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I.Spectral Theory

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Об эволюционных уравнениях, решаемых методом обратной задачи рассеяния. 1. Спектральная теория

Рассмотрены некоторые вопросы спектральной теории интегро-дифференциальных операторов, порождающих нелинейные эволюционные уравнения, интегрируемые методом обратной задачи рассеяния.

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On the Evolution Equations, solvable through the Inverse Scattering Method. 1. Spectral Theory

We consider some problems of the spectral theory of the integro-differential operators, generating non-linear evolution equations solvable through the inverse scattering problem.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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§1. INTRODUCTION

The inverse scattering method (ISM)[1,2] allowed one to distinguish a large class of nonlinear evolution equations (NLEE), describing completely integrable Hamiltonian systems[3,4]. The application of this method to the 1-dimensional Dirac operator[5,6]

$$\ell[w]y = \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}\right] y(x,\zeta) = i\zeta y(x,\zeta), \quad w = \begin{pmatrix} q \\ -r \end{pmatrix} \qquad (1.1)$$

enabled one to solve such important for the physics equations, as the nonlinear Schrödinger equation[5,6] the sine-Gordon equation[7,8] and others (see the review paper [9]).

The essence of the ISM consists in such a (nonlinear) change of variables, which reduces the NLEE to linear equations. This change is performed by using a related linear problem (for example, (1.1)) and consists in transition from the potential w of (1.1), determining the solution of the NLEE, to the scattering data of the operator $\ell[w]$ (1.1).

The idea of interpreting the ISM as a generalized Fourier transform was suggested for the first time in[8]. There it was also shown that the NLEE, solvable through the ISM for the system (1.1) are generated by the integro-differential operator L_+ (see formula (3.4)). The transition from the potential to the scattering data of the operator $\ell[w]$ is connected with the expansions over the eigenfunctions of L_+, the latter being the "squares" of the Jost solutions of (1.1). Important condition for the uniqueness and reversibility of this change of variables is the completeness relation for the eigenfunctions of the operator L_+[12].

Analogically the class of Backlund transformations (BT)[10] for these NLEE are generated by the integro-differential operator Λ_+[11], (see formula (3.2)), which is a natural generalization of the operator Λ_+. The eigenfunctions of the operator Λ_+ are the component-by-component products of solutions of two different Dirac systems (1.1); the completeness relation and the inversion formulae for the expansions over them are obtained in[13].

In this paper, which is an immediate continuation of[13], we construct the spectral theory of the operator Λ_+ (and consequently, - of the operator L_+). In paragraphs 2,3, and 4 we give the main notations and estimates, and also some of the necessary results of[13]. In the main paragraph 5 we define the Green function of the operator Λ_+ and use it to construct the operator calculus for Λ_+. In §6 we introduce a Lagrangian plane in the space of 2-component vector-functions, on which the operator L_+ is symmetric with respect to the skew-scalar product introduced in §3.

§2. In what follows we give some neseccary facts from the direct scattering problem for the system (1.1) (for proofs see[8]). We suppose, that the complex-valued functions q and $r \in \mathcal{S}$, where $\mathcal{S}$ is the Schwartz's space.

Let us introduce the Jost solutions $\phi^{\pm}(x,\zeta)=\begin{pmatrix}\phi_1^{\pm}\\ \phi_1^{\pm}\end{pmatrix}$, $\psi^{\pm}(x,\zeta)=\begin{pmatrix}\psi_1^{\pm}\\ \psi_2^{\pm}\end{pmatrix}$ of the Dirac system (1.1) with the asymptotic conditions

$$\lim_{x\to-\infty}\phi^{+}(x,\zeta)e^{i\zeta x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \lim_{x\to\infty}\psi^{+}(x,\zeta)e^{-i\zeta x}=\begin{pmatrix}0\\1\end{pmatrix},\quad \zeta\in\mathbf{C}^{+},\qquad (2.1)$$

$$\lim_{x\to-\infty}\phi^{-}(x,\zeta)e^{-i\zeta x}=\begin{pmatrix}0\\-1\end{pmatrix},\quad \lim_{x\to\infty}\psi^{-}(x,\zeta)e^{i\zeta x}=\begin{pmatrix}1\\0\end{pmatrix},\quad \zeta\in\mathbf{C}^{-},\qquad (2.2)$$

where $\quad \mathbf{C}^{+}\ (\mathbf{C}^{-})=\{\zeta\in\mathbf{C}\mid \operatorname{Im}\zeta>0\quad(\operatorname{Im}\zeta<0)\},\ \ \bar{\mathbf{C}}^{\pm}=\mathbf{C}^{\pm}\cup\boldsymbol{R}$.

The solutions $\phi^{\pm}(x,\zeta)$ and $\psi^{\pm}(x,\zeta)$ for any fixed x are analytic functions of $\zeta\in\mathbf{C}^{\pm}$, continuous for $\zeta\in\bar{\mathbf{C}}^{\pm}$. The following estimates hold for $\zeta\in\mathbf{C}^{+}$

$$|\phi_1^{+}(x,\zeta)e^{i\zeta x}|\le k,\ |\phi_2^{+}(x,\zeta)e^{i\zeta x}|\le k\int_{-\infty}^{x}dy\,|r(y)|\,e^{-2\operatorname{Im}\zeta(x-y)},\qquad (2.3)$$

$$|\psi_1^{+}(x,\zeta)e^{-i\zeta x}|\le k\int_{x}^{\infty}dy\,|q(y)|\,e^{-2\operatorname{Im}\zeta(y-x)},\ |\psi_2^{+}(x,\zeta)e^{-i\zeta x}|\le k,\qquad (2.4)$$

and for $\zeta\in\mathbf{C}^{-}$:

$$|\phi_1^{-}(x,\zeta)e^{-i\zeta x}|\le k\int_{-\infty}^{x}dy\,|q(y)|\,e^{-2\operatorname{Im}\zeta(y-x)},\ |\phi_2^{-}(x,\zeta)e^{-i\zeta x}|\le k,\qquad (2.5)$$

$$|\psi_1^-(x,\zeta)e^{i\zeta x}|\leq k,\ |\psi_2^-(x,\zeta)e^{i\zeta x}|\leq k\int_x^\infty dy\,|r(y)|\,e^{2\,\mathrm{Im}\,\zeta(y-x)} \qquad (2.6)$$

where k is independent of x and ζ constant,

For real $\zeta\in \boldsymbol{R}$ the following relations hold

$$\phi^\pm(x,\zeta)=\pm a^\pm(\zeta)\,\psi^\mp(x,\zeta)+b^\pm(\zeta)\psi^\pm(x,\zeta), \qquad (2.7)$$

where

$$a^\pm(\zeta)=W[\phi^\pm,\psi^\pm],\quad b^\pm(\zeta)=\mp W[\phi^\pm,\psi^\mp],$$

$$(2.8)$$

$$W[\phi,\psi]=\phi_1\psi_2-\phi_2\psi_1.$$

The coefficients $a^\pm(\zeta)$ are analytic functions in $\zeta\in \mathbf{C}^\pm$, continuous up to real axis $\boldsymbol{R}$, $a^\pm(\zeta)=1+O(\frac{1}{\zeta})$, for $\zeta\to\infty$; the coefficients $b^\pm(\zeta)$ are continuous functions for $\zeta\in\boldsymbol{R}$, $b^+(\zeta)\to 0$, $\zeta\to\pm\infty$. The "unitarity" condition has the form:

$$a^+(\zeta)a^-(\zeta)+b^+(\zeta)b^-(\zeta)=1,\quad \zeta\in \boldsymbol{R}. \qquad (2.9)$$

