sign} m}{2} z^{m-1}$; $g_1(z) = 2 \int^{z} dz' g(z')$

and the symplectic form:

$$\begin{aligned} \omega_f &= \frac{i}{2} \int_{-\infty}^{\infty} dx \; \widetilde{r_3 \delta w(y)} \wedge f(L_+) r_3 \delta w(y) \equiv \\ &\equiv -\frac{i}{2} [r_3 \delta_1 w, f(L_+) r_3 \delta_2 w] + \frac{i}{2} [r_3 \delta_2 w, f(L_+) r_3 \delta_1 w]. \end{aligned} \tag{4.4}$$

Here the skew-scalar product [,] is introduced in (2.6), and $\delta_1 w$, $\delta_2 w$ are two independent variations of the operator $\ell[w]$ coefficients.

The proof of the Jacobi identity of the closure and the compatibility of the forms ω_f can be done as in [6], recalculating ω_f in terms of the scattering data variations.

Using the standard method of classical mechanics [16] and the relation (3.11) we can verify that the Hamiltonian equations of motion, generated by H_g and ω_f

$$\omega_f (r_3 \frac{dw}{dt}, .) = \delta H_g \tag{4.5}$$

coincide with the NLEE (4.1).

Let us illustrate the way of recalculation of ω_f in terms of the scattering data variations by the simplest example with $f(z) \equiv 1$; let us denote $\omega_f |_{f \equiv 1} = \omega$. Let us insert into ω_0 the expansion of $r_3 \delta w$, analogic to (2.12); it is obtained from (2.12) by changing $\tilde{p}_t$ and $\tilde{q}_t$ by $\delta \tilde{p}$ and $\delta \tilde{q}$ respectively. This gives us:

$$\omega_0 = \frac{i}{2} \int_{-\infty}^{\infty} d\zeta \{[r_3\delta w, Q](\zeta) \wedge \delta\tilde{p}(\zeta) - [r_3\delta w, P](\zeta) \wedge \delta\tilde{q}(\zeta)\} + \tag{4.6}$$

$$+ \frac{i}{2} \sum_{k=1}^{N} \{[r_3\delta w, Q_k^{\pm}] \wedge \delta\tilde{p}_k^{\pm} - [r_3\delta w, P_k^{\pm}] \wedge \delta\tilde{q}_k^{\pm}\}.$$

The coefficients $[r_3\delta w, Q]$ and $[r_3\delta w, 0]$ can be calculated using the definition of the system $\{P,Q\}$ (see /5/) and the relations, analogous to (2.10). Thus, we obtain ω_0 directly in the canonical form:

$$\omega_0 = i \int_{-\infty}^{\infty} d\zeta\, \delta\tilde{p} \wedge \delta\tilde{q} + i \sum_{k=1}^{N} [\delta\tilde{p}_k^{+} \wedge \delta\tilde{q}_k^{+} + \delta\tilde{p}_k^{-} \wedge \delta\tilde{q}_k^{-}], \tag{4.7}$$

where $\{\tilde{p}, \tilde{p}_k^{\pm}, \tilde{q}, \tilde{q}_k^{\pm}\}$ are given in (2.13).

The calculation of ω_f in the general $f(\zeta) \neq 1$ case is analogous. Since $Q(x,\zeta)$, entering into (2.12) are not eigenfunctions of the operator L_+ (see lemma 5 in ref. /5/), technically it is somewhat simpler instead of (2.12) to use the expansion analogous to (2.9a) for $r_3\delta w$. The answer can be cast again in the canonical form (4.7), where instead of $\{\tilde{p}, \tilde{p}_k^{\pm}\}$ there will enter

$$\tilde{P}(\zeta, t) = f(\zeta)\tilde{p}(\zeta, t), \qquad \tilde{P}_j^{\pm} = \pm 2i f_1(\zeta_j^{\pm}), \tag{4.8}$$

where $f_1(z) = \int^z dz' f(z')$.

The Hamiltonian H_g may be expressed through the scattering data by inserting (3.8) into (4.3). This immediately gives

$$H_g = -i \int_{-\infty}^{\infty} d\zeta g(\zeta)\tilde{p}(\zeta) + \sum_{k=1}^{N} [g_1(\zeta_k^{+}) - g_1(\zeta_k^{-})]. \tag{4.9}$$

Obviously H_g depends only on the action-type variables $\{\tilde{P}, \tilde{P}_j^{\pm}\}$. Thus, we have shown, that the Hamiltonian system, related to the NLEE (4.1) is completely integrable.

