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QUADRATIC BUNDLE AND NONLINEAR EQUATIONS

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Герджиков В.С., Иванов М.И. Кулиш П.П. **E2 - 12590**

Квадратичный пучок и нелинейные уравнения

Описан класс нелинейных эволюционных уравнений, решаемых методом обратной задачи рассеяния для квадратичного пучка:

$$L_\lambda \psi \equiv [i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \lambda\begin{pmatrix}0 & q(x)\\ p(x) & 0\end{pmatrix} - \lambda^2]\,\psi(x,\lambda) = 0.$$

Показано, что все уравнения этого класса являются вполне интегрируемыми гамильтоновыми системами и приведен явный вид переменных действие-угол. При $q = \epsilon p^*$, $\epsilon = \pm 1$ рассматриваемый класс содержит такие физически интересные уравнения, как модифицированное нелинейное уравнение Шредингера $(iq_t + q_{xx} - i\epsilon(q^2q^*)_x = 0)$, массивную модель Тирринга и другие

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Gerdjikov V.S., Ivanov M.I., Kulish P.P. **E2 - 12590**

Quadratic Bundle and Nonlinear Equations

A class of nonlinear evolution equations, solvable through the inverse scattering method for the quadratic bundle

$$L_\lambda \psi \equiv [i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \lambda\begin{pmatrix}0 & q(x)\\ p(x) & 0\end{pmatrix} - \lambda^2]\,\psi(x,\lambda) = 0$$

is described. It is shown, that all the equations from this class are completely integrable Hamiltonian systems; the corresponding "action-angle" variables are explicitly calculated. For $q = \epsilon p^*, \epsilon = \pm 1$ this class contains such physically interesting equations like the modified nonlinear Schrödinger equation $(iq_t + q_{xx} - i\epsilon(q^2q^*)_x = 0)$, the massive Thirring model and others.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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§1. INTRODUCTION

In a number of physical papers concerning nonlinear optics, plasma physics (see, e.g., ref.[1]), an interest has been displayed towards the equation

$$iq_t + q_{xx} - i\epsilon(q^2q^*)_x = 0, \quad q_x = \frac{\partial q(x,t)}{\partial x}, \quad \epsilon = \pm 1 \tag{1.1}$$

which resembles the nonlinear Schrödinger equation ($iq_t + q_{xx} - 2\epsilon|q^2|q = 0$) and is called the modified nonlinear Schrödinger equation. As it has been noted in ref.[2], eq. (1.1) can be solved through the inverse scattering method (ISM), (see the review papers[3,4]), applied to the following linear problem:

$$L_\lambda \psi \equiv [i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \lambda\begin{pmatrix}0 & q(x)\\ p(x) & 0\end{pmatrix} - \lambda^2]\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}(x,\lambda) = 0 \tag{1.2}$$

with the additional condition* $q = \epsilon p^*$. The scattering problem for the operator bundle (1.2) is directly connected with the one, used for the solution of the massive Thirring model[5]. The operator bundle (1.2) can be obtained from the operator bundles used in ref.[5] by a transition to characteristic coordinates and subsequent gauge transformation. We will give more details at the end of this paper.

Since the spectral parameter λ enters quadratically into (1.2), it is natural that it should be called a quadratic bundle. The aim of this paper is to give an exchaustive description of the nonlinear evolution equations (NLEE), connected with this quadratic bundle, and their Hamiltonian structure. Formula (3.8) gives the compact form of these equations.

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* In this paper we suppose that the functions $q(x), p(x)$ are smooth and vanish rapidly enough when $|x| \to \infty$.

Our account follows the general scheme of the ISM. In § 2 we give the basic formulae for the scattering problem (1.2). The next paragraph contains the spectral decomposition of the integro-differential operator Λ, such that its eigenfunctions are the squared solutions of eq. (1.2). This decomposition is actively used: i) for the derivation in a compact form of the NLEE, connected with L_λ and their conservation laws (§ 3); ii) for the construction of the Hamiltonian structure of these equations and the calculation of the action-angle variables (see also ref. /6/).

In conclusion, using the notions of Hamiltonian structure hierarchy /7/, and gauge transformations /8/, we show how to single out physically interesting NLEE.

The authors are grateful to Academician I.T.Todorov, E.Kh.Khristov and V.A.Mikhailov for useful discussions.

§ 2. THE SCATTERING PROBLEM

In this paragraph we give the main facts /2,5/ about the scattering problem (1.2) with smooth potential $q(x), p(x)$, vanishing fast enough when $|x|\to\infty$. The Jost solutions of (1.2) are uniquely defined by the relations

$$L_\lambda \psi^\pm(x,\lambda) = 0, \qquad L_\lambda \phi^\pm(x,\lambda) = 0,$$

$$\lim_{x\to\infty} \psi^+(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda^2 x}, \quad \lim_{x\to-\infty} \phi^+(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda^2 x}, \tag{2.1}$$

$$\lim_{x\to\infty} \psi^-(x,\lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\lambda^2 x}, \quad \lim_{x\to-\infty} \phi^-(x,\lambda) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{i\lambda^2 x}.$$

Both pairs of the Jost solutions $\{\psi^+, \psi^-\}$ and $\{\phi^+, \phi^-\}$ form fundamental systems of solutions and are linearly related to each other:

$$\psi^\pm(x,\lambda) = \mp a^\pm(\lambda)\phi^\mp(x,\lambda) + b^\mp(\lambda)\phi^\pm(x,\lambda). \tag{2.2}$$

The coefficient functions in (2.2) can be expressed through the Wronskians of the Jost solutions

$$a^\pm(\lambda) = W[\phi^\pm, \psi^\pm], \quad b^\pm(\lambda) = \mp W[\phi^\pm, \psi^\mp],$$

$$W[\phi,\psi] = \phi_1\psi_2 - \phi_2\psi_1$$

and satisfy the "unitarity" condition $a^+a^- + b^+b^- = 1$, $\mathrm{Im}\,\lambda^2 = 0$. The existence and the uniqueness of the Jost solutions are proved by using the corresponding integral equations; we write down the equation for $\psi^+(x,\lambda)$ only:

$$\psi^+(x,\lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\lambda^2 x} - i\lambda \int_x^\infty dy \begin{pmatrix} e^{-i\lambda^2(x-y)} & 0 \\ 0 & e^{i\lambda^2(x-y)} \end{pmatrix} \begin{pmatrix} 0 & q(y) \\ -p(y) & 0 \end{pmatrix} \psi^+(y,\lambda), \tag{2.3}$$

$$\mathrm{Im}\,\lambda^2 > 0.$$

From eq. (2.3) and its analogs, we can derive also the analytic properties of the Jost solutions and to calculate their asymptotics at $\lambda\to\infty$; the corresponding results are collected in the table.