For simplicity we suppose, that the functions $a^+(\zeta)$ and $a^-(\zeta)$ have a finite number of simple zeroes located in $\mathbf{C}^+$ and $\mathbf{C}^-$ respectively, and besides, let $a^\pm(\zeta)\neq 0$ for $\zeta\in \boldsymbol{R}$. By $\sigma=\sigma^+\cup\sigma^-$

$$\sigma^\pm=\{\zeta_k^\pm\in\mathbf{C}^\pm\,|\,a^\pm(\zeta_k^\pm)=0,\quad k=1,\dots,N_\pm\},$$

we denote the discrete spectrum of the operator (1.1). From (2.8) we see that

$$\phi^\pm(x,\zeta_k^\pm)=b_k^\pm\psi^\pm(x,\zeta_k^\pm), \qquad (2.10)$$

where $b_k^\pm$ are non-vanishing constants.

Remark. If q and r are finite functions, then $a^\pm$, $b^\pm$, $\phi^\pm$, $\psi^\pm$ are analytic functions of ζ in the whole complex plane, and the relations (2.7)-(2.9) are defined for all $\zeta\in\mathbf{C}$. In a number of cases the proofs are given for finite q and r, and then are generalized by a limiting procedure.

§3. Let us denote by $\mathfrak{L}_i$, $i=1,2,\infty$, the linear normed space of vector functions $f(x)=\begin{pmatrix}f_1(x)\\ f_2(x)\end{pmatrix}$, where $f_1,f_2\in L_i(-\infty,\infty)$.

As usual,

$$||f||_{\mathfrak{L}_1}=\int_{-\infty}^{\infty}dx\,\{|f_1(x)|+|f_2(x)|\},\quad ||f||_{\mathfrak{L}_\infty}=\operatorname{ess\,sup}\{f_1(x),f_2(x)\}.$$

The norm in $\mathfrak{L}_2$ is defined through the scalar product

$$(f,g) = \int_{-\infty}^{\infty} dx(f_1(x)\bar{g}_1(x) + f_2(x)\bar{g}_2(x))$$

with the equality

$$||f||_{\mathfrak{L}_2} = (f,f)^{1/2} = (||f_1||_{L_2} + ||f_2||_{L_2})^{1/2}.$$

Let us introduce the skew-scalar product

$$[f,g] = \int_{-\infty}^{\infty} dx(f_1(x)g_2(x) - f_2(x)g_1(x)) = (f, B\bar{g}),\quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3.1}$$

where either $f, g \in \mathfrak{L}_2$ or $f \in \mathfrak{L}_1, g \in \mathfrak{L}_\infty$. Note that the bilinear form $[f,g]$ is non-degenerate, i.e., from $[f,g]=0$ for all $g \in \mathfrak{L}$ it follows that $f=0$. Denote by $\mathfrak{L}_i^{(\ell)}$, $\ell = 1,2,\ldots$ the everywhere dense in $\mathfrak{L}_i$ linear manifold

$$\mathfrak{L}_i^{(\ell)} = \{ f = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} \mid \frac{d^k f_1(x)}{dx^k}, \frac{d^k f_2(x)}{dx^k} \in L_i(-\infty,\infty),\ k=1,\ldots,\ell \},$$

where the functions $\frac{d^k f_{1,2}}{dx^k}$, $k=0,1,\ldots,\ell$ are absolutely continuous in every finite interval of the axis $(-\infty,\infty)$.

Let us introduce the above-mentioned in §1 integro-differential operator Λ_+ and the related to it operator Λ_-:

$$\Lambda_\pm = \frac{i}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{d}{dx} + \sum_{n=1}^{2}\begin{pmatrix} q_n(x) \\ -r_{3-n}(x) \end{pmatrix}\int_x^{\pm\infty} dy(r_n(y), q_{3-n}(y))\right\}, \tag{3.2}$$

Their action on each vector-function $f \in \mathfrak{L}_i^{(1)}$ is defined as follows:

$$(\Lambda_\pm f)_1(x) = \frac{i}{2}\{ \frac{df_1}{dx} + \sum_{n=1}^{2} q_n(x)\int_x^{\pm\infty} dy(r_n(y)f_1(y) + q_{3-n}(y)f_2(y))\},$$

$$(\Lambda_\pm f)_2(x) = \frac{i}{2}\{ -\frac{df_2}{dx} - \sum_{n=1}^{2} r_{3-n}(x)\int_x^{\pm\infty} dy(r_n(y)f_1(y) + q_{3-n}(y)f_2(y))\},$$

The coefficients $q_n(x)$ and $r_n(x)$ in (3.2) (like $q(x)$ and $r(x)$ above), are supposed to be functions of Schwartz type.

If the condition

$$q_1(x) \equiv q_2(x) = q(x),\quad r_1(x) \equiv r_2(x) = r(x) \tag{3.3}$$

holds, then the operator $\Lambda_+(\Lambda_-)$ goes into $L_+(L_-)$:

$$L_\pm = \frac{i}{2}\left\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{d}{dx} + 2\begin{pmatrix} q(x) \\ -r(x) \end{pmatrix}\int_x^{\pm\infty} dy(r(y), q(y))\right\}, \tag{3.4}$$

which generate the class of NLEE [8] related to the linear problem (1.1).

§4. Let us consider now two Dirac systems

$$\ell_n[w_n]y_n \equiv \left[\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\frac{d}{dx} + \begin{pmatrix} 0 & q_n(x) \\ -r_n(x) & 0 \end{pmatrix}\right]\begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix} = i\zeta\begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix}, \quad (n=1,2). \tag{4.1}$$