Suppose, that the ratio g/f satisfies the conditions of theorem 1; then it is easy to verify, that the equations of motion generated by H_g and ω_f in the scattering data space coincide with (3.6) with $\Omega = g/f$. Let us now make a few comments in the cases, when this supposition does not hold. In that case either equations (4.2), or the natural requirement that the integrals

$$\int_{-\infty}^{\infty} d\zeta f(\zeta)\rho_t^{\pm}(\zeta, t), \qquad \int_{-\infty}^{\infty} d\zeta g(\zeta)\rho^{\pm}(\zeta, t) \tag{4.10}$$

should be absolutely convergent for all t may impose additional restrictions on the scattering data.

1. Let in some interval $\delta_1 \subset R$, $f(\zeta) = g(\zeta) = 0$. Then from (4.8) and (4.9) we see, that ω_f and H_g do not depend on the values, the scattering data coefficients take on this interval.

2. If in the interval $\delta_2 \subset R$, $f(\zeta) = 0$, $g(\zeta) \neq 0$ then from (4.2) it follows, that $\rho^{\pm}(\zeta,t) = 0$ for all $\zeta \in \delta_2$. In this case it is clear, that the class of initial conditions for the NLEE should be constructed so, that to ensure $\rho^{\pm}(\zeta, 0) = 0$ for all $\zeta \in \delta_2$.

3. If in the interval $\delta_3 \subset R$, $f(\zeta) \neq 0$, $g(\zeta) = 0$ it follows from (4.2) that $\rho_t^{\pm}(\zeta,t) = 0$, for all $\zeta \in \delta_3$.

4. If $f(\zeta)$ and (or) $g(\zeta)$ have singularities at some point $\zeta = \mu \in R$, then from the convergence condition (4.10) it follows that $\rho_t^{\pm}(\zeta,t)$ and (or) $\rho^{\pm}(\zeta,t)$ should have at $\zeta = \mu$ zero of sufficiently high order. This is also an unexplicit restriction on the initial conditions of the corresponding nonlinear Cauchy problem.

5. If $f(\zeta_j^{+}) = 0$ for some $\zeta_j^{+} \in \sigma^{+}$, then from (4.2) we get

$$f'(\zeta_j^{+})\zeta_{j,t}^{+} = g(\zeta_j^{+}),$$

i.e., if $f'(\zeta_j^{+})g(\zeta_j^{+}) \neq 0$, then ζ_j^{+} depends on t. These are quite exotic NLEE, for which the functions f and g in the general case depend explicitly on t.

Before we go over to the concrete examples, let us make one more remark. Let us, by analogy with /5/, introduce in the space of vector-functions $\{q(x,t), r(x,t)\}$ the manifold $\mathfrak{M}(t)$:

$$\mathfrak{M}(t) = \{u(x,t) : [u(x,t), P(x,\zeta,t)] = 0 \quad \forall \quad t > 0, \ \forall \ \zeta \in R \quad \sigma\}.$$

It is easy to verify, that the restriction of the 2-forms ω_f on $\mathfrak{M}(t)$ vanish. Really, from theorem 4 in /5/ and from the definition of $\mathfrak{M}(t)$ it follows, that the restriction of $r_3\delta w$ on $\mathfrak{M}(t)$ is equivalent to the requirements $\delta\tilde{p} = 0$, $\delta\tilde{p}_k^{\pm} = 0$, which together with (4.7) and (4.8) lead to the desired result. Thus, according to the general definition (see, e.g., ref. /17/), $\mathfrak{M}(t)$ is the Lagrangian manifold of the NLEE (4.1).

Quite analogously as in /5/, we can formulate and prove for $\mathfrak{M}(t)$ all the properties and results concerning $\mathfrak{M}$. In particular $w(x,t) \in \mathfrak{M}(t)$ and consequently $\Omega(L_+)w = \Omega(L_-)w \in \mathfrak{M}(t)$

(see theorem 7 in ref. /5/). Using this and formula (3.11) it is not difficult to verify that the vector field, corresponding to the Hamiltonian $H_{\Omega(\zeta)}$ is tangent to $\mathfrak{M}(t)$, i.e., $r_3 w_t \in \mathfrak{M}(t)$. The conservation of all the quantities C_m, $\frac{dC_m}{dt}=0, m=\pm 1, \pm 2, \ldots$ leads to the same condition.