Table

	Asymptotics for $\lambda\to\infty$	Domain of analyticity with respect to λ
$\psi^+ e^{-i\lambda^2 x}$	$e^{i\zeta^+}\begin{pmatrix} \frac{q(x)}{2\lambda} \\ 1 \end{pmatrix}[1+O(\frac{1}{\lambda^2})]$	
$\phi^+ e^{i\lambda^2 x}$	$e^{i\zeta^-}\begin{pmatrix} 1 \\ \frac{p(x)}{2\lambda} \end{pmatrix}[1+O(\frac{1}{\lambda^2})]$	$\mathrm{Im}\,\lambda^2 > 0$
a^+	$e^{i\zeta}[1+O(\frac{1}{\lambda^2})]$	
$\psi^- e^{i\lambda^2 x}$	$e^{-i\zeta^+}\begin{pmatrix} 1 \\ \frac{p(x)}{2\lambda} \end{pmatrix}[1+O(\frac{1}{\lambda^2})]$	
$\phi^- e^{-i\lambda^2 x}$	$-e^{-i\zeta^-}\begin{pmatrix} \frac{q(x)}{2\lambda} \\ 1 \end{pmatrix}[1+O(\frac{1}{\lambda^2})]$	$\mathrm{Im}\,\lambda^2 < 0$
a^-	$e^{-i\zeta}[1+O(\frac{1}{\lambda^2})]$	

$$\zeta^+ = \frac{1}{2}\int_x^\infty dy\, qp, \quad \zeta^- = \frac{1}{2}\int_{-\infty}^x dy\, qp, \quad \zeta = \zeta^+ + \zeta^-$$

The continuous spectrum of L_λ is double-valued and fills up the contour Γ (see fig. 1); the discrete spectrum $\Delta = \Delta^+ \cup \Delta^-$ is located at the zeroes of $a^{\mp}(\lambda)$:

$$\Delta^\pm \equiv \{\lambda_j^\pm : a^\pm(\lambda_j^\pm)=0, \quad \mathrm{Im}\,\lambda_j^\pm \gtrless 0, \quad j=1,\dots,N^\pm\}.$$

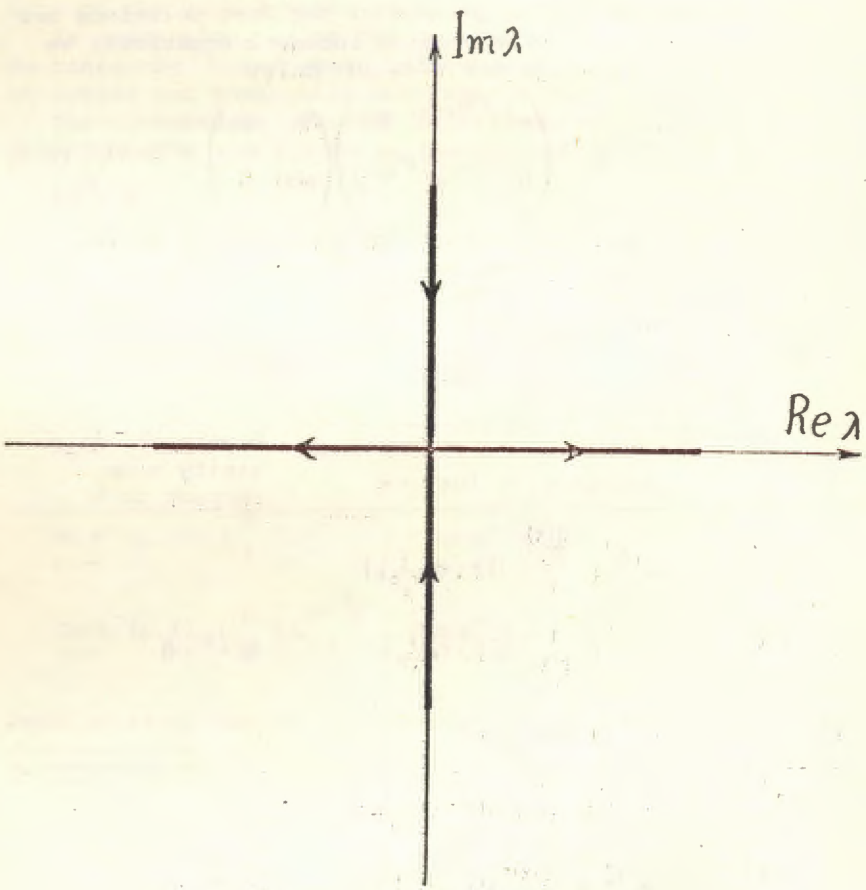

Fig. 1. Contur Γ.

For simplicity we suppose, that $a^\pm$ have a finite number $N^+=N^-=N$ of simple zeroes, located outside of Γ. Note, that from (1.2) it follows that

$$\psi^\pm(x,\lambda) = \mp\sigma_3\,\psi^\pm(x,-\lambda), \quad \phi^\pm(x,\lambda) = \pm\sigma_3\,\phi^\pm(x,-\lambda),$$

$$b^\pm(\lambda) = -b^\pm(-\lambda), \quad a^\pm(-\lambda) = a^\pm(\lambda). \tag{2.4}$$

If by $\lambda_j^+(\lambda_j^-), j=1,\dots,n$ we denote the eigenvalues in the first (second) quadrant of the spectral parameter λ plane, then from (2.4) we see, that $\lambda_{j+n}^\pm = -\lambda_j^\pm$, and $N=2n$. The set $T=T^+ \cup T^-$, where

$$T^\pm \equiv \{r^\pm(\lambda),\ \lambda \subset R_+ \cup i R_+,\ c_j^\pm,\ \lambda_j^\pm,\ j=1,\dots,n\}$$

$$r^\pm(\lambda) = \frac{b^\pm}{a^\pm}(\lambda), \quad c_j^\pm = \frac{b_j^\pm}{\dot a_j^\pm}, \quad \dot a_j^\pm = \frac{d a^\pm}{d\lambda}\Big|_{\lambda=\lambda_j^\pm},$$

$$b_j^\pm: \quad \phi^\pm(x,\lambda_j^\pm) = b_j^\pm\,\psi^\pm(x,\lambda_j^\pm), \quad j=1,\dots,n,$$

is called the scattering data for the problem (1.2). The coefficient functions $a^\pm, b^\pm$ are reconstructed from T using (2.4) and the dispersion relation

$$A(\lambda) = \frac{i}{2}\zeta - \frac{1}{2\pi i}\int_\Gamma \frac{d\mu\cdot\mu}{\mu^2-\lambda^2}\ln(1+r^+r^-(\mu)) + \sum_{j=1}^{n}\ln\frac{\lambda^2-\lambda_j^{+2}}{\lambda^2-\lambda_j^{-2}},$$

$$A(\lambda) = \pm\ln a^\pm(\lambda), \quad \mathrm{Im}\,\lambda^2 \gtrless 0; \quad A(\lambda) = \frac{1}{2}\ln\frac{a^+}{a^-}, \quad \mathrm{Im}\,\lambda^2 = 0. \tag{2.5}$$