Define the product $y_1 \circ y_2(x,\zeta)$ of the solutions $y_n(x,\zeta) = \begin{pmatrix} y_{n,1} \\ y_{n,2} \end{pmatrix}(x,\zeta)$ of eq. (4.1) by the formula

$$Y(x,\zeta) = y_1 \circ y_2(x,\zeta) \equiv \begin{pmatrix} y_{1,1}\, y_{2,1} \\ y_{1,2}\, y_{2,2} \end{pmatrix} \tag{4.2}$$

and let

$$\Psi^{\pm}(x,\zeta) = \psi_1^{\pm} \circ \psi_2^{\pm}(x,\zeta), \quad \Phi^{\pm}(x,\zeta) = \phi_1^{\pm} \circ \phi_2^{\pm}(x,\zeta), \quad \zeta \in \mathbf{C}^{\pm} \tag{4.3}$$

where $\phi_n^{\pm}$, $\psi_n^{\pm}$ are the Jost solutions (2.1) and (2.2) of the systems (4.1). Let us introduce the system of vector-functions $\{\Psi\}$:

$$\Psi^{\pm}(x,\zeta), \ \zeta \in R; \ \Psi^{\pm}_{n,k}(x) = \Psi^{\pm}(x,\zeta^{\pm}_{n,k}), \quad \zeta^{\pm}_{n,k} \in \sigma''_{\pm};$$
$$\Psi^{\pm}_{k}(x) = \Psi^{\pm}(x,\zeta^{\pm}_{k}), \quad \dot{\Psi}^{\pm}_{k}(x) = \frac{\partial \Psi^{\pm}(x,\zeta)}{\partial \zeta}\Big|_{\zeta=\zeta^{\pm}_{k}}, \quad \zeta^{\pm}_{k} \in \sigma'_{\pm} \tag{4.4}$$

and the system of vector-functions $\{U\}$:

$$U^{\pm}(x,\zeta) = \frac{1}{a^{\pm}(\zeta)}\Phi^{\pm}(x,\zeta), \qquad U^{\pm}_{n,k}(x) = \frac{1}{\dot{a}^{\pm}_{n,k}}\Phi^{\pm}_{n,k}(x),$$
$$U^{(1)\pm}_{k}(x) = \frac{2}{\ddot{a}^{\pm}_{k}}\Phi^{\pm}_{k}(x), \quad U^{(2)\pm}_{k}(x) = \frac{2}{\ddot{a}^{\pm}_{k}}\left[\dot{\Phi}^{\pm}_{k}(x) - \frac{\dddot{a}^{\pm}_{k}}{3\ddot{a}^{\pm}_{k}}\Phi^{\pm}_{k}(x)\right]. \tag{4.5}$$

Here the functions $a^{\pm}(\zeta) = a_1^{\pm}(\zeta)\, a_2^{\pm}(\zeta)$, $\zeta \in \mathbf{C}^{\pm}$; the sets $\sigma'_{\pm} = \sigma_1^{\pm} \cap \sigma_2^{\pm}$, $\sigma''_{2} = (\sigma_1^{\pm} \cup \sigma_2^{\pm}) \setminus \sigma'_{\pm}$,
where
$\sigma_n^{\pm} = \{\zeta^{\pm}_{n,k} \mid a_n^{\pm}(\zeta^{\pm}_{n,k}) = 0, \quad k = 1,\ldots,N_n^{\pm}\}$
is the discrete spectrum of the system (4.1); $\dot{a}^{\pm}_{n,k} = \dot{a}^{\pm}(\zeta^{\pm}_{n,k})$, etc.

In a standard way, using the following from (4.1) and (4.2) (4.2) identity

$$W[y_1 \circ y_2(x,\zeta),\ z_1 \circ z_2(x,\mu)] = \frac{1}{2i(\mu-\zeta)}\ \frac{d}{dx}\{\prod_{n=1}^{2} W[y_n(x,\zeta),\ z_n(x,\mu)]\} \qquad (4.6)$$

with y_n and z_n being solutions of (4.1) with ζ and μ respectively, we can verify that the system $\{U\}$ is biorthogonally conjugated to the system $\{\Psi\}$ with respect to the skew-scalar product (3.1). More exactly

$$[\Psi^{\pm}(\ \zeta),\ U^{\pm}(\ \mu)] = \ \pi\delta(\mu-\zeta), \qquad \zeta,\mu \in \boldsymbol{R},$$

$$[\Psi^{\pm}_{n,k}\ ,\ U^{\pm}_{m,\ell}\ ] = \frac{1}{2i}\,\delta_{mn}\,\delta_{k\ell}, \qquad \zeta^{\pm}_{n,k},\ \zeta^{\pm}_{m,\ell} \in \sigma''_{\pm}$$

$$[\dot{\Psi}^{\pm}_{k}\ ,\ U^{(1)\pm}_{\ell}\ ] = [\Psi^{\pm}_{k}\ ,\ U^{(2)\pm}_{\ell}\ ] = \frac{1}{2i}\delta_{k\ell}\ ,\ \zeta^{\pm}_{k},\zeta^{\pm}_{\ell}\ \in \sigma'_{\pm}$$

and all other products vanish; here the integral with respect to x in $[\Psi^{\pm}(\ \zeta),\ U^{\pm}(\ \mu)]$ should be understood in the sense of principal value.

Very important in this paper is the following

<u>Theorem 1</u> [13]. Let us be given two Dirac systems (4.1) for which the systems $\{\Psi\}$ (4.4) and $\{U\}$ (4.5) are constructed in the above-mentioned way. Then for each vector-function $f(x) \in \mathfrak{L}^{(1)}_{1}$ the following expansion formula holds

$$f(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta\{\ \Psi^{+}(x,\zeta)[f,U^{+}](\zeta) - \Psi^{-}(x,\zeta)\ [\ f\ ,U^{-}](\zeta)\ \} +$$
$$+2i\Sigma''\Psi^{\pm}_{n,k}\ (x)[f,U^{\pm}_{n,k}] + 2i\ \Sigma'\{\dot{\Psi}^{\pm}_{k}(x)\ [f,U^{(1)\pm}_{k}] + \Psi^{\pm}_{k}(x)[f,U^{(2)\pm}_{k}]\}, \qquad (4.7)$$

where in Σ'' the sum runs over all $\zeta^{+}_{n,k} \in \sigma''_{+}$ and $\zeta^{-}_{n,k} \in \sigma''_{-}$, and in Σ' - over all $\zeta^{+}_{k} \in \sigma'_{+}$ and $\zeta^{-}_{k} \in \sigma'_{-}$. Furthermore, if we define the systems $\{\Phi\}$ and $\{V\}$ by changing in (4.4) Ψ by Φ and in (4.5) - Φ by Ψ respectively, then the following expansion formula holds:

$$f(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} d\zeta\{\ \Phi^{+}(x,\zeta)[f,V^{+}](\zeta) - \Phi^{-}(x,\zeta)[f,V^{-}](\zeta)\ \} -$$
$$-2i\ \Sigma''\Phi^{\pm}_{n,k}(x)[f,V^{\pm}_{n,k}\ ] - 2i\Sigma'\{\ \dot{\Phi}^{\pm}_{k}\ (x)\ [\ f\ ,V^{(1)\pm}_{k}\ ] + \Phi^{\pm}_{k}(x)[f,V^{(2)\pm}_{k}]\}. \qquad (4.8)$$

The expansion coefficients $[f,U]$ and $[f,V]$ in (4.7) and (4.8) for every $f \in \mathfrak{L}_1$ are absolutely convergent integrals.

