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Submitted to ЯФ

Ильгенфриц Э.-М., Казаков Д.И., Мюллер-Пройскер М. **E2 - 12628**

Существует ли фазовый переход в янг-миллсовском инстантонном газе?

Уточняется анализ фазового перехода в инстантонном газе в теории Янга-Миллса, выполненный впервые Калланом, Дашеном и Гроссом. Показано, что подавление инстантонов во внешнем электрическом поле оказывается гораздо слабее при правильном учете вращательных степеней свободы. Фазовый переход по-прежнему возможен, но при больших значениях константы связи. Является ли он переходом первого рода /что важно для формирования мешка/, существенно зависит от анзатца среднего поля для инстантонных взаимодействий. Показано, что случай возникновения спонтанной намагниченности также возможен.

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Is there Really a Phase Transition in the Yang-Mills Instanton Gas?

The conditions for the existence of the first order phase transition bag-to-vacuum, suggested by Callan, Dashen, and Gross, are critically examined. The instanton suppression by external electric fields turns out to be much softer, when the global gauge degrees of freedom of the instantons are treated correctly. A nontrivial equation of state E = E(D) is possible only if relatively large couplings can be reconciled with the instanton gas. We point out an ambiguity in dealing with instanton interactions by mean-field methods, the alternative being the existence of spontaneous polarization instead of the wanted transition.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

Callan, Dashen, and Gross [1] have suggested a way ,how the MIT bag model [2] could evolve from QCD "first principles" as an approximation scheme. In order to accomplish this "derivation", the existence of different phases of the QCD vacuum and their equilibrium conditions must be spelled out. The mechanism suggested for this phase transition [1] has been based on the existence of the instanton solutions [3] of Euclidean Yang-Mills theory and their tunneling interpretation in constructing the Θ -vacua [4]. (We take always Θ =0.)

In the problem at hand there is another parameter, too. It is an external field coupled to free colour charges, the continuation of which to Euclidean space is denoted as F^{ext}. The Euclidean vacuum has been viewed [1] as a 4-dimensional medium of "magnetic" dipoles formed by an indefinite number of instantons and antiinstantons responding to the "static magnetic" field F^{ext}. The phase transition, if it exists, is driven by F^{ext}, which plays a role analogous to the volume in the van der Waals gas below the critical temperature.

In what sense does a phase transition eventually exist? Besides of lowering the vacuum energy density by tunneling, the instanton gas provides a (in general nonlinear) relation $F=\mu(F^{ext})\,F^{ext}$ due to its polarizability, such that it contributes a nonvanishing part to the Yang-Mills field F $(\mu>1)$. Since $F^{ext}=\tilde{H}$ +) and F are the Euclidean continuations of a set of Minkowski space fields $\vec{H}^a,\vec{D}^a$ and $\vec{B}^a,\vec{E}^a$, respectively,

$$F^a_{ik}=\varepsilon_{ikl}\,B^a_l \qquad F^a_{4k}=i\,E^a_k$$
$$F^{ext\,a}_{ik}=\varepsilon_{ikl}\,H^a_l \qquad F^{ext\,a}_{4k}=i\,D^a_k \tag{1}$$

there will be nonlinear relations between the Minkowski space fields. They will be called equations of state of the vacuum.

+) For shorthand we will write $F^{ext\,a}_{\mu\nu}=\tilde{H}^a_{\mu\nu}$ .Here $\tilde{H}^a_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}H_{\lambda\sigma}$ is dual to H. Analogously $F^a_{\mu\nu}=\tilde{B}^a_{\mu\nu}$. The complete formulation in terms of magnetic fields B and H is carried through in Ref. /1/.

In Ref. /1/ (however, on the basis of the previous, numerically wrong instanton density [5] carried over to SU(3)) a strongly nonlinear behaviour of $E=E(D)$ was found. $D=D(E)$ became nonunique. Hence, from that by means of a standard Maxwell construction a phase transition was pointed out. At a critical strength E_c of a static electric field a "vacuum" and a "bag" phase were found to coexist.

According to these calculations the "vacuum" phase was characterized by a permeability $\mu \gg 1$, stable below E_c and unable to support an external field $D \neq 0$. The "bag phase", stable for $E \gg E_c$, had permeability $\mu \gtrsim 1$ ($D \approx E$). The same mechanism has been studied following /1/ also in Ref. /6/.

Adhering to the instanton gas picture as such we reinvestigate the existence of this phase transition after the recent trivial numerical correction [5b] of the one-loop instanton density:

$$n_o(\rho) = C_{SU(N)} \frac{1}{\rho^4} x(\rho)^{2N} e^{-x(\rho)}, \quad x(\rho) = \frac{8\pi^2}{g^2(\rho)} \qquad (2)$$

with (see [7] for general N)

$$C_{SU(N)} = \frac{4.602}{\pi^2} \frac{e^{-1.678N}}{(N-1)!(N-2)!} \qquad . \qquad (3)$$

In Table 1 values of the fractional occupied space-time volume

$$f(x_c) = 2\int_0^{\rho_c} \frac{d\rho}{\rho} n_o(\rho) \frac{\pi^2}{2} \rho^4 \qquad (4)$$

are calculated corresponding to various cut-offs $x_c = x(\rho_c)$ and $x(\rho)$ varying according to the one- or two-loop approximation of the Gell-Mann-Low function /8/ for SU(2) and SU(3). The usual dilute gas criterion, $f(x_c) < 1$, has lost its restrictive power to fix the magnitude for x_c or ρ_c. In this paper we will therefore study the equation of state of the vacuum, $D=D(E)$, with respect to its dependence on the cut-off x_c and the behaviour of $x(\rho)$. One has to keep in mind, however, that in principle reasonable x_c are bounded from below by the requirement, that the one-loop approximation to the tunneling amplitude

$$\langle 1 | e^{-HT} | 0 \rangle = \int d^4x \frac{d\rho}{\rho} n_o(\rho) [1 + O(g^2)] \qquad (5)$$

is still good.

There is also another reason to reconsider the phase transition problem. In particular, we wish to make more explicit the relation between the equation of state of the (Minkowski) vacuum and the grand partition function of the (Euclidean)

Table 1:

Space-time fraction f, occupied by instantons, and susceptibility χ of the instanton gas, within the 1- and 2-loop approximation for $x(\rho)$

		SU(2)		SU(3)	
	x_c	f	$4\pi^2\chi$	f	$4\pi^2\chi$
1-loop	0	.35	3.50	.98	3.43
	5	.15	2.16	.75	2.97
	9	.019	.41	.20	1.11
	12	.0027	.071	.045	.31
2-loop	0	.21	2.23	.57	2.13
	5	.11	1.54	.47	1.91
	9	.017	.34	.15	.83
	12	.0025	.067	.038	.26

instanton gas. We do this in order to avoid any detailed reference /1/ to thermodynamical potentials of customary polarizable substances (which sometimes is misleading as we shall see below). In doing this we pay special attention both to the distribution in the global gauge degrees of freedom of the instantons and to the mean-field treatment of instanton interactions.