Let us now consider two examples of NLEE.

1. The sine-Gordon equation /13,14/ $u_{xt} + \sin u = 0$ is obtained from (4.1) for

$$f(\zeta) = -\frac{2}{\zeta}, \quad g(\zeta) = -\frac{i}{\zeta^2}, \quad r = -q = \frac{1}{2} u_x ,$$

where $u(x,t)$ is real valued function satisfying the boundary conditions $\lim_{x \to \infty} u(x,t) = o(\mathrm{mod}\, 2\pi)$. The involution relations

$$a^-(\zeta) = a^+(-\zeta) = a^{+*}(\zeta^*), \quad b^-(\zeta) = b^+(-\zeta) = b^{+*}(\zeta^*)$$

lead to the following structure of the scattering data on the discrete spectrum of the operator $\ell[w]$: i) complex eigenvalues $\zeta_a^{\pm}$, $-\zeta_a^{\pm *}$, $C_a^{\pm}$, $C^{\pm *}, a = 1, \ldots, n_1$; ii) purely imaginary eigenvalues $\zeta^{\pm}_{2n_1+j} = \pm i \rho_j^{\pm}$, $C^{+}_{2n_1+j} = -C^{-*}_{2n_1+j}$, $j = 1, \ldots, n$, $N = 2n_1 + n_2$. Using all these relations we get

$$\omega_{sG} = \int_{-\infty}^{\infty} dx\, \delta u_x \wedge \delta u, \qquad H_{sG} = 2 \int_{-\infty}^{\infty} dx (1 - \cos u),$$

$$H_{sG} = \int_0^{\infty} d\zeta \frac{1}{2\zeta} \hat{P}(\zeta) + 8 \sum_{a=1}^{n_1} \frac{\sin \beta_a}{|\zeta_a^+|} + 4 \sum_{j=1}^{n_2} \frac{1}{\rho},$$

where the action-angle variables equal /13,14/:

$$\hat{P}(\zeta) = -\frac{8}{\pi\zeta} \ln |a^+(\zeta)|, \qquad \hat{Q}(\zeta) = -\arg b^+(\zeta), \qquad 0 \leqq \zeta \leqq \infty,$$

$$\left. \begin{aligned} \hat{P}_{1a} &= 4 \ln |\zeta_a^+|, & \hat{Q}_{1a} &= 4 \ln b_a^+ , \\ \hat{P}_{2a} &= \arg \zeta_a^+ = \beta_a , & \hat{Q}_{2a} &= -16 \arg b_a^+ \end{aligned} \right\} \quad a = 1, \ldots, n_1$$

$$\hat{P}_j = \ln \rho_j , \quad \hat{Q}_j = 8 \ln |b_j^+| , \quad j = 1, \ldots, n_2 .$$

Note, that although the choice $f = 1$, $g(\zeta) = \frac{i}{2\zeta}$ in (4.1) with $-q = r = \frac{1}{2} u_x$ leads also to the sine-Gordon equation,

the corresponding ω_f and H_g are degenerated, i.e., $\omega_f |_{f=1, q=-r} \equiv 0$, $H_g |_{g=i/2z,\ q=-r} \equiv 0$.

2. The nonlinear Schrödinger equation /1,11,12/ $i v_t + v_{xx} + 2|v^2| v = 0$ is obtained from (4.1) with

$$f(\zeta) = 1, \quad g(\zeta) = 4i\zeta^2, \quad q = -r^* = v(x,t).$$

The involution relations have the form $a^-(\zeta) = a^{+*}(\zeta^*)$, $b^-(\zeta) = b^{+*}(\zeta^*)$, $\zeta_j^- = \zeta_j^{+*}$, $C_j^- = C_j^{+*}, j = 1, \ldots, N$ which leads to the following answers for H_{NLS} and ω_{NLS}:

$$\omega_{NLS} = -i \int_{-\infty}^{\infty} dx\, \delta q^* \wedge \delta q, \quad H_{NLS} = \int_{-\infty}^{\infty} dx \{ -|q_x|^2 + |q^4| \},$$

$$H_{NLS} = \int_{-\infty}^{\infty} d\zeta\, 4\zeta^2 \tilde{\tilde{P}}(\zeta) + \frac{1}{12} \sum_{j=1}^{N} \tilde{\tilde{P}}_{2j} (3\tilde{\tilde{P}}_{1j}^2 - \tilde{\tilde{P}}_{2j}^2),$$

where the action-angle variables have the form /11,12/

$$\tilde{\tilde{P}}(\zeta) = \frac{2}{\pi} \ln |a^+(\zeta)|, \qquad \tilde{\tilde{Q}}(\zeta) = \arg b^+(\zeta), \qquad -\infty < \zeta < \infty,$$

$$\left. \begin{aligned} \tilde{\tilde{P}}_{1j} &= -4 \operatorname{Re} \zeta_j^+, & \tilde{\tilde{Q}}_{1j} &= -\ln |b_j^+| , \\ \tilde{\tilde{P}}_{2j} &= -4 \operatorname{Im} \zeta^+, & \tilde{\tilde{Q}}_{2j} &= \arg b_j^+ . \end{aligned} \right\} \quad j = 1, \ldots, N .$$

§5. DESCRIPTION OF THE CLASS OF BACKLUND TRANSFORMATIONS OF THE NLEE

In this paragraph we limit ourselves only to such Bäcklund transformations (BT), which map the solution manifold of some NLEE (4.1) onto itself (for general definition see, e.g., ref. /18/). The most simple BT are given, for example, in the review /19/. We will write down only the BT for the sine-Gordon equation

$$\begin{aligned} BT_x &: (u_2 - u_1)_x = -4\rho_1 \sin \frac{u_1 + u_2}{2} ; \\ BT_t &: (u_1 + u_2)_t = \frac{1}{\rho_1} \sin \frac{u_2 - u_1}{2} . \end{aligned} \qquad (5.1)$$

It is not difficult to verify, that if $u_1(x,t)$ is a solution of the equation $u_{xt} + \sin u = 0$, then $u_2(x,t)$ also satisfies it.

The natural question of describing the action of the BT in the scattering data space $BT_x: T_1 \to T_2$ has been solved in [10]. There it has been shown, that the equations determining BT are determined only by the corresponding operators $\ell_1[w_1]$ and $\ell_2[w_2]$ in (2.1) and coincide for all NLEE of a given class*. Let us go to the explicit description of this relation on the basis of the expansions (2.4) and (2.5). Note, that the vector-functions $\Psi^{\pm}(x,\zeta)$ are eigenfunctions of the integro-differential operator Λ_+

$$\Lambda_+ = \frac{i}{2}\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \sum_{n=1}^{2}\begin{pmatrix}q_n(x)\\ -r_{3-n}(x)\end{pmatrix}\int_x^{\infty} dy\,(r_n(y), q_{3-n}(y))\right\} \quad (5.2)$$

From the uniqueness and invertibility of the expansions (2.4) and (2.5) there follows (see [10]):

Theorem 3. Let us be given the operators $\ell_n[w_n]$, $n=1,2$ (2.1), whose coefficients q_n, r_n are functions of Schwartz type, and let $\hat f(\zeta)$ and $\hat g(\zeta)$ be rational functions having no poles on the spectrum of the operator Λ_+. Then, in order that q_n and r_n satisfy the nonlinear relation (BT_x):

$$\hat f(\Lambda_+)w_+ + \hat g(\Lambda_+)w_- = 0 \quad (5.3)$$

it is necessary and sufficient that the scattering data T_n (2.2) satisfy the linear equations:

$$\hat f(\zeta)(\rho_2^{\pm} + \rho_1^{\pm}) \mp \hat g(\zeta)(\rho_2^{\pm} - \rho_1^{\pm}) = 0, \quad \zeta \in R$$

$$\hat f(\zeta_j^{\pm})(C_{1,j}^{\pm} + C_{2,j}^{\pm}) \mp \hat g(\zeta_j^{\pm})(C_{2,j}^{\pm} - C_{1,j}^{\pm}) = 0, \quad \zeta_j^{\pm} \in \sigma_1^{\pm} \cap \sigma_2^{\pm} \quad (5.4)$$