For $f \in \mathcal{L}_1^{(1)}$ the integrals over ζ in the right-hand side of (4.7) and (4.8) are absolutely and uniformly convergent with respect to $x \in (-\infty, \infty)$.

Remark. Theorem 1 in /13/ is proved for much weaker restrictions on $q_n(x)$ and $r_n(x)$, namely, it is enough that q_n, q_n' and r_n, r_n' are integrable on the real axis $-\infty < x < \infty$. Moreover, if the integrals in (4.7) and (4.8) are understood in the sense of principal value, then the expansion formulae hold for all $f \in \mathcal{L}_1$ with bounded variations in the neighbourhood of x. The supposition $q_n, r_n \in \mathcal{S}$ is justified by the subsequent application (see /14/) of these results to the theory of NLEE. The greater smoothness of the vector-function $f \in \mathcal{L}_1$, for which the expansion formula are written, allows one to differentiate and integrate, in the usual way, the necessary number of times under the integrals in (4.7) and (4.8). This is also somewhat simplifies the formulations and the proofs of the propositions, and enables us to keep the volume of the paper in the reasonable limits.

§5. In this paragraph we construct the Green functions for the operators Λ_+ and Λ_- (3.2), and show that formulae (4.7) and (4.8) respectively, create their spectral decompositions.

One verifies directly

Lemma 1. The operator Λ_- is conjugated to the operator Λ_+ with respect to the skew-scalar product (3.1), i.e.:

$$[\Lambda_+ f, g] = [f, \Lambda_- g]. \tag{5.1}$$

Remark. Since the form (3.1) is nondegenerate, eq. (5.1) uniquely determines the adjoint operator Λ_-, which is obviously given by the corresponding expression in (3.2). Note, that if we consider Λ_+ as an operator with domain of definition $D(\Lambda_+) = \mathcal{L}_2^{(1)}$ then $D(\Lambda_-) = \mathcal{L}_2^{(1)}$, and if $D(\Lambda_+) = \mathcal{L}_1^{(1)}$, then $D(\Lambda_-) = \mathcal{L}_\infty^{(1)}$, the operators $\Lambda_\pm : \mathcal{L}_1^{(1)} \to \mathcal{L}_i$, $(i = 1, 2, \infty)$ being closed.

An important relation between the vector-functions (4.3) $\Psi^\pm(x,\zeta)$ and $\Phi_\pm(x,\zeta)$ and the operator $\Lambda_\pm$ is established through the following

Lemma 2. The vector-functions $\Psi^\pm(x,\zeta)$ and $\Phi^\pm(x,\zeta)$ satisfy the equations:

$$\Lambda_+ \Psi^\pm(x,\zeta) = \zeta \Psi^\pm(x,\zeta), \quad \Lambda_- \Phi^\pm(x,\zeta) = \zeta \Phi^\pm(x,\zeta), \quad \zeta \in \bar{\mathbb{C}}^\pm \tag{5.2}$$

The proof is well known and follows from the asymptotics (2.1) and (2.2) and the following (see [18])

Lemma 2'. If the vector function $Y = y_1 \circ y_2$ is bounded for $x \to \infty$ ($x \to -\infty$), then

$$\Lambda_{\overset{+}{(-)}} Y(x,\zeta) = \zeta Y(x,\zeta) + \frac{i}{2} \begin{pmatrix} q_1(x) y_{1,2} y_{2,1}(z,\zeta) + q_2(x) y_{1,1} y_{2,2}(z,\zeta) \\ -r_1(x) y_{1,1} y_{2,2}(z,\zeta) - r_2(x) y_{1,2} y_{2,1}(z,\zeta) \end{pmatrix}_{z \to \overset{+}{(-)}\infty} \quad (5.3)$$

From the Lemmas 1 and 2 there follows directly,

Lemma 3. For every $f \in \mathcal{L}_1^{(\ell)}$, $\ell = 1,2,\ldots$ the following relations hold:

$$\zeta^{\ell} [f(x), \Phi^{\pm}(x,\zeta)] = [(\Lambda_+^{\ell} f)(x), \Phi^{\pm}(x,\zeta)]$$
$$\zeta^{\ell} [f(x), \Psi^{\pm}(x,\zeta)] = [(\Lambda_-^{\ell} f)(x), \Psi^{\pm}(x,\zeta)]. \quad (5.4)$$

Let us introduce, as in [1], the 2x2 matrix ($\tilde{Y} = (Y_2, -Y_1)$);

$$R_+(x,y,\zeta) = \frac{2i}{a(\zeta)} \{ \psi_1 \circ \psi_2(x,\zeta) \widetilde{\phi_1 \circ \phi_2}(y,\zeta) \theta(x-y) +$$
$$+ [\sum_{n=1}^{2} \psi_n \circ \phi_{3-n}(x,\zeta) \widetilde{\phi_n \circ \psi_{3-n}}(y,\zeta) - \phi_1 \circ \phi_2(x,\zeta) \widetilde{\psi_1 \circ \psi_2}(y,\zeta)] \theta(y-x) \}, \quad (5.5)$$

where for $\zeta \in \mathbf{C}^+$ in the r.h.s. of (5.5) $a = a^+$, $\psi = \psi^+$, $\phi = \phi^+$; for $\zeta \in \mathbf{C}^-$ $- a = a^-$, $\psi = \psi^-$, $\phi = \phi^-$; $\theta(x)$ is the Heviside function.

The main theorem in this paragraph is the following

Theorem 2. The integral operator

$$R_+(\zeta, f)(x) = \int_{-\infty}^{\infty} dy\, R_+(x,y,\zeta) f(y), \quad (f \in \mathcal{L}_i)$$

with the kernel (5.5) defines the resolvent

$$R_+(\zeta) = (\zeta I - \Lambda_+)^{-1}, \quad \zeta \in \rho(\Lambda_+) = \mathbf{C} \setminus \left(R \cup \sigma_1 \cup \sigma_2\right) \quad (5.6)$$

of the operator Λ_+, $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\rho(\Lambda_+)$ being the regularity region for the operator $R_+(\zeta)$. Moreover, the following estimates hold:

$$\| R_+(\zeta) \|_{[\mathcal{L}_i]} \overset{\text{def}}{=} \sup_{f \in \mathcal{L}_i, \|f\|_{\mathcal{L}_i} \le 1} \| R_+(\zeta, f) \|_{\mathcal{L}_i} \le \frac{k}{|a(\zeta)| \cdot |\text{Im}\,\zeta|}, \quad (i = 1, \infty) \quad (5.7)$$

$$\|R_+(\zeta)\|_{[\mathfrak{L}_2]} \leqq \frac{k}{|\alpha(\zeta)|} \max\left(\frac{1}{|\mathrm{Im}\,\zeta|^2}, \frac{1}{|\mathrm{Im}\,\zeta|}\right), \tag{5.8}$$

where k is a constant, $k>0$.