The eigenfunctions of the operator bundle L_λ satisfy the completeness relation

$$\delta(x-y) = \frac{1}{2\pi}\int_\Gamma d\lambda\cdot\lambda\Big[\frac{\psi^+(x,\lambda)\,\phi^{+T}(y,\lambda)}{a^+(\lambda)} - \frac{\psi^-(x,\lambda)\,\phi^{-T}(y,\lambda)}{a^-(\lambda)}\Big]\sigma_1 -$$

$$-\sum_{j=1}^{N}\left[i\lambda_j^+ c_j^+ \psi_j^+(x)\psi_j^{+T}(y) + i\lambda_j^- c_j^- \psi_j^-(x)\psi_j^{-T}(y)\right]\sigma_1 ,$$

where

$$\psi_j^{\pm}(x)=\psi^{\pm}(x,\lambda_j^{\pm}).$$

For the solution of the inverse scattering problem we should introduce Volterra transformation operators

$$e^{i\zeta^+\sigma_3}\psi^{\pm}(x,\lambda)=e^{i\zeta_0^+\sigma_3}\psi_0^{\pm}(x,\lambda)+\int_x^{\infty}dy\,K(x,y)e^{i\sigma_3\zeta_0^+(y)}\psi_0^{\pm}(y,\lambda), \qquad (2.6)$$

where $K=K_1(x,y)+\lambda K_2(x,y)$ and $\psi_0^{\pm}(x,\lambda)$ are the Jost solution of the operator bundle (1.2) with potential $Q_0(x)=\begin{pmatrix} 0 & q_0 \\ p_0 & 0\end{pmatrix}$, $\zeta^+=\frac{1}{2}\int_x^{\infty}dy\,p\,q(y)$, $\zeta_0^+=\frac{1}{2}\int_x^{\infty}dy\,q_0p_0(y)$. Using the standard methods of the inverse scattering problem /3,4/ we can obtain the differential equations for K_1 and K_2 and the Gelfand-Levitan-Marchenko equation as well. But since into the definition (2.6) there enter the phase factors ζ_0^+, ζ^+ and into K there enters the spectral parameter λ, the corresponding equations for $q_0, p_0 \neq 0$ are very complicated. Since our aim is to analyze the general structure of the NLEE connected with L_λ, rather than the properties of their solutions, we will not write down these equations. The solution of the inverse scattering problem in the case $q_0=p_0=0$ and $q=\epsilon p^*$, is given in /2,5/.

§ 3. EXPANSIONS OVER THE "SQUARED" SOLUTIONS OF (1.2)

Using the method in /9/ we will derive the completeness relation for the "squared" solutions of (1.2). Let us introduce the systems of vector-functions:

$$\{\Psi\} \equiv \{\Psi^{\pm}(x,\lambda),\ \lambda \in \mathbf{R}_+ \cup i\,\mathbf{R}_+,\ \Psi_j^{\pm}(x),\ \dot{\Psi}_j^{\pm}(x),\ j=1,\dots,n\},$$

$$\{\Phi\} \equiv \{\Phi^{\pm}(x,\lambda),\ \lambda \in \mathbf{R}_+ \cup i\,\mathbf{R}_+,\ \Phi_j^{\pm}(x),\ \dot{\Phi}_j^{\pm}(x),\ j=1,\dots,n\},$$

where

$$\Psi^{\pm} = \psi^{\pm}\circ\psi^{\pm}(x,\lambda),\quad \Phi^{\pm}=\phi^{\pm}\circ\phi^{\pm}(x,\lambda),\quad \phi\circ\psi = \begin{pmatrix}\phi_1\psi_1\\ \phi_2\psi_2\end{pmatrix},$$

$\Psi_j^{\pm}(x) = \Psi^{\pm}(x,\lambda_j^{\pm})$ and so on. The elements of the system $\{\Psi\}$ are the eigenfunctions and adjoint functions of the integrodifferential operator Λ:

$$(\Lambda-\lambda^2)\Psi^{\pm}(x,\lambda)=0,\ (\Lambda-\lambda_j^{\pm 2})\Psi_j^{\pm}(x)=0,\ (\Lambda-\lambda_j^{\pm 2})\dot{\Psi}_j^{\pm}(x)=2\lambda_j^{\pm}\Psi_j^{\pm}(x), \qquad (3.1)$$

where

$$\Lambda=\frac{i}{2}(1+iI_+)\sigma_3\frac{d}{dx},\quad I_+=\begin{pmatrix} q\\ p\end{pmatrix}(x)\int_x^{\infty}dy\,(p,-q)(y). \qquad (3.2)$$

Analogically, the elements of $\{\Phi\}$ are the eigenfunctions and adjoint functions of the integro-differential operator Λ^+.

$$(\Lambda^+-\lambda^2)\Phi^{\pm}(x,\lambda)=0,\ (\Lambda^+-\lambda^{\pm 2})\Phi_j^{\pm}(x)=0,\ (\Lambda^+-\lambda_j^{\pm 2})\dot{\Phi}_j^{\pm}(x)=2\lambda_j^{\pm}\Phi_j^{\pm}(x),$$

$$\Lambda^+=\frac{i}{2}(1-iI_-)\sigma_3\frac{d}{dx},\quad I_-=\begin{pmatrix} q\\ p\end{pmatrix}(x)\int_{-\infty}^{x}dy\,(p,-q)(y).$$

The operator Λ^+ is adjoint to the operator Λ with respect to the scalar product $\left(u=\begin{pmatrix}u_1\\u_2\end{pmatrix},\ v=\begin{pmatrix}v_1\\v_2\end{pmatrix}\right)$

$$(u,v)=\int_{-\infty}^{\infty}dx\,\tilde{u}(x)\sigma_3\frac{d}{dx}v(x),\qquad \tilde{u}=(u_2,-u_1).$$

Let us introduce the analogue of the Green function for the operator Λ /9/:

$$G(x,y,\lambda)=\frac{2i}{a^2}\{\Psi(x,\lambda)\tilde{\Phi}(y,\lambda)\theta(x-y)+$$
$$+[2\phi\circ\psi(x,\lambda)\widetilde{\phi\circ\psi}(y,\lambda)-\Phi(x,\lambda)\tilde{\Psi}(y,\lambda)]\theta(y-x)\},$$

where $\tilde{\Phi}=(\Phi_2,-\Phi_1)=\Phi^T(-i\sigma_2)$. Here and in what follows formula of this type means $G=G^+$ when $\mathrm{Im}\lambda^2>0$ and $G=G^-$ when $\mathrm{Im}\lambda^2<0$; the corresponding expression for $G^+(G^-)$ is obtained by adding everywhere in the expression for G the supperscript + (-). From (3.1) and (1.2) it follows that G satisfies the equation

$$(\Lambda-\lambda^2)G^{\pm}(x,\ y,\lambda)=-(1+iI_+)\delta(x-y).$$

Note, that the integral operator $(1+iI_+)$ has an inverse $(1+iI_+)^{-1}=1-iI_+$ (see (3.2)).