The paper is organized as follows. In section 2 we write down the grand partition function of the interacting instanton gas and indicate how to extract from it physical information in terms of Minkowski space fields. In section 3 we contrast our instanton distribution in an external electric field with that of Ref. /1/ and point out wherefrom this difference arises in the formulation used there. In section 4 we study the nonlinear equation of state induced by an instanton gas with neglect of interactions. If and only if large effective coupling constants ($g^2/8\pi^2 \sim .2$) are admitted, instabilities occur which lead to a first-order phase transition. In section 5 we motivate a mean-field treatment of instanton interactions different from that used in Ref. /1/. In section 6 we present and discuss the equations of state corresponding to both mean-field methods. Depending on the strength of the instanton interaction the possibility of spontaneous polarization instead of a first-order phase transition turns out. Section 7 contains the conclusions.

2. Grand partition function of the instanton gas in an external field and the equation of state

We consider the vacuum-to-vacuum amplitude by taking the Euclidean functional integral around multiinstanton configurations being only approximate stationary points of the action:

$$A_\mu^a(x)=\sum_{i=1}^{N_++N_-}\frac{2}{g}R^{(i)a}{}_\alpha\,\bar{\eta}^{\varepsilon_i}_{\alpha\mu\nu}\frac{(x-x_i)_\nu\,\rho_i^2}{(x-x_i)^2[(x-x_i)^2+\rho_i^2]}\,. \qquad (6)$$

Here the standard collective coordinates of the N_+ (N_-) (anti-) instantons are used: positions x_i, sizes ρ_i, $\varepsilon_i=+1$ (-1). The orthogonal matrices $R^{(i)}$ represent independent global gauge rotations

$$A^{(i)}\rightarrow g_i^{-1}A^{(i)}g_i\;,$$

$$R^{(i)a}{}_\alpha\,T_a=g_i^{-1}T_\alpha\,g_i \qquad (a,\alpha=1,2,..,N^2-1)\,.$$

g runs over SU(N)/T(N) where T(N) is the stability group of the instanton /7/. The factor (3) includes the volume of SU(N)/T(N), such that $\int dR$ means the normalized measure over this subgroup. The η symbols introduced in Ref. /5/ are used here as follows: $\bar{\eta}^{\varepsilon}_{\alpha\mu\nu}=\bar{\eta}_{\alpha\mu\nu}$ $(\eta_{\alpha\mu\nu})$ for instantons (antiinstantons).

Choosing (6) as classical configurations to expand about, we may write the vacuum-to-vacuum amplitude as the grand partition function of an interacting (anti-)instanton gas. In our particular case we consider this gas under the influence of a weakly varying external field $F^{ext}=\tilde{H}$. For later use we introduce also chemical potentials $\xi_\varepsilon(x,\rho,R)$, in general depending on all collec-

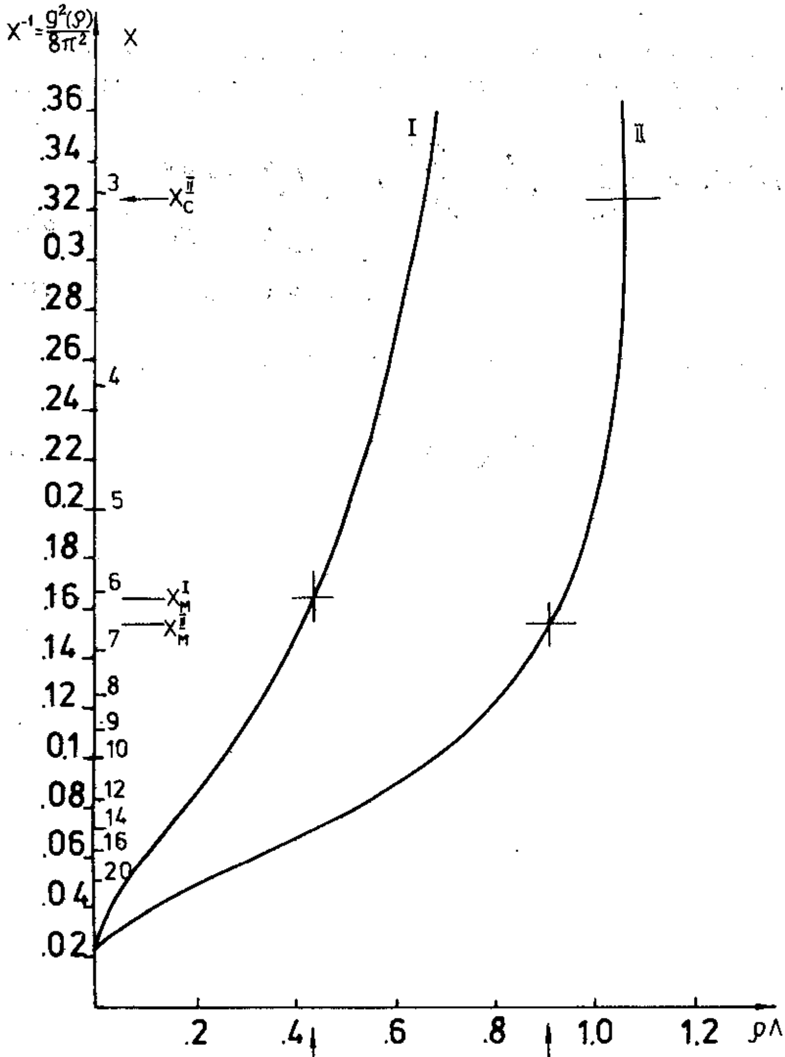


Fig. 1: Dependence of the effective coupling constant $x^{-1} = \frac{g^2(\rho)}{8\pi^2}$ on the instanton size in the case of SU(2).