Proof. First we will obtain the estimates (5.7) and (5.8). For definiteness, let us choose $\mathrm{Im}\,\zeta>0$. Let us write down $R_+(x,y,\zeta) = R_+^{(0)} + \sum_{n=1}^{2} R_+^{(n)}$;

$$R_+^{(n)}(x,y,\zeta) = \frac{2i}{\alpha^+(\zeta)} \psi_n^+ \circ \phi_{3-n}^+ (x,\zeta)\, \phi_n^+ \circ \psi_{3-n}^+ (y,\zeta)\, \theta(y-x).$$

From (2.3) and (2.4) we get the following estimates for the matrix elements of $R_+^{(0)}$:

$$|R_{+,k\ell}^{(0)}(x,y,\zeta)| \leqq \frac{k}{|\alpha^+(\zeta)|} e^{-2\mathrm{Im}\,\zeta|x-y|}, \quad (k,\ell=1,2)$$

and for the matrix element of $R_+^{(n)}$:

$$|R_{+,11}^{(n)}| \leqq \frac{k}{|\alpha^+(\zeta)|} H_1(q_n,x) H_2(r_n,y);$$

$$|R_{+,12}^{(n)}| \leqq \frac{k}{|\alpha^+(\zeta)|} H_1(q_n,x) H_1(q_{3-n},y)$$

$$|R_{+,21}^{(n)}| \leqq \frac{k}{|\alpha^+(\zeta)|} H_2(r_{3-n},x) H_2(r_n,y);$$

$$|R_{+,22}^{(n)}| \leqq \frac{k}{|\alpha^+(\zeta)|} H_2(r_{3-n},x) H_1(q_{3-n},y),$$

where

$$H_1(f,x) = \int_x^\infty dt\,|f(t)|\, e^{-2\,\mathrm{Im}\,\zeta(t-x)},$$

$$H_2(f,x) = \int_{-\infty}^x dt\,|f(t)|\, e^{-2\,\mathrm{Im}\,\zeta(x-t)}.$$

From here the estimate (5.7) for $i=\infty$ follows directly as a consequence of the following simple estimates:

$$\int_x^\infty dy\, e^{-2\mathrm{Im}\zeta(y-x)}|f(y)| \leq \frac{1}{2\,\mathrm{Im}\zeta}\|f\|_{L_\infty}$$

$$\int_x^\infty dy\, H_\ell(q_n, y)|f(y)| \leq \frac{\|f\|_{L_\infty}}{2\,\mathrm{Im}\zeta}\int_{-\infty}^\infty dx|q_n(x)|,$$

$$\|H_\ell(q_n, x)\|_{L_\infty} \leq \int_{-\infty}^\infty dx\,|q_n(x)|, \qquad \ell = 1,2.$$

The estimates (5.7) for $i=1$ and (5.8) are obtained analogously if we take into consideration (see, e.g., ref.[15]), that for any function $f(x) \in L_i, i=1,2$ the function $H_\ell(f,x) \in C(-\infty,\infty)$, $H_\ell(f,x) \to 0$ when $|x| \to \infty$ and $\|H_\ell(f)\|_{L_i} \leq k(\mathrm{Im}\zeta)^{-1}\|f\|_{L_i}$, $(i,\ell = 1,2)$. We note only, that for $i=2$ one should also use the Cauchy-Bunjakovski (in L_2) inequality and the inequality $\|f_1\|_{L_2} + \|f_2\|_{L_2} \leq \sqrt{2}\,\|f\|_{\mathfrak{L}_2}$.

The proof of (5.6), i.e., the verification of the equalities

$$(\zeta I - \Lambda_+)R_+(\zeta, f)(x) = f(x), \quad R_+(\zeta, (\zeta I - \Lambda_+)\, f)(x) = f(x), \quad \zeta \in \rho(\Lambda_+) \qquad (5.9)$$

will be done in detail for the case $f \in \mathfrak{L}_1^{(1)}$ only. For this we need the following

<u>Lemma 4.</u> (About the integral representation of the resolvent R_+). In the notations of theorem 1 for every $f \in \mathfrak{L}_1$ the vector-function $R_+(\zeta, f)(x)$ can be represented in the form:

$$R_+(\zeta, f)(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{d\mu}{\zeta-\mu}\{\Psi^+(x,\mu)[f, U^+](\mu) - \Psi^-(x,\mu)[f, U^-](\mu)\}$$

$$+ 2i\Sigma'\left\{\frac{\partial}{\partial\mu}\left(\frac{\Psi^\pm(x,\mu)}{\zeta-\mu}\right)\Big|_{\mu=\zeta_k^\pm}[f, U_k^{(1)\pm}] + \Psi_k^\pm(x)\frac{[f, U_k^{(2)\pm}]}{\zeta-\zeta_k^\pm}\right\} + \qquad (5.10)$$

$$+ 2i\Sigma''\Psi_{n,k}^\pm(x)\frac{[f, U_{n,k}^\pm]}{\zeta-\zeta_{n,k}^\pm}.$$

<u>Proof.</u> Let us consider the contour integral

$$I_N(x,\zeta) = \frac{1}{2\pi i}\left\{\oint_{\gamma_N^+}\frac{d\mu}{\zeta-\mu}R_+(\mu, f)(x) - \oint_{\gamma_N^-}\frac{d\mu}{\zeta-\mu}R_+(\mu, f)(x)\right\}, \qquad (5.11)$$

where the contours $\gamma_N^\pm$ are shown on the <u>figure.</u> Let us

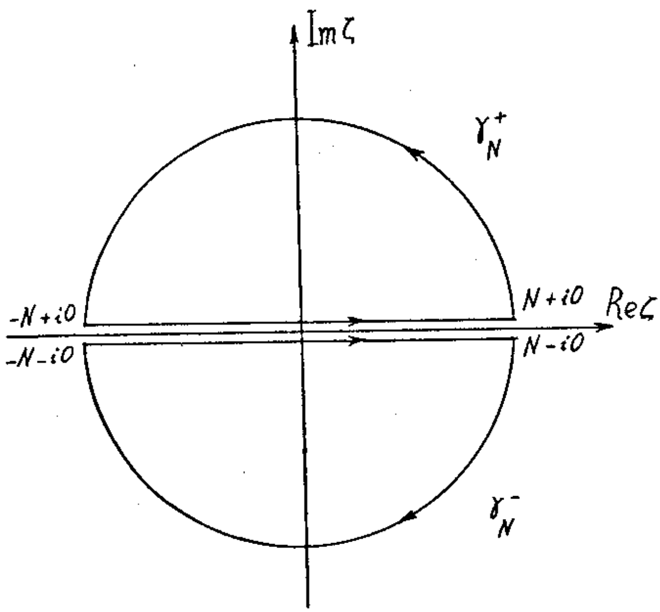

Fig. Contours $\gamma_N^{\pm}$

equate the value of $I_N(x,\zeta)$, obtained by the Cauchy theorem with the value of $I_N(x,\zeta)$, obtained by integration along the contours $\gamma_N^+ \cup \gamma_N^-$. Going to the limit $N \to \infty$ we get (5.10). We drop out the calculational details since they are completely analogical to the ones in the proof of theorem 1 in /13/. We note only, that the integral term in the right-hand side of (5.10) comes out of the integral

$$\frac{1}{2\pi i}\int_{-N}^{N} d\mu\,[R_+(\mu+i0,f)(x) - R_+(\mu-i0,f)(x)]/(\zeta-\mu)$$

using the relations (2.7), (2.8) and (2.10); the proof of the equality $\lim_{N\to\infty}(\int_{s_N^+} - \int_{s_N^-})R_+(\mu,f)d\mu/(\zeta-\mu) = 0$, where s_N^+ and s_N^- are the semiarcs of the circle with radius N (see the figure), is a direct consequence of Jordan's lemma and the well known (see /8/) Jost solutions' asymptotics $\phi^{\pm}$, $\psi^{\pm}$ for $\zeta \to \infty$.