The completeness relation for the system $\{\Psi\}$ is obtained by applying the method of contour integration for the function G. Let us consider the integral

$$J=\frac{1}{2\pi i}\oint_{\gamma_1 \cup \gamma_3} d\lambda\, \lambda G^{+}(x,y,\lambda)-\frac{1}{2\pi i}\oint_{\gamma_2 \cup \gamma_4} d\lambda\lambda\, G^{-}(x,y,\lambda),$$

where the contours $\gamma_i, i=1,\dots,4$ are given in fig. 2.

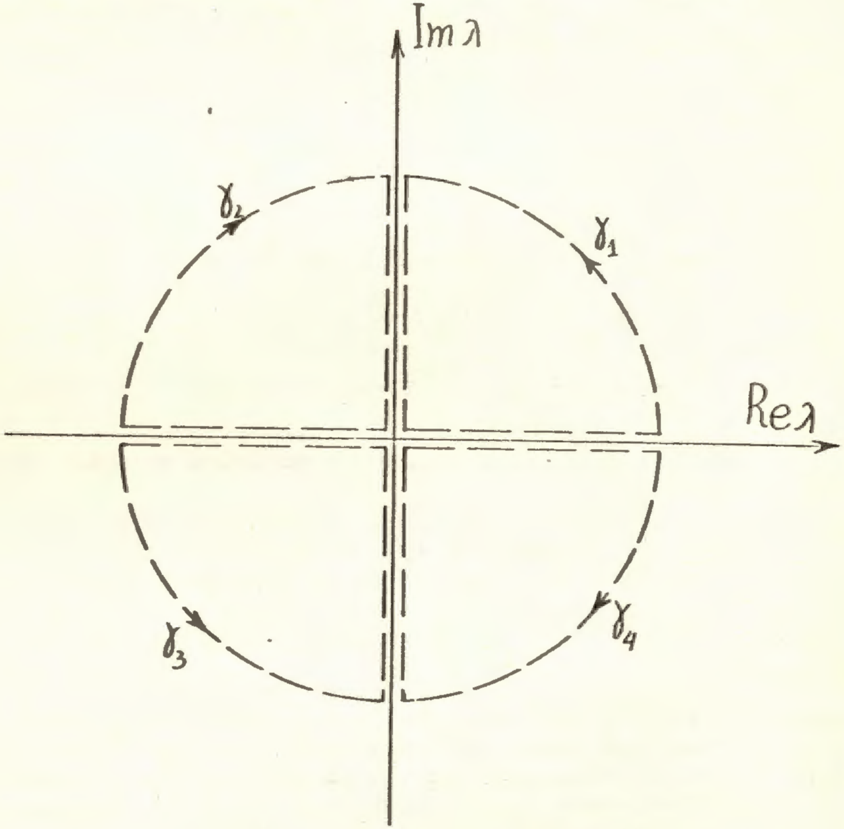

Fig. 2. Countours $\gamma_j, j=1,\dots,4$.

On the one hand, using the Caushy theorem, J equals the sum of the pole residues inside the contours γ_i. On the other hand, $J=J'+J''$, where J' is the integral along the contour Γ, and J''-along the arcs of the infinite circle. J'' can be calculated explicitly using the Jost solutions' asymptotics at $\lambda\to\infty$, given in the table. Equating both expressions for J after some rearrangements, we obtain:

$$(1+iI_+)\delta(x-y)=-\frac{1}{\pi}\int_{\Gamma} d\lambda\cdot\lambda\Big[\frac{\Psi^+(x)\tilde{\Phi}^+(y)}{(a^+)^2}-\frac{\Psi^-(x)\tilde{\Phi}(y)}{(a^-)^2}\Big]+4i\sum_{j=1}^{n}(X_j^+ + X_j^-),$$

$$X_j(x,y)=\frac{1}{(\dot{a}_j)^2}\Big\{\Big(1-\frac{\lambda_j\ddot{a}_j}{\dot{a}_j}\Big)\Psi_j(x)\tilde{\Phi}_j(y)+\lambda_j(\dot{\Psi}_j(x)\tilde{\Phi}_j(y)+\Psi_j(x)\dot{\tilde{\Phi}}_j(y))\Big\}.$$

From here it is possible to obtain the expansions of the vector-functions $w(x)=\binom{q}{p}(x)$ and $\sigma_3\delta w(x)$ over the system $\{\Psi\}$:

$$w(x)=\frac{i}{\pi}\int_{\Gamma} d\lambda\,(r^+\Psi^+ + r^-\Psi^-)(x,\lambda)+4\sum_{j=1}^{n}(c_j^+\Psi_j^+(x)-c_j^-\Psi_j^-(x)), \quad (3.3)$$

$$-(1+iI_+)\sigma_3\delta w(x)=\frac{i}{\pi}\int_{\Gamma} d\lambda\,(\delta r^+\Psi^+ - \delta r^-\Psi^-)(x,\lambda)+4\sum_{j=1}^{n}(Y_j^+ + Y_j^-), \quad (3.4)$$

$$Y_j(x)=\delta c_j\Psi_j(x)+c_j\delta\lambda_j\dot{\Psi}_j(x).$$

Here by $\delta w=\binom{\delta q}{\delta p}$ we have denoted the variation of the potential, and by $\delta r^{\pm}$, $\delta c_j^{\pm}$, $\delta\lambda_j^{\pm}$, the corresponding variations of the scattering data. The expansion coefficients in (3.3) and (3.4) can be calculated using the relations

$$\int_{-\infty}^{\infty} dx\,\tilde{\Phi}(x,\lambda)\,w(x)=\frac{i}{\lambda}\phi_1\phi_2(x,\lambda)\Big|_{x=-\infty}^{\infty},$$

$$\int_{-\infty}^{\infty} dx\,\tilde{\Phi}(x,\lambda)\,\sigma_3\delta w(x)=\frac{i}{\lambda}(\delta\phi_2\phi_1-\phi_2\delta\phi_1)(x,\lambda)\Big|_{x=-\infty}^{\infty}, \quad (3.5)$$

which follow from (1.2) and

$$[i\sigma_3 \frac{d}{dx} + \lambda Q - \lambda^2]\delta\psi + \lambda\delta Q \psi = 0,$$

and inserting into the right-hand sides of (3.5) the known asymptotics of the Jost solutions and their variations at $x \to \pm\infty$ (see (2.1) and (2.2)).