Curve I: one-loop approximation

Curve II: two-loop approximation

Shown are also the values x_M^I and x_M^{II} which give maximal contributions to the polarization in the transition region (compare Fig. 3).

tive coordinates of a single (anti-)instanton. We refer to an Euclidean 4-volume V and write the grand partition function associated with V

$$Z[\tilde{H},\zeta_\pm,V] = \sum_{N_+,N_-} \frac{1}{N_+!N_-!} \prod_{j=1}^{N_++N_-} \int_V d^4x_j \int^{\rho_c} \frac{d\rho_j}{\rho_j} \int dR^{(j)} \cdot n_0(\rho_j)\, e^{\zeta_{\varepsilon_j}(x_j,\rho_j,R^{(j)})}\, e^{2\pi^2 D^j \tilde{H}} \cdot \exp(+V_{int}) \cdot \tag{7}$$

Here a dipole-like interaction /4/ of an (anti-)instanton having the dipole moment

$$D^{ja}_{\mu\nu} = \frac{\rho_j^2}{g} R^{(j)a}{}_{\alpha}\, \bar{\eta}^{\varepsilon_j}_{\alpha\mu\nu} \tag{8}$$

with an external field has been written out explicitly:

$$-\delta S = 2\pi^2 D^a_{\mu\nu} \tilde{H}^a_{\mu\nu} . \tag{9}$$

The standard instanton density is given by (2) and defined via (5). The density (2) has been calculated in Refs. /5,7/ and involves both the Jacobian from the functional measure of the zero modes to the configuration space measure $d^4x \frac{d\rho}{\rho} dR$ and the instanton action in $\exp(-x(\rho))$ renormalized by the integral over nonzero modes. According to common believe, also the unrenormalized Jacobian factor $(\frac{8\pi^2}{g^2})^{2N}$ becomes renormalized into $x(\rho)^{2N}$. For the ρ dependence of the effective coupling constant $x^{-1}(\rho)$ we shall take and compare the one- and two-loop expressions (see Fig.1) following from the corresponding approximations for the Gell-Mann- Low function /8/,

$$-\frac{d\rho}{\rho} = \frac{3}{11N}\, dx \qquad \text{(one loop)}$$

or

$$-\frac{d\rho}{\rho} = \frac{3}{11N}\left(1+\frac{34}{22}N\frac{1}{x}\right)^{-1} dx \cdot \qquad \text{(two loop)} \tag{10}$$

The interaction term V_{int} in (7) has to absorb all corrections occurring at finite distances between the (anti)instantons. These include i) corrections to the classical action due to its nonadditivity and to the fact that (6) is no local minimum of the action, ii) the logarithm of the appropriate determinants of propagators in the multiinstanton background field (up to the sum of logarithms of individual instanton determinants), and iii) corrections from the Jacobian with respect to all collective coordinates. All this is systematically developed in Ref./9/, but far from allowing a treatment of the partition function (7). However, useful for our purposes is the result /9/ that none of the mentioned contributions to V_{int} +) falls less rapidly with distance than $\varrho_i^2\varrho_k^2/(x_i-x_k)^4$. This justifies, for the time being, to concentrate on the leading part of the classical interaction, $V_{int}=-S_{int}$, with

$$
\begin{aligned}
S_{int} &\approx -\frac{1}{4}\sum_{i\neq k}\int \mathcal{F}^{(i)a}_{\mu\nu}(x-x_i)\,\mathcal{F}^{(k)a}_{\mu\nu}(x-x_k)\,d^4x \\
&\approx -\frac{1}{2}\sum_{i\neq k} 2\pi^2 D^{(i)a}_{\mu\nu}\,\mathcal{F}^{(k)a}_{\mu\nu}(x_i-x_k) \qquad (11)\\
&\approx \sum_{i\neq k} 4\pi^2 D^{(i)a}_{\mu\nu}\,D^{(k)a}_{\nu'\mu'}\,T_{\nu\nu'}\,T_{\mu\mu'}/(x_i-x_k)^4 .
\end{aligned}
$$

($\mathcal{F}^{(i)a}_{\mu\nu}=\partial_\mu A^{(i)a}_\nu - \partial_\nu A^{(i)a}_\mu$ is the Abelian part of the i-th (anti)instantons field, $\mathcal{F}^a_{\mu\nu}\approx\frac{4}{x^4}T_{\mu\mu'}T_{\nu\nu'}D^a_{\nu'\mu'}$ where $T_{\mu\nu}=\delta_{\mu\nu}-2x_\mu x_\nu/x^2$.) This is just the classical dipole-dipole interaction /1,4/.

From the partition function (7) we shall obtain instanton distributions

$$
\langle n_\varepsilon(x,\varrho,R)\rangle = \left.\frac{\delta\ln Z[\tilde{H},\zeta_\pm,V]}{\delta\zeta_\varepsilon(x,\varrho,R)}\right|_{\zeta_\pm=0} \qquad (12)
$$

(or more-instanton distributions if required) influenced by the external field, or directly the magnetization density

+) Apart from a logarithmic one induced by massless fermions, which we will disregard following Ref. /1/

$$4\pi^2 \tilde{M}^a_{\mu\nu} = \frac{\partial}{\partial H^a_{\mu\nu}} \; \frac{1}{V} \ln Z[\tilde{H}, \zeta_\pm, V]/_{\zeta_\pm = 0} \; . \qquad (13)$$

The macroscopic field $F^a_{\mu\nu} = \tilde{B}^a_{\mu\nu}$ summarizes the external and the averaged dipole fields of the polarized (anti)instantons:

$$\tilde{B}^a_{\mu\nu} = \tilde{H}^a_{\mu\nu} + 4\pi^2 \tilde{M}^a_{\mu\nu} = \mu(\tilde{H})\,\tilde{H}^a_{\mu\nu} \, . \qquad (14)$$

This relation, when expressed via (13) and then via (1) written in terms of the Minkowski space fields $\vec{E}^a$, $\vec{B}^a$ and $\vec{D}^a$, $\vec{H}^a$, is the equation of state of the vacuum we are looking for.

3. Free energy density and instanton suppression

For this discussion it is sufficient to consider the instanton gas without interaction V_{int}. Then we obtain from (7) immediately

$$\frac{1}{V} \ln Z[\tilde{H}, \zeta_\pm, V] = \sum_\varepsilon \int \frac{d\rho}{\rho} dR \; n_0(\rho)\, e^{2\pi^2 \overset{\varepsilon}{D} \tilde{H}} \, e^{\zeta_\varepsilon(\rho, R)} \qquad (15)$$

and trivially by (12)

$$\langle n_\varepsilon(\rho, R) \rangle = n_0(\rho) \exp(2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu} \tilde{H}^a_{\mu\nu}) . \qquad (16)$$

This is the average instanton density, from which a distribution $\langle n_\varepsilon(\rho) \rangle$ will be obtained by integrating over dR. With (13) we find

$$4\pi^2 \tilde{M}^a_{\mu\nu} = \sum_\varepsilon \int \frac{d\rho}{\rho} dR \;\; 4\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu} \;\; n_0(\rho)\, e^{2\pi^2 \overset{\varepsilon}{D}{}^b_{\lambda\sigma} \tilde{H}^b_{\lambda\sigma}} \, . \qquad (17)$$