The lemma is proved.

End of proof of theorem 2. Let in (5.10) $f \in \mathcal{L}_1^{(1)}$. From (5.4) and the estimates (2.4)-(2.6) it follows, that the subintegral expression in (5.10) is continuously differentiable with respect to x vector-function, which falls off like $O(\frac{1}{\zeta^2})$ for $\zeta\to\infty$ uniformly with respect to $x\in(-\infty,\infty)$. Let us apply to both sides of (5.10) the operator $(\zeta I-\Lambda_+)$; because of the above-mentioned arguments, it can be introduced under the integral. From (5.2) and the expansion formula (4.7) we get the first of the equalities in (5.9). The second equality in (5.9) is obtained also from (4.7) noting that from (5.1) and (5.2) we have $[(\zeta I-\Lambda_+)f(x),\ \Phi^{\pm}(x,\mu)]=(\zeta-\mu)[f(x),\ \Phi^{\pm}(x,\mu)]$.

The proof of the equalities (5.9) in the general case, when in the first of them $f\in\mathfrak{L}_i$, and in the second $f\in\mathfrak{L}_i^{(1)}$, $(i=1,2,\infty)$, is completed directly by inserting (5.5) and (3.2) into (5.9). The calculations are lengthy and we omit them. We note only, that one should use lemmas 2 and 2', the relations (2.7) and the known asymptotics of the Jost solutions $\psi_n^{\pm}(x,\zeta)$ for $x\to\pm\infty$. Finally, one should use the remark at the end of §2. The theorem is proved.

Let us list some simple corollaries of this theorem. Using lemma 1 we have:

Corollary 1. The resolvent $R_-(\zeta)=(\zeta I-\Lambda_-)^{-1}$ of the operator Λ_- is defined with the help of $R_+(x,y,\zeta)$(5.5) from the equality

$$[R_+(\zeta,f),g]=[\ f,R_-(\zeta,g)\ ],$$

where either $f,g\in\mathfrak{L}_2$, or $f\in\mathfrak{L}_{1,(\infty)}$, $g\in\mathfrak{L}_{\infty,(1)}$ is an integral operator $R_-(\zeta,f)(x)=\int_{-\infty}^{\infty}dy\,R_-(x,y,\zeta)f(y)$, where

$$R_-(x,y,\zeta)=-BR_+^T(y,x,\zeta)B,\qquad B=\begin{pmatrix}0&1\\-1&0\end{pmatrix}.$$

Corollary 2. From (5.6) and the estimates (5.7) and (5.8) we get, because of the well-known Hilbert identity $R(\zeta)-R(\mu)=(\mu-\zeta)R(\zeta)R(\mu)$ that the operator $R_+(\zeta)$ is in analytic operator-valued function of $\zeta\in\rho(\Lambda_+)$; furthermore

$$\frac{\partial^k R_+(\zeta)}{\partial\zeta^k}=(-1)^k k!(\zeta I-\Lambda_+)^{-k-1},\quad k=0,1,\ldots \tag{5.12}$$

where the derivatives exist in the sense of the operator norm $[\mathfrak{L}_i]$, $i=1,2,\infty$.

Note also, that from theorem 2 and lemma 4 we get

Corollary 3. The spectrum $\sigma(\Lambda_+)$ of the operator Λ_+ is double-valued and continuous for $\zeta\in\boldsymbol{R}$; its discrete part consists of the set $\sigma_+''\cup\sigma_-''$ of simple eigenvalues $\zeta_{n,k}^{\pm}$ and of the set $\sigma_+'\cup\sigma_-'$ of doubly degenerated eigenvalues $\zeta_k^{\pm}$. From corollary 1 it is clear that $\sigma(\Lambda_-)=\sigma(\Lambda_+)$.

At the end of this paragraph we formulate the theorem, which plays the main role in the presentation in [14] of the theory of NLEE and their Backlund transformations.

Theorem 3. Let

$$\Omega(\zeta)=\sum_{s=0}^{\ell} b_s\zeta^s+\sum_{m,k}\frac{c_{m,k}}{(\zeta-\mu_m)^k} \tag{5.13}$$

be a rational function with poles $\mu_m\in\rho(\Lambda_+)$ i.e., lying in the regularity region of the operator $R_+(\zeta)$ and let $\Omega(\Lambda_+)$ be the operator

$$\Omega(\Lambda_+)=\sum_{s=0}^{\ell} b_s\Lambda_+^s+\sum_{m,k} c_{m,k}\,(\Lambda_+-\mu_m I)^{-k}. \tag{5.14}$$

Then, using the notations of theorem 1, for every $f\in\mathfrak{L}_1^{(\ell+1)}$ the following expansion holds:

$$\begin{aligned}\Omega(\Lambda_+)f(x)=&-\frac{1}{\pi}\int_{-\infty}^{\infty}d\zeta\,\Omega(\zeta)\{\Psi^+(x,\zeta)[f,U^+](\zeta)-\Psi^-(x,\zeta)[f,U^-](\zeta)\}+\\&+2i\Sigma'\{\frac{\partial}{\partial\zeta}(\Omega(\zeta)\Psi^{\pm}(x,\zeta))|_{\zeta=\zeta_k^{\pm}}[f,U_k^{(1)\pm}]+\Omega(\zeta_k^{\pm})\Psi_k^{\pm}(x)[f,U_k^{(2)\pm}]\}+\\&+2i\Sigma''\Omega(\zeta_{n,k}^{\pm})\Psi_{n,k}^{\pm}(x)[f,U_{n,k}^{\pm}],\end{aligned} \tag{5.15}$$

where the integrals are absolutely and uniformly convergent.

Proof. From lemma 3 and the estimates (2.3)-(2.6) it follows, that the subintegral expressions in the right-hand side of (4.7) for every $f\in\mathfrak{L}_1^{(s+1)}$ are infinitely differentiable vector-functions of x, whicl fall off like $O(|\zeta|^{-s-1})$ for $\zeta\to\pm\infty$ uniformly with respect to $x\in(-\infty,\infty)$. Then, applying the operator Λ_+ s times to both sides of (4.7) we obtain because of (5.2) formula (5.15) in the case of $\Omega(\zeta)=\zeta^s$, $\Omega(\Lambda_+)=\Lambda_+^s$. Analogously, differentiating both sides of (5.10) $k-1$ times with respect to ζ we get, using the identity (5.12), that equation (5.15) holds for $\Omega(\zeta)=(\zeta-\mu_m)^{-k}$, $\Omega(\Lambda_+)=(\Lambda_+-\mu_m I)^{-k}$. The theorem is proved.