$$\hat f(\zeta_{1,k}^{\pm}) \pm \hat g(\zeta_{1,k}^{\pm}) = 0, \qquad \zeta_{1,k}^{\pm} \in \sigma_1^{\pm} \setminus (\sigma_1^{\pm} \cap \sigma_2^{\pm}).$$

$$\hat f(\zeta_{2,k}^{\pm}) \mp \hat g(\zeta_{2,k}^{\pm}) = 0, \qquad \zeta_{2,k}^{\pm} \in \sigma_2^{\pm} \setminus (\sigma_1^{\pm} \cap \sigma_2^{\pm}).$$

Let us consider the simplest case of BT_x (5.3) when $\hat f(\zeta) = f_0 + f_1\zeta$, and $\hat g(\zeta) = g_0 + g_1\zeta$ are linear functions; f_0, f_1, g_0, g_1 are complex numbers satisfying $f_1^2 \neq g_1^2$ and

* We can distibguish subclasses in the class of NLEE, imposing involution relations on the operator $\ell[w]$ (see ref. [10]). The general problem of reduction of a given class of NLEE has been considered in [20].

$\mathrm{Im}(f_0-g_0)/(g_1-f_1) > 0$. In that case (5.3) gives:

$$\frac{i}{2}r_3\frac{d}{dx}(f_1 w_+ + g_1 w_-) + w_+(f_0 + \frac{ig_1}{4}Z(x)) + w_-(g_0 - \frac{if_1}{4}Z(x)) = 0, \quad (5.5)$$

where $Z(x) = \int_x^{\infty} dy\,\tilde w_+(y)w_-(y) = 2\int_x^{\infty} dy\,(q_2 r_2 - q_1 r_1)$. Let us show that $Z(x)$ can be expressed locally through q_n and r_n. Multiply (5.5) from the left by $f_1\tilde w_+(x) + g_1\tilde w_-(x)$ and integrate both sides of (5.5) with respect to dx. Thus, we obtain for $Z(x)$ a quadratic equation, giving the following solution:

$$Z(x) = -i\frac{a_2}{a_1} + \frac{i}{a_1}\sqrt{a_2^2 - 4a_1 V(x)},$$

$$V(x) = g_1^2\tilde w_- r_1 w_-(x) + 2f_1 g_1 \tilde w_- r_1 w_+(x) - f_1^2 \tilde w_+ r_1 w_+(x), \quad (5.6)$$

$$a_1 = f_1^2 + g_1^2, \quad a_2 = 4(f_1 g_0 - g_1 f_0), \quad r_1 = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}.$$

With (5.6) the BT_x (5.5) becomes local. From the equations (5.4) we see, that BT_x (5.5) adds two new eigenvalues $\zeta_2^{\pm}$ to the spectrum of the operator $\ell_1[w_1]$, $\zeta_2^{\pm} = -(f_0 \mp g_0)/(f_1 \mp g_1)$ and multiplies $\rho_1^{\pm}(\zeta)$ and $C_{1,k}^{\pm}$ by a fraction-linear functions:

$$\rho_2^{\pm}(\zeta) = K^{\pm}(\zeta)\rho_1^{\pm}(\zeta), \quad C_{2,k}^{\pm} = K^{\pm}(\zeta_k^{\pm})C_{1,k}^{\pm}, \quad K^{\pm}(\zeta) = -\frac{\hat f(\zeta) \mp \hat g(\zeta)}{\hat f(\zeta) \pm \hat g(\zeta)}. \quad (5.7)$$

In the particular cases the BT_x (5.5) goes into the well-known BT_x [10,19]. Here we note two such cases.