For the purpose of the equation of state it is therefore sufficient to consider the average density (16), expressed in terms of the external field in Minkowski space. In the case of SU(2) and for an external electric field $D^a_k = \delta^{a3} \delta_{k3} D$ we have then

$$\langle n_\varepsilon(\rho, R) \rangle = n_0(\rho) \exp\left(i\varepsilon\, 4\pi^2 R^3_3 \, \rho^2 D / g(\rho)\right) \qquad (18)$$

and

$$\langle n_\varepsilon(\rho) \rangle = n_0(\rho) \frac{\sin \xi}{\xi} \;\; , \quad \xi = 4\pi^2 \rho^2 D / g(\rho) . \qquad (19)$$

In spite of the complexvaluedness and oscillating behaviour of these quantities, being single instanton amplitudes, we will continue to use the terms "distribution" or "density". The use of the free energy density below is legitimate as long as real

Euclidean fields are considered. The final relations between the fields, however, are equivalent to those derived from the functional integral (7) and may be continued to Minkowski space.

The instanton density in Ref. /1/ corresponding to (19) for the case of SU(2) would be

$$\langle n_\varepsilon(\rho)\rangle = n_0(\rho)\, exp\left(-\frac{\pi^2}{3}x(\rho)\rho^4 D^2\right) \tag{20}$$

showing exponential suppression instead of an oscillating one in (19). We will show how this difference arises.

The free energy density of the instanton gas at given $\tilde{H}$ is defined as a functional of $n_\varepsilon(\rho, R)$:

$$F[n_\pm, \tilde{H}] = -\frac{1}{V} \ln Z[\tilde{H}, \zeta_\pm, V] + \sum_\varepsilon \int \frac{d\rho}{\rho} dR\, n_\varepsilon(\rho,R)\zeta_\varepsilon(\rho,R), \tag{21}$$

where $\zeta_\varepsilon(\rho, R)$ is eliminated by

$$\frac{\delta}{\delta\zeta_\varepsilon(\rho,R)}\frac{1}{V}\ln Z[\tilde{H},\zeta_\pm,V] = n_\varepsilon(\rho,R) = n_0(\rho)e^{\zeta_\varepsilon + 2\pi^2 \overset{\varepsilon}{D}\tilde{H}}. \tag{22}$$

Thus one obtains

$$\begin{aligned} F[n_\pm, \tilde{H}] = {} & \sum_\varepsilon \int \frac{d\rho}{\rho} dR\, n_\varepsilon(\rho,R)\left[\ln\frac{n_\varepsilon(\rho,R)}{n_0(\rho)} - 1\right] \\ & - \sum_\varepsilon \int \frac{d\rho}{\rho} dR\, n_\varepsilon(\rho,R)\, 2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu} \tilde{H}^a_{\mu\nu}. \end{aligned} \tag{23}$$

By minimization the average density (16) is reobtained. Assume however, that one wants to express the free energy density as functional of the distributions $n_\varepsilon(\rho)$. Such a reduced free energy density functional makes only sense if it is constructed as minimum of (23) with respect to distributions $w_\varepsilon(\rho \mid R)$, defined by $n_\varepsilon(\rho, R) = n_\varepsilon(\rho)w_\varepsilon(\rho \mid R)$ and being subject to the normalization condition $\int dR\, w_\varepsilon(\rho \mid R) = 1$. Performing the desired minimization, one obtains

$$w_\varepsilon(\rho \mid R) = exp(-\lambda_\varepsilon(\rho))\, exp(2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu}\tilde{H}^a_{\mu\nu})$$

with a Lagrange multiplier $\lambda_\varepsilon(\rho)$, which because of normalization is the logarithm of the group integral $\int dR\, exp\, 2\pi^2 \overset{\varepsilon}{D}\tilde{H}$. The reduced free energy density functional is found to be

$$\begin{aligned} F[n_\pm(\rho), \tilde{H}] = {} & \sum_\varepsilon \int \frac{d\rho}{\rho} n_\varepsilon(\rho)\left[\ln\frac{n_\varepsilon(\rho)}{n_0(\rho)} - 1\right] \\ & - \sum_\varepsilon \int \frac{d\rho}{\rho} n_\varepsilon(\rho) \ln \int dR\, exp(2\pi^2 \overset{\varepsilon}{D}{}^a_{\mu\nu}\tilde{H}^a_{\mu\nu}). \end{aligned} \tag{24}$$

Only for small external fields, namely such that $\pi^2 x(\rho)\rho^4 H^2/(N^2-1) \ll 1$ over a range ρ where $n_\varepsilon(\rho)$ is appreciable, the last expression can be approximated by

$$F[n_\pm(\rho),\tilde{H}] = F[n_\pm(\rho),0] \tag{25}$$
$$-\frac{\pi^2}{2}\sum_\varepsilon \int \frac{d\rho}{\rho} n_\varepsilon(\rho)\frac{1}{N^2-1}\, x(\rho)\rho^4 \tilde{H}^a_{\mu\nu}(\tilde{H}^a_{\mu\nu}-\varepsilon H^a_{\mu\nu}).$$

Introducing then the susceptibility

$$\chi[n_\pm] = \frac{1}{2}\sum_\varepsilon \int \frac{d\rho}{\rho} n_\varepsilon(\rho)\frac{1}{N^2-1}\, x(\rho)\rho^4, \tag{26}$$

noticing, that for an external electric field D (in Minkowski space) it holds

$$\tilde{H}^a_{\mu\nu}(\tilde{H}^a_{\mu\nu}-\varepsilon H^a_{\mu\nu}) = -2D^2, \tag{27}$$

and adding the energy density $\mathcal{E}_0 = \frac{1}{2}D^2$, a free energy density

$$F^*[n_\pm(\rho),D] = F[n_\pm(\rho),0] + \frac{1}{2}DE$$

with

$$E = (1+4\pi^2\chi[n_\pm])D \tag{28}$$

is derived, which is formally the free energy density of a dielectric /10/ and has been the starting point of the instanton thermodynamics in Ref. /1/. It is correct only to second order in D, as well as the instanton density (20) , formally obtained by minimization of (28). Up to this order the density (20) does not differ from (19). Effects depending upon the "exponential suppression" must be considered with care. In the following sections we study the consequences of an instanton density like (18).