Remark. We stress, that the definition of the operator $\Omega(\Lambda_+)$ (5.14) is not apriori related to the expansion formulae (4.7), (5.10) and to the choice of the spaces $\mathfrak{L}_i (i=1,2,\infty)$. Really, the positive powers $\Lambda_+^s, s=1,2,\ldots$, can be defined as usual $\Lambda_+^s=\Lambda_+(\Lambda_+)^{s-1}$, $D(\Lambda_+^s)=\mathfrak{L}_i^{(s)}$; the operator

$(\Lambda_+ - \mu I)^{-k}$, $k=1,2,\ldots$, $\mu \in \rho(\Lambda_+)$ is, because of the estimates (5.7), (5.8), a bounded operator in $\mathfrak{L}_i$ with a domain of definition $D((\Lambda_+ - \mu I)^{-k}) = \mathfrak{L}_i, (i=1,2,\infty)$. Note also, that through (5.15) the correspondence between the rational functions $\Omega(\zeta)$ (5.13) and the operators $\Omega(\Lambda_+)$ (5.14) can naturally be expanded (see, e.g., [16]) to some classes of functions $\mathcal{F}(\zeta)$, analytic in the neighbourhood of the spectra $\sigma(\Lambda_+)$, and rising not very fast for $\zeta \to \pm\infty$, taking as a definition of the operator $\mathcal{F}(\zeta)$ the operator, obtained from (5.15) by changing the function $\Omega(\zeta)$ to $\mathcal{F}(\zeta)$.

§6. In this paragraph we shall consider in more detail the properties of the operator $L_\pm$ (3.4). Besides the conditions (4.3), in this paragraph we suppose that $b^+(\zeta)b^-(\zeta) \neq 0, -\infty<\zeta<\infty$, where $b^\pm(\zeta)$ are defined in (2.8). We omit here the cases, when for some values of $\zeta \in \boldsymbol{R}$, $b^+b^-(\zeta)=0$; they lead only to some technical complications in our next considerations.

Let us construct like in [13] the system

$$P(x,\zeta) = \frac{1}{\pi}(\sigma^+(\zeta)\Phi^+(x,\zeta) + \sigma^-(\zeta)\Phi^-(x,\zeta)), \quad \sigma^\pm = \frac{b^\mp}{a^\pm}, \quad \zeta \in \boldsymbol{R},$$

$$Q(x,\zeta) = \frac{1}{2b^+(\zeta)\,b^-(\zeta)}(\sigma^+(\zeta)\Phi^+(x,\zeta) - \rho^+(\zeta)\Psi^+(x,\zeta)), \quad \zeta \in \boldsymbol{R}, \qquad (6.1)$$

$$P_k^\pm(x) = \mp 2iM_k^\pm \Phi_k^\pm(x), \quad Q_k^\pm(x) = \mp \frac{1}{2}(C_k^\pm \dot{\Psi}_k^\pm(x) - M_k^\pm \dot{\Phi}_k^\pm(x)),$$

where $C_k = b_k^\pm/\dot{a}_k^\pm$, $M_k^\pm = (b_k^\pm \dot{a}_k^\pm)^{-1}$; the functions $a^\pm(\zeta)$, $b^\pm(\zeta)$ are defined in (2.8). From (2.7), (2.10) and (2.12) it follows that the vector-functions $P(x,\zeta)$, $Q(x,\zeta)$ and $P_k^\pm(x)$ can be written down in the form:

$$P(x,\zeta) = \frac{1}{\pi}(\rho^+(\zeta)\Psi^+(x,\zeta) + \rho^-(\zeta)\Psi^-(x,\zeta)), \quad P_k^\pm(x) = \mp 2iC_k\Psi_k^\pm(x),$$

$$Q(x,\zeta) = \frac{1}{2b^+(\zeta)b^-(\zeta)}(\rho^-(\zeta)\Psi^-(x,\zeta) - \sigma^-(\zeta)\Phi^-(x,\zeta)). \qquad (6.2)$$

From (6.1) and (6.2) it follows, that for any $f \in \mathfrak{L}_1$ the following identity holds (see [13]):

$$\frac{1}{\pi}\frac{1}{(a^+(\zeta))^2}\{\Phi^+(x,\zeta)[f,\Psi^+](\zeta)-\Psi^+(x,\zeta)[f,\Phi^+](\zeta)\}-$$
$$-\frac{1}{\pi}\frac{1}{(a^-(\zeta))^2}\{\Phi^-(x,\zeta)[f,\Psi^-](\zeta)-\Psi^-(x,\zeta)[f,\Phi^-](\zeta)\}= \tag{6.3}$$
$$=Q(x,\zeta)[f,P](\zeta)-P(x,\zeta)[f,Q](\zeta).$$

Using lemma 3 we obtain, that for every $f\in\mathfrak{L}_1^{(\ell)}$ the following estimate for $\zeta\to\pm\infty$

$$Q(x,\zeta)[f,P](\zeta)-P(x,\zeta)[f,Q](\zeta)=O(\frac{1}{|\zeta|^{\ell}}), \tag{6.4}$$

holds uniformly with respect to $x\in(-\infty,\infty)$. An important role in our next considerations plays the following

Theorem 4. The constructed according to (6.1) system $\{P,Q\}$ is a simplectic basis in $\mathfrak{L}_1^{(1)}$, i.e., for every vector-function the following expansion formula holds:

$$f(x)=\int_{-\infty}^{\infty}d\zeta\{Q(x,\zeta)[f,P](\zeta)-P(x,\zeta)[f,Q](\zeta)\}+$$
$$+\Sigma\{Q_k^{\pm}(x)[f,P_k^{\pm}]-P_k^{\pm}(x)[f,Q_k^{\pm}]\}. \tag{6.5}$$

Furthermore

$$[Q(x,\zeta),P(x,\mu)]=\delta(\zeta-\mu),\qquad \zeta,\mu\in\boldsymbol{R},$$
$$[Q_k^{\pm},P_\ell^{\pm}]=\delta_{k\ell} \tag{6.6}$$

and all the other skew-scalar products vanish.

Let us remind (see [13]), that the expansion formula (6.5) is obtained by summing the expansion formulae (4.7) and (4.8) using (6.3) and (2.10). The relations (6.6), which are intuitively clear from (6.5), can be verified using the identity (4.6).