The expansions (3.3) and (3.4) enable us to give a complete description of the class of NLEE, connected with the operator bundle L_λ (1.2). Really, if in (3.4) we limit ourselves to variations of the type:

$$\delta w = w(x,t+\delta t) - w(x,t) = \frac{\partial w}{\partial t}(x,t)\delta t + O((\delta t)^2),$$

where t is an external parameter, we get

$$-(1+iI_+)\sigma_3 \frac{\partial w}{\partial t} = \frac{i}{\pi} \int_\Gamma d\lambda \, (r_t^+ \Psi^+ - r_t^- \Psi^-)(x,\lambda) + 4 \sum_{j=1}^{n} (Z_j^+ + Z_j^-),$$

$$Z_j(x,t) = \frac{dc_j}{dt} \Psi_j(x,t) + c_j \frac{d\lambda_j}{dt} \dot{\Psi}_j(x,t). \qquad (3.6)$$

Comparing the expansions (3.3) and (3.6) and using (3.1), we obtain that the following statement holds:

Let $q(x,t)$ and $p(x,t)$, entering in the potential of the operator bundle (1.2) and the meromorphic functions $f(z)$ and $g(z)$ be such, that the integrals

$$\int_\Gamma d\lambda \, f(\lambda^2) \frac{\partial r^\pm(\lambda,t)}{\partial t}, \quad \int_\Gamma d\lambda \, g(\lambda^2) r^\pm(\lambda,t) \qquad (3.7)$$

are absolutely convergent for all t. Then, if q and p satisfy the NLEE:

$$f(\Lambda)(1+iI_+)\sigma_3 \frac{\partial w}{\partial t} + g(\Lambda) w = 0, \qquad (3.8)$$

the corresponding scattering data satisfy the linear equations*:

* Here we limit ourselves to the functions $f(z)$ and $g(z)$, which are not singular and do not vanish in a neighbourhood of the spectrum of L_λ; otherwise the convergence of the integrals (3.7) and eqs. (3.9) may impose restrictions on the scattering data T (see also foot-note on page 20).

$$f(\lambda^2) \frac{\partial r^\pm(\lambda,t)}{\partial t} = \pm g(\lambda^2) r^\pm(\lambda,t),$$

$$f(s_j^\pm) \frac{dc_j^\pm(t)}{dt} = \pm g(s_j^\pm) c_j^\pm(t), \quad \frac{ds_j^\pm}{dt} = 0, \quad s_j^\pm = \lambda_j^{\pm 2}. \qquad (3.9)$$

The inverse statement also holds. Namely, let two functions $f(z)$ and $g(z)$ and an operator-bundle (1.2) with the scattering data $T(t)$ be such, that the integrals (3.7) are absolutely convergent for all t. Then, if the scattering data $T(t)$ satisfy the linear relations (3.9), the corresponding potentials $q(x,t)$ and $p(x,t)$ of the bundle (1.2) will satisfy the NLEE (3.8).

Let us now discuss the conservation laws of the NLEE (3.8). From (3.9) and (2.5) it follows that $\frac{dA(\lambda,t)}{dt} = 0$. Thus, the function $A(\lambda)$ can be considered as a generating functional of the conservation laws of the NLEE (3.8). In the appendix we will prove that $\exp(A(\lambda))$ is the regularized determinant of the operator-bundle (1.2). As conserved quantities, we can choose, e.g., the expansion coefficients of $A(\lambda)$ over the powers of λ^2 (or over the inverse powers of λ^2):

$$A(\lambda) = \frac{i\zeta}{2} + \sum_{m=1}^{\infty} \frac{C_m}{\lambda^{2m}} = \sum_{m=0}^{\infty} C_{-m} \lambda^{2m}. \qquad (3.10)$$

Using the dispersion relation (2.5), we easily express C_m in terms of the scattering data T:

$$C_m = -\frac{|m|}{m} \{ \frac{i}{2\pi} \int_\Gamma d\mu \, \mu^{2m-1} \ln(1 + r^+ r^-(\mu)) + \sum_{j=1}^{n} \frac{1}{m} (s_j^{+m} - s_j^{-m}) \}, \qquad (3.11)$$

where $m = \pm 1, \pm 2, \ldots$. It is possible to obtain also the recurrent relations [2], which allow us to express C_m as functionals of q and p. In particular, it is clear from these relations that all the densities of C_m with $m > 0$ depend locally on q, p and their derivatives with respect to x. Analogically as in ref. [6] (see the appendix) we can obtain a compact expression for C_m of the type:

$$C_m = \frac{1}{2|m|}\left\{ i\int_{-\infty}^{\infty} dx\,(p,q)\Lambda^m \begin{pmatrix} q \\ p \end{pmatrix} + 4\int_{-\infty}^{\infty} dx \int_x^{\infty} dy\,(p,-q)\Lambda^{m+1}\begin{pmatrix} q \\ p \end{pmatrix}\right\}, \qquad (3.12)$$

where the operator Λ is defined in (3.2). Let us write down the first four C_m for $m=\pm 1, \pm 2$:

$$C_2 = \frac{1}{8}\int_{-\infty}^{\infty} dx\,[\, i(q_{xx}p + qp_{xx}) - i(qp)^3 + \frac{3}{2}qp(q_x p - qp_x)\,],$$

$$C_1 = \frac{1}{4}\int_{-\infty}^{\infty} dx\,[q_x p - qp_x - i(qp)^2],$$

$$C_{-1} = \int_{-\infty}^{\infty} dx\,(\tilde q \tilde p_x - \tilde p \tilde q_x),$$

$$C_{-2} = \int_{-\infty}^{\infty} dx\,[4i\tilde q\tilde p - (\tilde q_x p - \tilde q \tilde p_x)\tilde q \tilde p],$$

where $\tilde q(x) = \int_x^{\infty} dy\, q(y)$, $\tilde p(x) = \int_x^{\infty} dy\, p(y)$. These expressions are in agreement with the answers obtained in ref. /2,5/. Let us write down also the relation (see the appendix):

$$\delta C_m = -\frac{i}{2}\,\frac{m}{|m|}\int_{-\infty}^{\infty} dx\,(\delta p, \delta q)\Lambda^m \begin{pmatrix} q \\ p \end{pmatrix}, \qquad (3.13)$$

which we will need in deriving the Hamiltonian structure of the NLEE (3.8).

§ 4. HAMILTONIAN STRUCTURE AND COMPLETE INTEGRABILITY

In this paragraph we show that all NLEE of the type (3.8) are completely integrable Hamiltonian systems and explicitly calculate the corresponsing "action-angle" variables.