1. BT_x for the nonlinear Schrödinger equation is obtained for*

$$\hat f(\zeta) = -ic_1, \quad \hat g(\zeta) = \zeta - c_0, \quad q_n = -r_n^* = v_n(x,t), \quad (5.8)$$

*See the footnote on p.18.

where $c_0, c_1 > 0$. In that case $Z(x) = 4c_1 - 2\sqrt{4c_1^2 + |v_2 - v_1|^2}$ and (5.5) goes into the well-known BT_x /19/

$$\frac{d}{dx}(v_2 - v_1) = -2ic_0(v_2 - v_1) + (v_2 + v_1)\sqrt{4c_1^2 + |v_2 - v_1|^2}.$$

It adds to the spectrum of the operator $\ell_1[w_1]$ two eigenvalues $\zeta_2^{\pm} = c_0 \pm ic_1$, and the function $K^{\pm}(\zeta)$ in (5.7) equals

$$K^{\pm}(\zeta) = \frac{\zeta - \zeta_2^{\mp}}{\zeta - \zeta_2^{\pm}}.$$

2. BT_x for the sine-Gordon equation is obtained for *

$$\hat f(\zeta) = i\rho_1, \quad \hat g(\zeta) = \zeta, \quad q_n = -r_n = -\frac{1}{2}u_{n,x}, \qquad (5.9)$$

with $\rho > 0$. This gives $Z(x) = -4\rho_1 + \sqrt{16\rho_1^2 - (u_{2x} - u_{1x})^2}$ and (5.5) transfers to

$$\frac{d}{dx}(u_{2x} - u_{1x}) + \frac{1}{2}(u_{2x} + u_{1x})\sqrt{16\rho_1^2 - (u_{2x} - u_{1x})^2} = 0.$$

Let us multiply both sides of this equation by $dx/\sqrt{16\rho_1^2 - (u_{2x} - yu_{1x})^2}$ and integrate. This immediately gives the first of the equations (5.1). In this case $\zeta_2^{\pm} = \pm i\rho_1$ and $K^{\pm}(\zeta) = \frac{\zeta \mp i\rho_1}{\zeta \pm i\rho_1}$.

APPENDIX

Let us give here a short derivation of the relations (3.7), (3.10) and (3.12).

The relation (3.7) is obtained from the explicit expression for the resolvent $R(x,y,\zeta)$ of the operator ℓ: $(\ell - \zeta)R(x,y,\zeta) = \delta(x-y)$, where $R = R^{\pm}(x,y,\zeta)$, $\mathrm{Im}\,\zeta \gtrless 0$, $R = R^+ - R^-$, $\mathrm{Im}\,\zeta = 0$,

$$R^{\pm}(x,y,\zeta) = \frac{i}{a^{\pm}(\zeta)}\{\psi^{\pm}(x,\zeta)\phi^{\pm T}(y,\zeta)\theta(x-y) + \phi^{\pm}(x,\zeta)\psi^{\pm T}(y,\zeta)\theta(y-x)\}r_1. \qquad (A.1)$$

Let us insert (A.1) into the relation

* The involution relations impose additional restrictions on the functions $\hat f$ and $\hat g$ (see /10/). For example, if $q_n = \pm r_n^*$ the compatibility condition for the system (5.4) requires that $\hat f(\zeta^*)\hat g(\zeta) = -\hat f(\zeta)\hat g^*(\zeta^*)$. Thus, the choice of the functions $\hat f(\zeta)$ and $\hat g(\zeta)$ in (5.8) and (5.9) is the most general one, compatible with the given involution.

$$\frac{d}{d\zeta}\,\mathrm{Tr}\ln(\ell - \zeta)(\ell_0 - \zeta)^{-1} = \mathrm{Tr}[(\ell - \zeta)^{-1} - (\ell_0 - \zeta)^{-1}] = \int_{-\infty}^{\infty} dx\,\mathrm{tr}[R(x,x,\zeta) - R_0(x,x,\zeta)],$$

where R_0 is the resolvent of the operator $\ell_0 = \ell|_{q=r=0}$. For $\mathrm{Im}\,\zeta > 0$

$$\frac{d}{d\zeta}\,\mathrm{Tr}\ln[(\ell - \zeta)(\ell_0 - \zeta)^{-1}] = -i\int_{-\infty}^{\infty} dx\left[\frac{(\phi_1^+\psi_2^+ - \phi_2^+\psi_1^+)(x,\zeta)}{a^+(\zeta)} - 1\right] =$$

$$= -\int_{-\infty}^{\infty} dx\,\frac{d}{dx}\left\{\frac{W[\phi^+,\dot\psi^+](x,\zeta)}{a^+(\zeta)} - ix\right\} = \frac{d}{d\zeta}\ln a^+(\zeta) \qquad (A.2)$$