4. The equation of state for the noninteracting instanton gas

In the absence of instanton interactions only the external field acts on each individual dipole.The magnetization of the dipole gas is given by (17). In a field weak enough for a linear response we expand the exponential and obtain

$$\tilde{M}^a_{\mu\nu} = \chi\tilde{H}^a_{\mu\nu} = \sum_\varepsilon \int \frac{d\rho}{\rho}\, dR\, n_0(\rho)\, 2\pi^2 \frac{\rho^4}{g^2(\rho)} R^a_\alpha\, \bar{\eta}^\varepsilon_{\alpha\mu\nu} \cdot R^b_\beta\, \bar{\eta}^\varepsilon_{\beta\lambda\sigma}\tilde{H}^b_{\lambda\sigma}, \tag{29}$$

i.e. with (for general N)

$$\int dR\, R^a_\alpha R^b_\beta = \frac{1}{N^2-1}\delta^{ab}\delta_{\alpha\beta},$$

we have

$$\chi = \frac{1}{N^2-1}\int^{\rho_c}\frac{d\rho}{\rho}\, n_0(\rho)\, x(\rho)\,\rho^4 . \qquad (30)$$

This susceptibility is dependent both on the cutoff $x_c=x(\rho_c)$ and the assumed dependence $x(\rho)$. In Table 1 there are the corresponding values of $4\pi^2\chi$ given for SU(2) and SU(3), for the one- and two- loop approximation $x(\rho)$ (compare (10)), and for several values of x_c. Due to the numerical factor (3) these values are considerably smaller than previous estimates. We see from this Table that permeabilities $\mu=1+4\pi^2\chi$, large enough to allow a strongly paramagnetic behaviour of the "vacuum" phase, occur only if one accepts large couplings as $x^{-1}=\frac{g^2}{8\pi^2}\approx.2$ to be consistent with the instanton gas picture. Even then a phase transition due to the instanton gas mechanism alone will never produce a jump in μ between the two phases large enough to enforce really bag-like boundary conditions. Maybe other mechanisms help in this respect (merons ?). This attitude has been taken also in Ref./1/. We are here merely looking for instabilities in the equation of state due to the instanton mechanism.

In order to study the nonlinear response of the Euclidean instanton medium to an Minkowskian external electric field, we integrate the full expression (17). Unfortunately we are unable to do the group integration for SU(3). In the case of SU(2) we get for $\tilde{H}^a_{4k}=\delta^{a3}\delta_{k3}\,iD$ and $\tilde{M}^a_{4k}=\delta^{a3}\delta_{k3}\,iP$ ("polarization") the relation

$$E = D + 4\pi^2 P = D + \frac{2}{D}\int^{\rho_c}\frac{d\rho}{\rho}\, n_0(\rho)\left(\frac{\sin\xi}{\xi} - \cos\xi\right) \qquad (31)$$

with $\quad \xi = 4\pi^2\rho^2 D/g(\rho)$.

In Fig.2 we show this equation of state for different x_c using the one- and two-loop dependence $x(\rho)$ (compare (10) and Fig.1). For $x_c=12$ there is almost no influence of instantons on $E=E(D)\approx D$. For $x_c=0$ there is an instability visible in the "one-loop" equation of state around $\frac{E_c}{\Lambda^2}\approx 3$. At this critical field strength both phases are clearly distinguished by their respec-

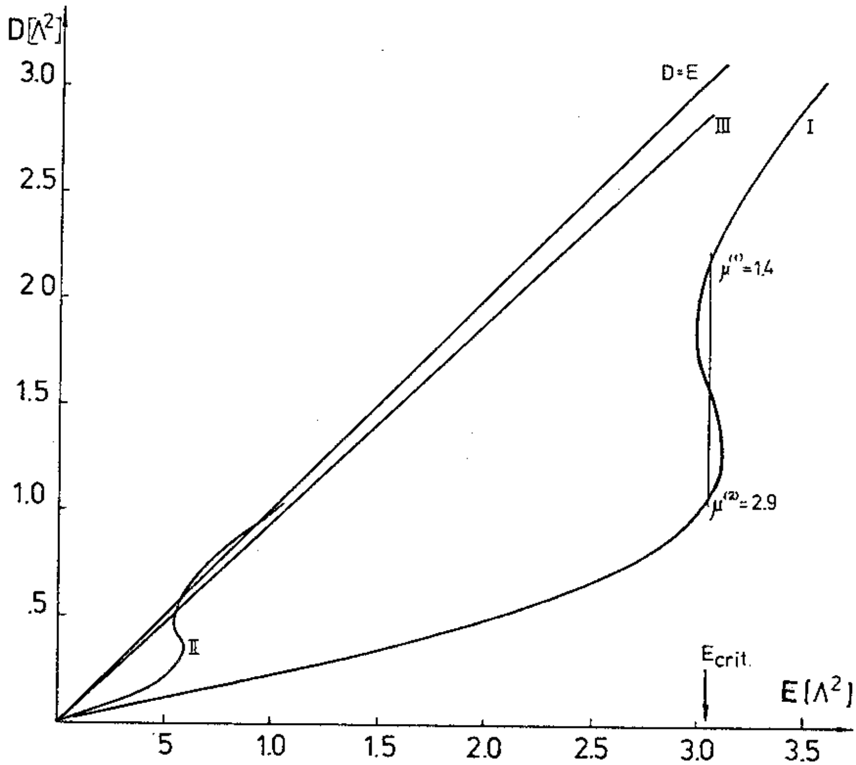


Fig. 2: Equation of state of the noninteracting instanton gas depending on the cut-off x_c and the assumed $x(\varrho)$.
Curve I: $x_c=0$, one-loop approximation for $x(\varrho)$
Curve II: $x_c=0$, two-loop approximation for $x(\varrho)$
Curve III: $x_c=12$ for both forms of $x(\varrho)$.