Let us choose a Hamiltonian and a 2-form of the type:

$$H_g = \sum_{m\neq 0} h_m C_m = i\int_{-\infty}^{\infty} dx\,(p,q)\, g_1(\Lambda)\begin{pmatrix} q \\ p \end{pmatrix} + 4\int_{-\infty}^{\infty} dx \int_x^{\infty} dy\,(p,-q)\Lambda g_1(\Lambda)\begin{pmatrix} q \\ p \end{pmatrix}, \qquad (4.1)$$

$$\Omega_f = \frac{-i}{2}\int_{-\infty}^{\infty} dx\,(\delta p, \delta q)\wedge f(\Lambda)(1 + iI_+)\begin{pmatrix} \delta q \\ -\delta p \end{pmatrix}, \qquad (4.2)$$

where

$$g(z) = \sum_{m\neq 0} h_m \frac{|m|}{2m} z^m, \qquad g_1(z) = \int^z \frac{dz'}{z'} g(z'). \qquad (4.3)$$

Among the forms Ω_f, the most simple in a certain sense is /2/:

$$\Omega_{-1} = \frac{1}{2}\int_{-\infty}^{\infty} dx\,(\delta p, \delta q)\wedge \int_x^{\infty} dy \begin{pmatrix} \delta q \\ \delta p \end{pmatrix}(y)$$

with $f(z) = \frac{1}{z}$. The form Ω_{-1} is closed, non-degenerate and skew-symmetric. The same properties for Ω_f can be proved after writing Ω_f in terms of the "action-angle" variables (see below), which allows us to claim that Ω_f are symplectic 2-forms. The construction of the manifold of symplectic forms Ω_f, connected with the operator bundle L_λ (1.2) is analogous to the one, described in ref. /7/.

Let us consider the Hamiltonian equations of motion:

$$\Omega_f\left(\sigma_3 \frac{dw}{dt}, \cdot\,\cdot\right) = \delta H_g .$$

Using (3.13), (4.1) and (4.2) we see, that these equations coincide with the NLEE (3.8).

Now, let us express Ω_f and H_g through the scattering data. In Ω_f we insert the expansion (3.4) for $(1+iI_+)\sigma_3\delta w$ into (4.2). Then, using (3.1) it is easy to calculate the action of the operator $f(\Lambda)$ on $(1+iI_+)\sigma_3\delta w$, after which one should express integrals of the type $\int_{-\infty}^{\infty} dx\,(\delta p, \delta q)\Psi(x,\lambda)$ in terms of the scattering data. This is done by using relations, analogous to (3.5). After some algebraic transformations, we obtain

$$\Omega_f = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{ds}{s} f(s)\,\delta \ln\frac{b^+}{b^-}\wedge \delta\ln(1 + r^+ r^-) +$$

$$+ 4\sum_{j=1}^{n}[\, f(s_j^+)\delta\ln b_j^+ \wedge \delta\ln\lambda_j^+ + f(s_j^-)\delta\ln b_j^- \wedge \delta\ln\lambda_j^-\,] = \qquad (4.4)$$

$$= \int_{-\infty}^{\infty} ds\, \delta P(\lambda) \wedge \delta Q(\lambda) + \sum_{j=1}^{n} [\delta P_j^+ \wedge \delta Q_j^+ + \delta P_j^- \wedge \delta Q_j^-],$$

where $s = \lambda^2$, $s_j^{\pm} = \lambda_j^{\pm}$, and P and Q we may choose, for example, in the form:

$$P(\lambda) = \frac{f(s)}{\pi s} \ln(1 + r^+ r^-), \qquad Q(\lambda) = -\frac{i}{2} \ln \frac{b^+}{b^-}, \qquad -\infty < s < \infty$$

$$P_j^{\pm} = \mp 4 i f_1(s_j^{\pm}), \qquad Q_j^{\pm} = \mp i \ln b_j^{\pm}, \tag{4.5}$$

$$f_1(z) = \int^{z} \frac{dz'}{2z'} f(z').$$

The calculations for H_g may be performed using (4.1) and (3.11)

$$H_g = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{ds}{s} g(s) \ln(1 + r^+ r^-) - 2 \sum_{j=1}^{n} [g_1(s_j^+) - g_1(s_j^-)]. \tag{4.6}$$

The Hamiltonian H_g depends only on the half of the canonical variables (of action-type) $\{P(\lambda), P_j^{\pm}\}$. The systems with such Hamiltonians are completely integrable; and the canonically conjugated momenta and coordinates $\{P(\lambda), P_j^+, Q(\lambda), Q_j^{\pm}\}$ are known as the "action-angle" variables. The equations of motion, corresponding to the 2-form Ω_f (4.4) and the Hamiltonian H_g (4.6) have the form

$$\frac{dP(\lambda,t)}{dt} = 0, \qquad f(\lambda^2) \frac{dQ(\lambda,t)}{dt} = -ig(\lambda^2),$$

$$\frac{dP_j^{\pm}}{dt} = 0, \qquad f(s_j^{\pm}) \frac{dQ_j^{\pm}}{dt} = -ig(s_j^{\pm}).$$

It is not difficult to see, that this system is equivalent to (3.9).

Let us consider three important examples.

1. The modified nonlinear Schrödinger equation (MNLS):

$$iq_t + q_{xx} - i\epsilon (q^2 q^*)_x = 0, \qquad \epsilon = \pm 1 \tag{4.7}$$

is obtained from (3.8) with:

$$f(z) = \frac{\epsilon}{z}, \qquad g(z) = 4i\epsilon z, \qquad q = \epsilon p^*. \tag{4.8}$$

The last condition $q = \epsilon p^*$ puts restrictions on the scattering data, namely.

$$a^-(\lambda) = a^{+*}(\lambda^*), \qquad b^-(\lambda) = -\epsilon b^{+*}(\lambda^*),$$
$$\lambda_j^+ = \lambda_j^{-*}, \qquad b_j^- = \epsilon b_j^{+*}. \tag{4.9}$$

Inserting (4.8) into (4.1) and (4.2), we get

$$\Omega_{MNLS} = \Omega_{-1}|_{q=\epsilon p^*} = + \int_{-\infty}^{\infty} dx [\delta q \wedge \int_x^{\infty} dy\, \delta q^*(y) + \delta q^* \wedge \int_x^{\infty} dy\, \delta q(y)],$$
$$H_{MNLS} = 4i\epsilon C_1|_{q=\epsilon p^*} = \int_{-\infty}^{\infty} dx [i(q q_x^* - q_x q^*) + \epsilon |q^4|], \tag{4.10}$$

which generate the MNLS equation (4.7) (see [2]). The expressions for Ω_{MNLS} and H_{MNLS} in terms of the scattering data are obtained by inserting (4.8) into (4.4)-(4.6). After some simplifications for the action-angle variables of (4.7), we get:

$$\tilde{P}(\lambda) = -\frac{2}{\pi} \frac{\epsilon \ln|a^+|}{s^2}, \qquad \tilde{Q}(\lambda) = \arg b^+, \qquad -\infty < s < \infty,$$

$$\tilde{P}_{1j} = \frac{4\epsilon \cos 2\alpha_j}{|s_j^+|}, \qquad \tilde{Q}_{1j} = \ln|b_j^+|, \tag{4.11}$$

$$\tilde{P}_{2j} = \frac{4\epsilon \sin 2\alpha_j}{|s_j^+|}, \qquad \tilde{Q}_{2j} = \arg b_j^+,$$

where $\alpha_j = \arg \lambda_j^+$, $0 < \alpha_j < \pi/2$, $s_j^+ = \lambda_j^{+2}$, and for the Hamiltonian:

$$H_{MNLS} = 4\{\int_{-\infty}^{\infty} ds\cdot s^2\,\tilde{P}(s) + 16\sum_{j=1}^{n} \frac{\tilde{P}_{2j}}{\tilde{P}_{1j}^2 + \tilde{P}_{2j}^2}\}. \qquad (4.12)$$

Thus, the complete integrability of the MNLS equation is evident.