we get the relation (3.7) with $A(\zeta)$ defined by (2.3); for $\mathrm{Im}\,\zeta < 0$ everything goes analogously. In the derivation of (A.2) we have used the identity $(W[\phi,\psi] = \phi_1\psi_2 - \phi_2\psi_1)$:

$$\frac{d}{dx}W[\phi,\dot\psi](x,\zeta) = i(\phi_1\psi_2 + \phi_2\psi_1)(x,\zeta)$$

and the well-known asymptotics of the Jost solutions for $x \to \pm\infty$; at the end we take into account the remark at the end of §2 in ref. /5/. We shall do the derivation of (3.9) in two stages. First we shall show that

$$\frac{dA(\zeta)}{d\zeta} = -2i\int_{-\infty}^{\infty} dx \int_{x}^{\infty} dy\,\tilde w(y)\,B(y,\zeta), \qquad (A.3)$$

where $B(x,\zeta) = \pm\frac{\phi^{\pm}\circ\psi^{\pm}(x,\zeta)}{a^{\pm}\zeta}$, $\mathrm{Im}\,\zeta \gtrless 0$, , $B(x,\zeta) = \frac{\phi^+\circ\psi^+(x,\zeta)}{a^+(\zeta)} - \frac{\phi^-\circ\psi^-(x,\zeta)}{a^-(\zeta)}$

for $\mathrm{Im}\,\zeta = 0$, and then we shall obtain that

$$B(x,\zeta) = \frac{i}{2}(L_+ - \zeta)^{-1}\,w(x). \qquad (A.4)$$

We shall prove (A.3) only for $\mathrm{Im}\,\zeta > 0$. Let us insert in the first line of (A.2) the relation

$$\frac{(\phi_1^+\psi_2^+ + \phi_2^+\psi_1^+)(x,\zeta)}{a^+(\zeta)} - 1 = -2i\int_{x}^{\infty} dy\,w(y)\,\frac{\phi^+\circ\psi^+(x,\zeta)}{a^+(\zeta)},$$

which is obtained from (1.1) and the definitions of the Jost solutions (see /5/). This immediately gives (A.3).

The relation (A.7) is derived by the contour integration method applied to the integral

$$J_N(x,\zeta) = \frac{1}{2\pi i}\left\{\oint_{\gamma_N^+}\frac{d\xi}{\xi - \zeta}\,\frac{\phi^+\circ\psi^+(x,\xi)}{a^+(\xi)} - \oint_{\gamma_N^-}\frac{d\xi}{\xi - \zeta}\,\frac{\phi^-\circ\psi^-(x,\xi)}{a^-(\xi)}\right\},$$

where the contours $\gamma_N^{\pm}$ are given in the figure in ref. [5]. Equating the answers for $J_N(x,\zeta)$ obtained by the Causchy theorem and by direct integration along the contours in the limit $N \to \infty$ we get $(\mathrm{Im}\,\zeta > 0)$:

$$\frac{\psi^{+} \circ \phi^{+}(x,\zeta)}{a^{+}(\zeta)} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{d\xi}{\xi - \zeta} (\rho^{+}\Psi^{+} + \rho^{-}\Psi^{-})(x,\xi) -$$

$$- \sum_{k=1}^{N} \left(\frac{C_k^{+}\Psi_k^{+}(x)}{\zeta_k^{+} - \zeta} - \frac{C_k^{-}\Psi_k^{-}(x)}{\zeta_k^{-} - \zeta} \right) = \frac{i}{2} (L_{+} - \zeta)^{-1} w(x).$$

Here we have used the known expansion (2.8a) for $w(x)$ over the system $\{\Psi\}$ and theorem 3 in [5]. Thus (A.4), and consequently (3.10), are proved.

The relation (3.12) follows from the known formula:

$$\delta A(\zeta) = -\frac{i}{2} \int_{-\infty}^{\infty} dx\, \widetilde{\tau_3\, \delta w}(x)\, B(x,\zeta)$$

and from (A.4) (see [9]).

After the completion of this manuscript the authors came across the paper [21], in which close results are obtained. The authors are grateful to Dr. P.P.Kulish for calling their attention to reference [21].

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Received by Publishing Department
on August 17 1979.