The following lemma is easily obtained from (6.1), (6.2) and from lemmas 2 and 2':

Lemma 5. The vector-functions $P(x,\zeta)$ and $Q(x,\zeta)$ satisfy the equations:

$$L_{\pm}P(x,\zeta)=\zeta P(x,\zeta),\quad L_{\pm}Q(x,\zeta)=\zeta Q(x,\zeta)+\frac{i}{2}w(x),\quad(\zeta\in\boldsymbol{R}\cup\sigma). \tag{6.7}$$

Let us define in $\mathfrak{L}_1^{(\ell)}$ the subspace:

$\mathfrak{M} = \{f \in \mathfrak{L}_1 \mid [f,P](\zeta) = 0,\ \zeta \in \boldsymbol{R};\ [f,P_k^{+}] = 0,\ \zeta_k \in \sigma\}$
and put $\mathfrak{M}^{(\ell)} = \mathfrak{M} \cap \mathfrak{L}_1^{(\ell)}$ $(\ell = 1,2,\ldots)$.

Remark. Since, as a consequence of the expansion formula (6.5) for every $f \in \mathfrak{M}^{(1)}$, the following relation

$$f(x) = -\int_{-\infty}^{\infty} d\zeta\, P(x,\zeta)[f,Q](\zeta) - \Sigma\, P_k^{\pm}(x)\,[f,Q_k^{\pm}] \tag{6.8}$$

holds, then from (6.6) it follows, that for every $f,g \in \mathfrak{M}^{(1)}$ we have $[f,g] = 0$, i.e., that the linear manifold $\mathfrak{M}$ defines a Lagrangian plane in the space $\mathfrak{L}_1$ /17/ with respect to the skew-scalar product (3.1).

Lemma 6. The linear manifold $\mathfrak{M}^{(\ell)}$ is an invariant manifold for the operators L_+ and L_-, i.e., $L_{\pm} f \in \mathfrak{M}^{(\ell-1)}$ for every $f \in \mathfrak{M}^{(\ell)}$.

Proof. By multiplying the equality $[f,P](\zeta) = 0$ by ζ and $[f,P_k^{\pm}] = 0$ by $\zeta_k^{\pm}$ we obtain because of (6.7) the equalities $[f, L_+ P(x,\zeta)] = 0$ and $[f, L_{\pm} P_k^{\pm}] = 0$. Furthermore, applying lemma 1 we get $[L_{\pm} f, P](\zeta) = 0$ and $[L_{\pm} f, P_k^{\pm}] = 0$, what had to be proved.

Let us denote by L the operator L_+ (either L_-) with the domain of definition $\mathfrak{M}^{(2)}$, i.e.,

$$Lf(x) \overset{\text{def}}{=} L_+(L_-)f(x), \qquad f \in \mathfrak{M}^{(2)}. \tag{6.9}$$

The definition (6.9) is justified by the following

Theorem 5. For every $f \in \mathfrak{M}$

$$L_+ f(x) = L_- f(x) = -\int_{-\infty}^{\infty} d\zeta\,\zeta P(x,\zeta)[f,Q](\zeta) - \Sigma\,\zeta_k^{\pm} P_k^{\pm}(x)[f,Q_k^{\pm}]. \tag{6.10}$$

Furthermore the domain of definition of the operator L (6.9) is contained in $\mathfrak{M}^{(1)}$.

Proof. Since formula (6.8) holds for any $f \in \mathfrak{M}$, then using (6.4) we obtain the equality (6.10) directly from (6.7). The last equality is a direct consequence of lemma 5. The theorem is proved.

Remark. From the definition of $\mathfrak{M}^{(\ell)}$ and from lemma 1 it follows, that the operator L (6.9) is symmetric with respect to the skew-scalar product (3.1), i.e., for any $f,g \in \mathfrak{M}^{(2)}$ we have $[Lf,g] = [f,Lg]$.

Theorem 6. The operator

$$R(\zeta,f)(x)=\frac{1}{2}[R_{+}(\zeta,f)(x)+R_{-}(\zeta,f)(x)] \tag{6.11}$$

defined on $\mathfrak{L}_1$, where

$$R_{\pm}(\zeta,f)=(\zeta I-L_{\pm})^{-1} \tag{6.12}$$

determines on the linear manifold $\mathfrak{M}$ the resolvent of the operator L (6.9), i.e.,

$$R(\zeta)=(\zeta I-L)^{-1}, \qquad (\zeta\in\rho(L)=\mathbf{C}\setminus(\boldsymbol{R}\cup\sigma). \tag{6.13}$$

Furthermore for any $f\in\mathfrak{M}$ the expansion

$$R(\zeta,f)=-\int_{-\infty}^{\infty}\frac{d\mu}{\zeta-\mu}P(x,\mu)[f,Q](\mu)-\Sigma P_k^{\pm}(x)\frac{[f,Q^{\pm}]}{\zeta-\zeta_k^{\pm}} \tag{6.14}$$

holds.

Proof. We first note, that from the estimate (5.7) for $R_{+}(\zeta)$ and an analogic one for $R_{-}(\zeta)$ it follows, that if $f\in\mathfrak{L}$ then the vector-function $R_{+}(\zeta,f)\in\mathfrak{L}_1$. Furthermore from (6.12) and the equalities (6.7) and (5.1) we obtain, that for any $f\in\mathfrak{L}_1$ the following identity

$$(\zeta-\mu)[R(\zeta,f)(x),P(x,\mu)]=[f(x),P(x,\mu)],\ \zeta\in\rho(L),\ \mu\in\boldsymbol{R}\cup\sigma$$

holds. Hence, if $f\in\mathfrak{M}$ then $R(\zeta,f)\in\mathfrak{M}$, and from here using (6.9) and (6.12) we obtain (6.13). In order to get the expansion (6.14) we sum equality (5.10) with an analogic one for $R_{-}(\zeta,f)$. From (6.3) and (2.10) it follows, that for any $f\in\mathfrak{L}_1$

$$R(\zeta,f)(x)=\int_{-\infty}^{\infty}\frac{d\mu}{\zeta-\mu}\{Q(x,\mu)[f,P](\mu)-P(x,\mu)[f,Q](\mu)\}+$$

$$+\Sigma\frac{1}{\zeta-\zeta_k^{\pm}}\{Q_k^{\pm}(x)[f,P_k^{\pm}]-P_k^{\pm}(x)[f,Q_k^{\pm}]\}.$$

The theorem is proved.

At the end of this paragraph we note, that as a consequence of theorems 3 and 5 we obtain the following

Theorem 7. Let us for a given rational function $\Omega(\zeta)$ (5.13) construct the operator $\Omega(L_{\pm})$ (6.14). Then on the linear manifold $\mathfrak{M}^{(\ell)}$

$$\Omega(L_{+})f(x)=\Omega(L_{-})f(x)=\Omega(L)f(x)=-\int_{-\infty}^{\infty}d\zeta\,\Omega(\zeta)P(x,\zeta)[f,Q](\zeta)-$$

$$-\Sigma\,\Omega(\zeta_k^{\pm})P_k^{\pm}(x)[f,Q_k^{\pm}].$$

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