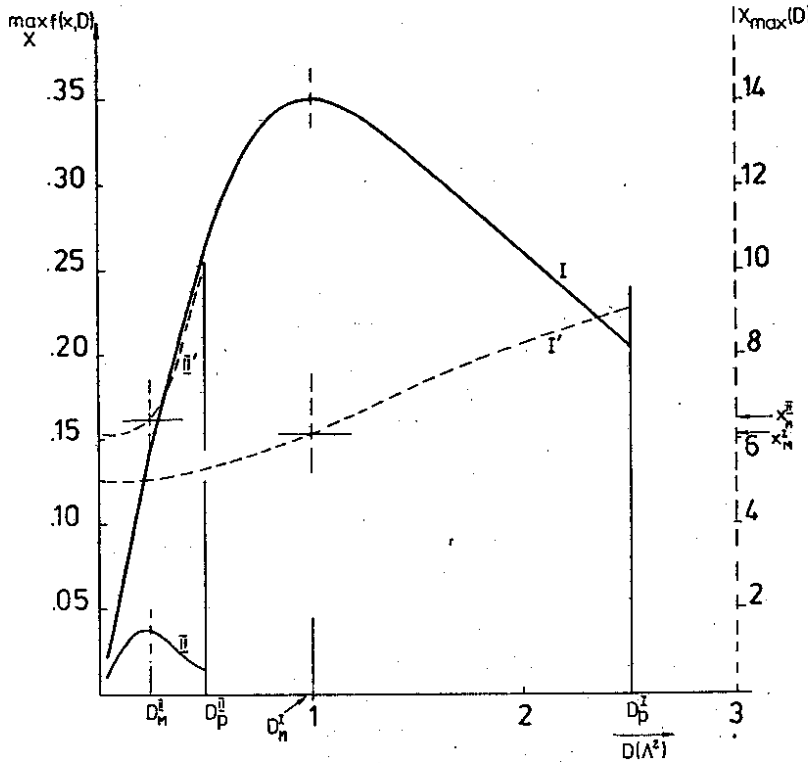


Fig. 3: Maximum contribution to the polarization of the noninteracting gas per unit x (solid curves, left ordinate) and corresponding peak values x_{max} (dashed curves, right ordinate) versus external field D.
Curves I,I': one-loop approximation,
Curves II,II': two-loop approximation for $x(\varrho)$.
Shown is only the range $D < D_P$ where no strong cancellations occur. The maxima D_M correspond to the maxima of polarization in Fig.2. The corresponding x_{max} values are marked as x_M.

tive permeabilities $\mu^{(1)}=1.4$ and $\mu^{(2)}=2.9$. If the two-loop effective coupling constant $x(\rho)$ is adopted, the corresponding somewhat lower susceptibilities in Table 1 lead to expect less pronounced effects. But from Fig. 2 one realizes moreover that cancellations due to the oscillating density spoil the polarization contribution to (31) already at lower D. The instability is shifted to weaker critical fields E_c on the scale of Λ^2.

It is instructive to learn from Fig.3 which values of x give maximal contributions to the polarization integral in (31) at different external field strengthes (these are denoted as $x_{max}(D)$) and what the corresponding maximum value of the integrand ($\max_x f(x,D)$) is. This is shown for the one- and two-loop approximations for $x(\rho)$, respectively. Only those ranges in D are shown in either case, where strong cancellations due to the oscillating density not yet occur. Over these regions $x_{max}(D)$ moves from 5 to 10 with increasing D both for the one- and two-loop $x(\rho)$. Notice, however, the different range in D over which this happens in both variants. Comparing with Fig. 2 it becomes obvious that for those D which allow for the largest polarization (at the lower end of the instability regions in Fig. 2) also the maximum of the integrand, $\max_x f(x,D)$, attains largest values in Fig. 3, where these field strengths are marked by "D_M". The corresponding locations of the maxima $x_M = x_{max}(D_M)$ are however rather insensitive to the $x(\rho)$ used: $x_M \approx 6...7$.

This consideration shows why the occurrence of the phase transition is so strongly dependent on x_c. If the integration range over x from x_c to ∞ encloses $x_M \approx 6...7$, then such instabilities are met. The fact that we found x_M essentially independent on the dependence $x(\rho)$ adopted, suggests a rough scaling formula

$$\rho^2(x_M)\, D_M \approx .2 \tag{32}$$

for extrapolating our results to other functions $x(\rho)$. The reason why $\rho^2(x_M)$ provides a natural unit for D_M lies in the argument $\xi = \sqrt{2x(\rho)} \approx \rho^2 D$ of the oscillating part in (31).

We have seen that already the noninteracting instanton gas induces instabilities in the equation of state $D=D(E)$ such that the wanted first-order phase transition is conceivable. They occur only if the cut-off x_c is considerably lower compared with previous estimates, enclosing $x_{max} \approx 6...7$, where essential polarization effects come from. (In the case of SU(3), where we have been unable to do the group integral in (17), we expect $x_{max} \approx$ 8...9 .) Whether two-loop corrections to the tunneling amplitude indicated in (5) turn out to be relatively small even at corresponding large coupling, leaving the instanton gas picture intact, has not been investigated up to now as far as we know.

Now it remains to be seen how the inclusion of instanton interactions modifies the equation of state studied so far.

5. Effective field treatment of the instanton interaction

Inspired by Onsager's /11/ treatment of strongly dielectric media, in Ref. /1/ the effective field acting on a given (anti) instanton has been identified with

$$\tilde{H}^{eff} = \frac{2\mu}{1+\mu}\tilde{H} . \qquad (33)$$

(This is just the field inside a spherical cavity where $\mu_c = 1$, cut out of the continuum having arbitrary permeability μ.) This ansatz excludes from the beginning the possibility of spontaneous polarization of the instanton medium.

Another ansatz, popular in the mean-field theory of ferromagnetism /12/,

$$\tilde{H}^{eff} = \tilde{H} + \lambda 4\pi^2 \tilde{M} , \qquad (34)$$

should be considered as well. In the next section we compare the resulting equations of state for both forms of the effective field. A more thorough treatment of the partition function of the interacting instanton gas is beyond the scope of the present paper.

In the following we are going to specify the mean-field ansatz (34) based on a variational approximation to the logarithm of the partition function. We add an auxiliary term $\hat{\zeta}_\pm$ to the chemical potential in the free partition function and notice the inequality /13/

$$\ln Z[\tilde{H}, \zeta_\pm, V] \geq \ln Z_0[\tilde{H}, \zeta_\pm + \hat{\zeta}_\pm, V] - \langle S_{int} + \sum_\varepsilon \int n_\varepsilon \hat{\zeta}_\varepsilon \rangle_0 , \qquad (35)$$

where

$$\ln Z_0[\tilde{H}, \zeta_\pm + \hat{\zeta}_\pm, V] = \sum_\varepsilon \int_V d^4x \int \frac{d\rho}{\rho} \int dR \, n(\rho) \, e^{\zeta_\varepsilon + \hat{\zeta}_\varepsilon + 2\pi^2 D^{\varepsilon} \tilde{H}} , \qquad (36)$$

and $\langle\cdots\rangle_0$ denotes the average with respect to the corresponding ensemble, where $\hat{\xi}_\varepsilon$ simulates the interaction. Maximizing the r.h.s. of (35) under variation of $\hat{\xi}_\varepsilon$ yields the equation

$$\hat{\xi}_\varepsilon(x,\rho,R) = \sum_{\varepsilon'} 2\pi^2 \mathcal{D}^{\varepsilon a}_{\mu\nu} \int \mathcal{F}^{\varepsilon' a}_{\mu\nu}(x-x')\, n_0(\rho')\cdot e^{\xi_{\varepsilon'}+\hat{\xi}_{\varepsilon'}+2\pi^2 \mathcal{D}^{\varepsilon'}\tilde{H}}\, d^4x' \frac{d\rho'}{\rho'} dR', \tag{37}$$