The next equation in this series

$$q_t = [q_{xx} - 3i\epsilon\,|q^2|\,q_x - \frac{3}{2}|q^4|\,q]_x \qquad (4.13)$$

is obtained from (3.8) with $f(z) = \frac{\epsilon}{z}$, $g(z) = -8\epsilon z^2$, and $q = \epsilon p^*$.

Equation (4.13) is generated by the same 2-form, as the MNLS equation, and therefore, has the same action-angle variables (4.11). The Hamiltonian of (4.13) equals:

$$H = \int_{-\infty}^{\infty} dx\,\{q_{xx}q^* + q^*_{xx}q - |q^6| - \frac{3i\epsilon}{2}(q_x q^* - q^*_x q)|q^2|\} =$$

$$= -8\epsilon\{\epsilon\int_{-\infty}^{\infty} ds\cdot s^3\tilde{P}(s) + 64\sum_{j=1}^{n}\frac{\tilde{P}_{1j}\tilde{P}_{2j}}{(\tilde{P}_{1j}^2 + \tilde{P}_{2j}^2)^2}\}.$$

2. The massive Thirring model (MTM). The MTM and the associated with it operator bundle $\tilde{L}_\lambda$ in characteristic coordinates has the form [5] $(m=2)$:

$$\begin{aligned} -iu_{1t} + 2u_2 + 2u_1|u_2^2| &= 0, \\ -iu_{2x} + 2u_1 + 2u_2|u_1^2| &= 0, \end{aligned} \qquad (4.14)$$

$$\tilde{L}_\lambda\tilde{\psi} \equiv [\,i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + 2|u_1^2| + \lambda\begin{pmatrix}0 & u_1^*\\ -u_1 & 0\end{pmatrix} -$$

$$- \lambda^2\,]\,\tilde{\psi} = 0. \qquad (4.15)$$

The operator bundle $\tilde{L}_\lambda$ is unitarily equivalent to the operator bundle L_λ (1.2) with $q = -p^* = u_1^* \exp(2i\int_x^\infty dy\,|u_1^2|)$, i.e., the classes of NLEE associated with the operator bundles L_λ and $\tilde{L}_\lambda$ are gauge equivalent [8].

The MTM is obtained from (3.8) with (see foot-note* on page 20)

$$f(z) = \frac{1}{2}, \quad g(z) = \frac{i}{z}, \quad q = -p^*. \qquad (4.16)$$

Since $\Lambda^{-1}w = 2i\int_x^\infty dy\,\sigma_3 w(y)$ from (3.8) and (4.16) we get

$$q_t - 4\tilde{q} - 4iq\,|\tilde{q}^2| = 0, \quad \tilde{q} = \int_x^\infty dy\,q(y), \qquad (4.17)$$

and, in particular, $\int_x^\infty dy\,|q^2|_t = 4|\tilde{q}^2|$. Let us introduce now

$$u_1 = \frac{1}{2}q^* e^{+2i\phi}, \quad \phi = \int_x^\infty dy\,|u_1^2|, \qquad (4.18)$$

$$u_2 = 2e^{2i\phi(x)}\int_x^\infty dy\,u_1(y)e^{-2i\phi(y)}. \qquad (4.19)$$

Then, equation (4.17) goes into the first of the equations (4.14); the second equation of the MTM (4.14) is obtained by differentiating (4.19) with respect to x. The corresponding 2-form and Hamiltonian equal

$$\Omega_{MMT} = +2i\int_{-\infty}^{\infty} dx\,\delta u_1^* \wedge \delta u_1 = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{ds}{s}\,\delta\arg b^+(\sqrt{s}) \wedge \delta\ln|a^+(\sqrt{s})| +$$

$$+ 4\sum_{j=1}^{n}[\,\delta\ln|b_j^+| \wedge \delta\ln|\lambda_j^+| - \delta\beta_j^+ \wedge \delta\alpha_j^+\,],$$

$$H_{MMT} = 2\int_{-\infty}^{\infty} dx\,(u_1u_2^* + u_2u_1^*) = \qquad (4.20)$$

$$= -\frac{2}{\pi}\int_{-\infty}^{\infty}\frac{ds}{s^2}\ln|a^+| + 4\sum_{j=1}^{n}\frac{\sin 2\alpha_j}{|s_j^+|},$$

where $\beta_j^+ = \arg b_j^+$. The expressions for Ω_{MMT} and H_{MMT} in terms of u_1 and u_2 together with the requirement $u_2 \to 0$, $x \to \infty$ generate eq. (4.14). The action-angle variables and the dependence of H on the "action" variables, which can be easily obtained from (4.20), are in agreement with the results of Kuznetzov and Mikhailov /5/.

3. Let us note one more relativistically-invariant model, found by V.A.Mikhailov:

$$h_{1xt} - 2ih_1h_2h_{1x} + m^2h_1 = 0,$$
$$h_{2xt} + 2ih_1h_2h_{2x} + m^2h_2 = 0. \tag{4.21}$$

The system of equations (4.21) can be obtained from (3.8), if we put *

$$f = -\frac{m^2}{2z}, \qquad g = \frac{im^4}{4z^2},$$
$$q = -i\frac{\sqrt{2}}{m}h_{1x}, \quad p = -i\frac{\sqrt{2}}{m}h_{2x}. \tag{4.22}$$

Then the Hamiltonian and the 2-form equal:

$$H_{MM} = -\frac{im^4}{4}C_{-2} = \int_{-\infty}^{\infty} dx\,[\,ih_1h_2(h_{2x}h_1 - h_2h_{1x}) + m^2h_1h_2\,],$$
$$\Omega_{MM} = -\int_{-\infty}^{\infty} dx\,[\delta h_{1x}\wedge\delta h_2 + \delta h_{2x}\wedge\delta h_1\,]. \tag{4.23}$$

For $m=2$ and after the reduction $h_1=h_2^*=h$ (4.21) goes into (4.17), and the model becomes gauge equivalent to the MTM. But, as it is seen from (4.23), the Hamiltonian H_{MM} and the 2-form Ω_{MM} do not coincide with H_{MTM} and Ω_{MTM}. This model illustrates how a given NLEE can be represented as a Hamiltonian system by different choices of H_g and Ω_f /7/; it is characteristic that for all such choices the ratio $g(z)/f(z)$ is one and the same ($2/iz$ in our case)**.