and the following approximation to $\ln Z$,

$$\begin{aligned}\ln Z[\tilde{H},\xi_\pm,V] \approx \ln Z_0[\tilde{H},\xi_\pm+\hat{\xi}_\pm,V] \\ -\frac{1}{4}\sum_{\varepsilon_1\varepsilon_2}\int d1\, d2\, n_0(1)\, e^{\xi(1)+\hat{\xi}(1)+2\pi^2 \mathcal{D}_1\tilde{H}} \\ \cdot n_0(2)\, e^{\xi(2)+\hat{\xi}(2)+2\pi^2\mathcal{D}_2\tilde{H}} \\ \cdot \int \mathcal{F}(x-x_1)\,\mathcal{F}(x-x_2)\, d^4x .\end{aligned} \tag{38}$$

According to (12) we obtain by variation with respect to ξ_ε the average instanton density, where the dependence of $\hat{\xi}$ on ξ due to (37) must be taken into consideration:

$$\langle n_\varepsilon(x,\rho,R)\rangle = n_0(\rho)\exp\left(2\pi^2 \mathcal{D}^{\varepsilon a}_{\mu\nu}\tilde{H}^a_{\mu\nu} + \hat{\xi}_\varepsilon(\rho,R)\right)\Big|_{\xi_\pm=0}. \tag{39}$$

Comparing this with (37) we are lead to identify

$$\hat{\xi}_\varepsilon(x,\rho,R)\big|_{\xi_\pm=0} = 2\pi^2 \mathcal{D}^{\varepsilon a}_{\mu\nu}\, 2\pi^2 \tilde{M}^a_{\mu\nu}, \tag{40}$$

where M is the magnetization (17). Then we may write (39) as

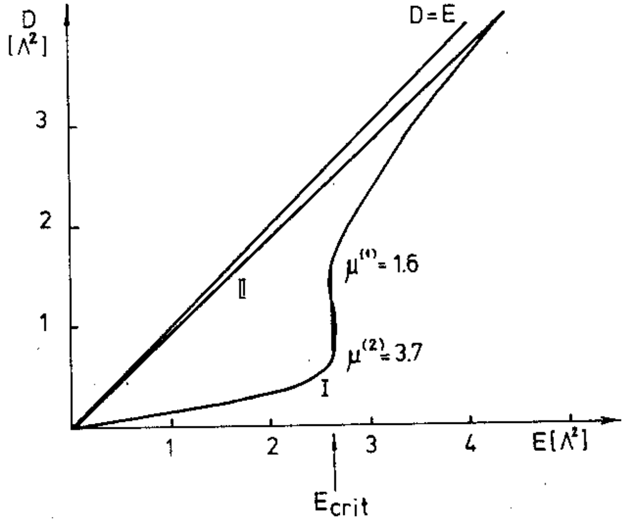
$$\langle n_\varepsilon(x,\rho,R)\rangle = n_0(\rho)\exp\left(2\pi^2 \mathcal{D}^{\varepsilon a}_{\mu\nu}(\tilde{H}^a_{\mu\nu}+2\pi^2\tilde{M}^a_{\mu\nu})\right), \tag{41}$$

with an effective field of the form (34) where $\lambda = \frac{1}{2}$.

Since we have considered only the dipole-dipole interaction (11) being representative for the leading contributions to the interaction term V_{int} in (7), we will not insist strongly on $\lambda = \frac{1}{2}$, but certainly $\lambda \leqslant 1$ is a reasonable choice to investigate for the resulting equation of state. Work is in progress to go beyond the mean-field approximation.

6. Instanton gas with interactions

Here we consider the equation of state only in the one-loop approximation for $x(\rho)$. We are mainly interested in how the inclusion of the instanton interaction influences the equation of state obtained from the free instanton gas.



<u>Fig</u>. 4: Equation of state for an effective field $D^{eff} = \frac{2\mu}{1+\mu} D$ ($E=\mu D$) in the one-loop approximation for $x(\rho)$. Curve I: $x_c = 0$, Curve II: $x_c=12$

First we consider an effective field (33) according to the point of view of Ref. /1/, $D^{eff} = \frac{2\mu}{1+\mu} D$, and construct (with

$$\xi = 4\pi^2 \rho^2 D^{eff}/g(\rho))$$

$$\begin{aligned} 4\pi^2 P &= (\mu-1) D = \frac{\mu^2-1}{2\mu} D^{eff} \\ &= 8\pi^2 \int \frac{d\rho}{\rho^5} u_0(\rho) \frac{\rho^2}{g(\rho)} \left(\frac{\sin\xi}{\xi^2} - \frac{\cos\xi}{\xi} \right). \end{aligned} \tag{42}$$

We define

$$\eta(D^{eff}) = \frac{\mu^2-1}{2\mu} = \frac{8\pi^2}{D^{eff}} \int \frac{d\rho}{\rho^5} u_0(\rho) \frac{\rho^2}{g(\rho)} \left(\frac{\sin\xi}{\xi^2} - \frac{\cos\xi}{\xi} \right) \tag{43}$$

and obtain from this

$$\mu(D^{eff}) = \eta + \sqrt{1+\eta^2},$$

and

$$D = \frac{1+\mu}{2\mu} D^{eff}, \qquad (44)$$

$$E = \frac{1+\mu}{2} D^{eff}.$$

In Fig.4 the corresponding equation of state is shown depending on x_c. Again large couplings x^{-1} have to be included in order to obtain a nontrivial behaviour. By comparison with Fig.2 we notice, that the inclusion of interactions via D^{eff} effects a shift of the instability to somewhat smaller E_c and raises the permeability of the stronger paramagnetic phase, making the transition more pronounced, which is induced already by the noninteracting gas.