* With the choice (4.16) for $g(z)$ condition (3.7) holds if for some $\delta>0$: $\lambda^{-1-\delta}r^{\pm}(\lambda)\xrightarrow[\lambda\to 0]{}\mathrm{const}$. The choice (4.21) requires $\lambda^{-8-\delta}r^{\pm}(\lambda)\xrightarrow[\lambda\to 0]{}\mathrm{const}$.

** By a change of variables $t=cr$ in the relation (3.9) we always can go from $g(z)/f(z)$ to $cg(z)/f(z)$, $c=\mathrm{const}$.

The action-angle variables, obtained from Ω_{MM} (4.21) with $h_1=h_2^*$ (it corresponds to $q=-p^*$), are given by formula (4.12). At last the relation

$$H_{MM} = \frac{m^4}{4}\{-\int_{-\infty}^{\infty}\frac{ds}{s}P(s) + \frac{1}{4}\sum_{j=1}^{n}\tilde{P}_{1j}\tilde{P}_{2j}\}$$

gives the explicit dependence of the Hamiltonian H_{MM} on the action-type variables (4.11).

APPENDIX

Here we show that $\exp(A(\lambda))$ is the functional determinant of the operator bundle L_λ and derive formulae (3.12) and (3.13).

By definition the regularized functional determinant of the operator bundle L_λ is given by $\exp[\mathrm{Tr}\ln(L_\lambda L_{0\lambda}^{-1})]$, where $L_{0\lambda}$ is a quadratic bundle (1.2) with $q=p=0$. Using the expression for the Green function of (1.2)

$$R(x,y,\lambda) = \frac{i}{a}[\psi(x,\lambda)\phi^T(y,\lambda)\theta(x-y) + \phi(x,\lambda)\psi^T(y,\lambda)\theta(y-x)]\sigma_1 \tag{A.1}$$
$$(L_\lambda R = \delta(x-y)),$$

we get for $\mathrm{Im}\lambda^2>0$

$$\frac{d}{d\lambda}\mathrm{Tr}\ln L_\lambda L_{0\lambda}^{-1} = \mathrm{Tr}[-2\lambda(L_\lambda^{-1}-L_{0\lambda}^{-1}) + Q(x)L_\lambda^{-1}] =$$
$$= -2i\lambda\int_{-\infty}^{\infty}dx\left(\frac{\psi_1^+\phi_2^+ + \psi_2^+\phi_1^+}{a^+}(x,\lambda)-1\right) + i\int_{-\infty}^{\infty}dx\,(p,q)\frac{\phi^+\circ\psi^+}{a^+}(x,\lambda). \tag{A.2}$$

Using the Wronskian relation

$$i\frac{dW[\dot{\phi},\psi](x,\lambda)}{dx} = -2\lambda(\phi_1\psi_2+\phi_2\psi_1) + (p,q)\phi\circ\psi(x,\lambda), \tag{A.3}$$

we get that the r.h.s. of (A.2) equals $\frac{dA}{d\lambda}$ for $\mathrm{Im}\lambda^2>0$. Analogical considerations for $\mathrm{Im}\lambda^2<0$ and $\mathrm{Im}\lambda^2=0$ allow us to show, that

$$\mathrm{Tr}\ln L_\lambda L_{0\lambda}^{-1} = A(\lambda).$$

Using once more (A.3), for the r.h.s. of (A.2), we get

$$\frac{dA}{d\lambda} = i\int_{-\infty}^{\infty} dx(p,q)\,B(x,\lambda) + 4\lambda^2\int_{-\infty}^{\infty}dx\int_{x}^{\infty}dy(p,-q)\,B(y,\lambda) \tag{A.4}$$

with

$$B(x,\lambda)=\pm\frac{\phi^{\pm}\circ\psi^{\pm}}{a^{\pm}}(x,\lambda),\qquad \mathrm{Im}\,\lambda^2 \gtrless 0,$$

and $B(x,\lambda)=\frac{\phi^{+}\circ\psi^{+}}{a^{+}}(x,\lambda)-\frac{\phi^{-}\circ\psi^{-}}{a^{-}}(x,\lambda)$, when $\mathrm{Im}\lambda^2=0$. In order to get (3.12) it is enought to show, that

$$B(x,\lambda)=\frac{\lambda}{2}(\Lambda-\lambda^2)^{-1}\binom{q}{p}. \tag{A.5}$$

The relation (A.5) can be obtained by the contour integration method, applied to the integral

$$J_1=\frac{1}{2\pi i}\oint_{\gamma_1\cup\gamma_3}\frac{d\mu\cdot\lambda}{\mu^2-\lambda^2}\,\frac{\phi^{+}\circ\psi^{+}}{a^{+}}(x,\lambda)-\frac{1}{2\pi i}\oint_{\gamma_2\cup\gamma_4}\frac{d\mu\cdot\lambda}{\mu^2-\lambda^2}\,\frac{\phi^{-}\circ\psi^{-}}{a^{-}}(x,\lambda).$$

Equating the values of J_1 ,obtained by the Caushy theorem with the value of J_1 obtained by direct integration along the contour, we get for $\mathrm{Im}\lambda^2>0$

$$\frac{\phi^{+}\circ\psi^{+}}{a^{+}}(x,\lambda)=\frac{\lambda}{2}\Big\{\frac{i}{\pi}\int_{\Gamma}\frac{d\mu}{\mu^2-\lambda^2}\,(r^{+}\Psi^{+}+r^{-}\Psi^{-})(x,\mu)+$$

$$+4\sum_{j=1}^{n}\Big[\frac{c_j^{+}}{\lambda_j^{+2}-\lambda^2}\Psi_j^{+}(x)-\frac{c_j^{-}}{\lambda_j^{-2}-\lambda^2}\Psi_j^{-}(x)\Big]\Big\}=\frac{\lambda}{2}(\Lambda-\lambda^2)^{-1}\binom{q}{p}(x).$$

Here we have used the expansion (3.3) for $w=\binom{q}{p}$ and the property (3.1) of the operator Λ. The proof goes analogically for $\mathrm{Im}\lambda^2<0$ and $\mathrm{Im}\lambda^2=0$. Thus, relation (A.5), and therefore (3.12), are proved.

Relation (3.13) follows directly from the relation

$$\delta A(\lambda)=i\lambda\int_{-\infty}^{\infty}dx(\delta p,\delta q)\,B(x,\lambda)$$

and the formula (A.5).

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