Now we turn to the alternate, perhaps more likely form of the mean-field interaction (34), i.e. $D^{eff} = D + \lambda 4\pi^2 P$,the consequences of which we study for $\lambda \leq 1$. First we notice,that if $\lambda 4\pi^2 \chi > 1$, there arises the possibility of spontaneous polarization. From Table 1 can be seen that this is not improbable, provided x_c is taken small enough to allow for a nontrivial equation of state at all. Again we consider the polarization integral

$$4\pi^2 P = 8\pi^2 \int \frac{d\rho}{\rho} u_0(\rho) \frac{\rho^2}{g(\rho)} \left(\frac{\sin \xi}{\xi^2} - \frac{\cos \xi}{\xi} \right) \qquad (45)$$

with $\xi = 4\pi^2 \rho^2 D^{eff}/g(\rho)$ and calculate

$$D = D^{eff} - \lambda 4\pi^2 P(D^{eff}),$$
$$E = D^{eff} + (1-\lambda) 4\pi^2 P(D^{eff}). \qquad (46)$$

In Fig. 5 we show the behaviour of D(E) for $x_c = 12$ (trivial for all $\lambda \leq 1$) and $x_c=0$ for $\lambda = \frac{1}{4}$, $\frac{1}{2}$,1. For undercritical λ the equation of state becomes similar to the case considered before, with D^{eff} according to (33) (Onsager's ansatz, /1/). The other curves, including the favoured case $\lambda = \frac{1}{2}$, show a spontaneous (remnant) polarization at D=0 , but no phase transition of first

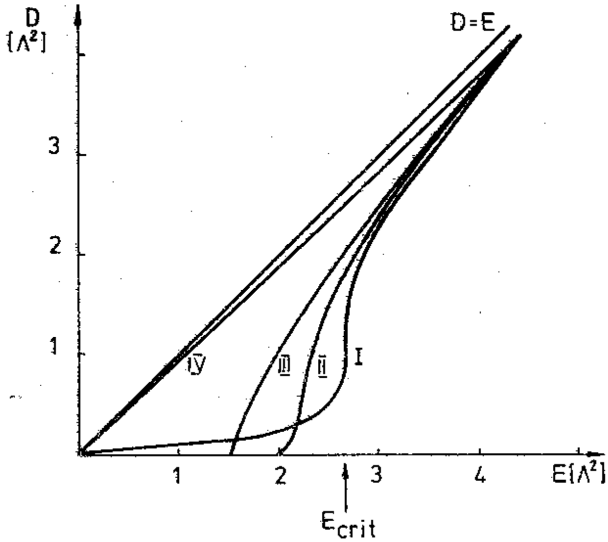


Fig. 5: Equation of state for an effective field $D^{eff} = D + \lambda 4\pi^2 P$ ($E = D + 4\pi^2 P$) in the one loop approximation for $x(\varrho)$.
Curve I: $x_c=0, \lambda = \frac{1}{4}$ Curve III: $x_c=0, \lambda=1$
Curve II: $x_c=0, \lambda = \frac{1}{2}$ Curve IV: $x_c=12, \lambda \leq 1$

order. From the point of view of deriving the coexistence of different phases this is not desirable.

7. Concluding remarks

Essentially, we have studied the influence of two corrections onto the phase transition scenario of Callan, Dashen, and Gross[1]. First, there is the numerical change in the instanton density. The result are considerably smaller values of the permeability μ, i.e., the externally perturbed instanton gas behaves not so strongly paramagnetic as estimated before. We did not consider the role of other nonperturbative field configurations, concentrating only on possible instabilities due to the instanton gas.

We have to emphasize the following. Whether a nontrivial equation of state is possible at all thanks to the presence of instantons, depends on whether the integral over instanton sizes encloses the region where the effective coupling constant $x^{-1} = g^2/8\pi^2$ reaches values of 0.1 ... 0.2 . Otherwise, due to the innocent correction factor 2^{-2N} /5b/ in the density, instantons could not mark itself qualitatively in the equation of state of the Minkowski vacuum !

Second, there is a change on principle. We have seen, that the strong instanton suppression in external electric fields, argued for and exploited in Ref./1/, must be replaced by an oscillating distribution. In order to study this exactly we had to restrict ourselves to SU(2) for technical reasons, i.e. our inability to do the group integrals in general. Nevertheless, in section 3 we have pointed out in principle, how the separation of that part of the free energy density related to the group orientation degrees of freedom of the instantons must be done, and why the authors of /1/ fail in this respect.

The oscillating behaviour of the instanton distribution becomes manifest in cancellations destroying the polarization response of the Euclidean instanton medium. Therefore we see the phase transition, even when considering the noninteracting instanton gas.

The x values contributing most to the polarization of the instanton gas, which we mentioned above, are almost insensitive to the dependence $x(\varrho)$ assumed.The corresponding instanton sizes ϱ provide the natural scale for the maximal external field, $D \propto \varrho^{-2}$, that can be reached in the strong paramagnetic phase.

Whether coupling constants as large as mentioned can be reconciled with the instanton gas picture, must be decided calculating higher loop corrections to the one instanton tunneling amplitude.

From the calculated permeabilities of the coexisting phases we see that a phase transition by the instanton mechanism will probably not be as precipitous as to enforce bag-like boundary conditions between the two phases.

Furthermore we have pointed out an ambiguity concerning the mean-field approximations for dealing with instanton interactions. Either they do not change essentially the behaviour of the noninteracting gas, or the possibility of spontaneous polarization appears, instead of the first order phase transition we looked for.

The problem of instanton interactions deserves thorough study. In this respect we did not pretend to any rigour here. But if there is really a first-order phase transition from bag to vacuum, we doubt that instantons will play a major role in this phenomenon.

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References

1. C.Callan, R.Dashen, D.Gross, Phys. Rev. D19, 1826 (1979); Phys. Letters 78B, 307 (1978).
2. A. Chodos, R.Jaffe, K.Johnson, C.Thorn, V.Weisskopf, Phys.Rev. D9,3471 (1974); P.Hasenfratz, J.Kuti, Phys.Reports 40C, 75 (1978).
3. A.Belavin, A.Polyakov, P.Schwartz, Y.Tyupkin, Phys.Letters 59B, 85 (1975).
4. C.Callan, R.Dashen, D.Gross, Phys.Rev. D17, 2717 (1978).
5. G. 't Hooft, a) Phys.Rev. D14, 3432 (1976), b) Phys.Rev. D18, 2199 (1978).
6. M.Ida, Dynamical properties of the vacuum in QCD, Kyoto preprint RIFP - 354 (1979).
7. C.Bernard, Gauge zero modes, instanton determinants, and QCD calculations, Los Angeles preprint UCLA 79/TEP/3 (1979).
8. A.A.Vladimirov, O.V.Tarasov, Yadernaya Fizika 25, 1104 (1977).
9. H.Levine, L.G.Yaffe, Phys,Rev. D19, 1225 (1979).
10. L.D.Landau, E.M.Lifshitz, Theoretical Physics VIII, Electrodynamics of continua, Moscow 1959 (in Russian).
11. L.Onsager, J.Amer.Chem.Soc. 58, 1486 (1936).
12. H.E.Stanley, Introduction to phase transition and critical phenomena, Oxford 1971.
13. Adapted from: R.F.Peierls, Phys.Rev. 54, 918 (1958).

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