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Submitted to ЯФ

Скачков Н.Б., Соловцов И.Л. **E2 - 12525**

Описание спектра масс и траекторий Редже мезонов на основе релятивистского двухчастичного квазипотенциального уравнения

Трехмерное релятивистское двухчастичное квазипотенциальное уравнение, записанное в релятивистском конфигурационном представлении, применено к описанию спектра масс и траекторий Редже мезонов. Квазипотенциальное уравнение решается методом ВКБ. Найдено релятивистское модифицированное ВКБ условие квантования, позволяющее простым образом исследовать траектории Редже мезонов. Получены формулы для лептонных ширин распадов, проведено сравнение с экспериментом.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1979

Skachkov N.B., Solovtsov I.L. **E2 - 12525**

A Description of the Mass Spectrum and Regge Trajectories for Mesons on the Basis of the Relativistic Two-Particle Quasipotential Equation

A three-dimensional relativistic two-particle equation in the relativistic configurational representation (RCR) is applied to describe the mass spectrum and Regge trajectories of mesons. This quasipotential equation is solved by the WKB method. A modified relativistic WKB condition of quantization is found which essentially simplifies the study of meson Regge trajectories. Formulae for the lepton widths are derived.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

After the discovery of J/Ψ mesons the interest has raised in describing of the meson mass spectrum according to the positronium scheme. First studies along this line used mainly (and quite successfully) the nonrelativistic Schrödinger equation (with potential $V(r)=\sigma r$). However, it was shown that after spin variables are introduced through the Breit-Fermi potential, this nonrelativistic (sémi-relativistic) model becomes inconsistent. It turned out that the contribution of relativistic corrections for higher radial excitations is large ($v^2/c^2\approx 0.4$), and for light vector ρ,ω- mesons it is even comparable with the contribution of the nonrelativistic part of the Hamiltonian.

In this paper we will look for the mass spectrum and Regge trajectories of mesons in the framework of the relativistic three-dimensional formalism based on the relativistic two-particle equations of the quasipotential type[1]. We shall use that of equations considered in ref.[2], which coincides with the Kadyshevsky equation[3] and, after transition to the relativistic configurational representation[4] (and some redefinition of normalization of the quasipotential and wave function[4,5,6]) has the following form (at fixed orbital moment ℓ)[4,5]:

$$[\mathrm{ch}(i\lambda\frac{\partial}{\partial r})+\frac{\lambda^2\ell(\ell+1)}{r(r+i\lambda)}\exp(i\lambda\frac{\partial}{\partial r})-X(r)]\Phi_\ell(r)=0 \tag{1.1}$$

$$X(r)=\frac{M-V(r)}{2mc^2};\quad \lambda=\frac{\hbar}{mc}.$$

Here M is the mass of a bound state of two spinless particles (quarks) with equal masses $m_1=m_2=m$ and momenta p_1 and p_2

$$M^2=\mathcal{P}^2=(p_1+p_2)^2. \tag{1.2}$$

The partial expansion of the wave function (WF) of a system with a given eigenvalues of mass M, $\Psi_M(\vec{r})$ over radial WF $\Phi_\ell(r)$ has the form

$$\Psi_M(\vec{r}) = 4\pi \sum_{\ell=0}^{\infty}(2\ell+1)i^\ell \frac{\Phi_\ell(r)}{r} P_\ell(\cos\theta_{\vec{n}}), \quad \vec{n} = \vec{r}/r . \tag{1.3}$$

In this paper, based on the relativistic quasipotential equation (1.1), we define the mass spectrum for any orbital momentum ℓ * and find the Regge trajectories for mesons. We use here the method proposed in [9] for finding solutions to eq. (1.1) in the quasiclassical approximation and in the second part of the paper generalize the modified WKB method of quantum mechanics [10] to the relativistic case. In the third part we derive formulae for the lepton widths of meson decays for the potentials of quarks confinement of the type $V_1(r) = \sigma r^s$, $s > 0$; $V_2(r) = -\frac{\kappa_s^2}{r} + \sigma r^s$. In the fourth part the comparison is made with the experimental data on masses of meson resonances.

2. RELATIVISTIC ANALOG OF THE "MODIFIED WKB METHOD" AND REGGE TRAJECTORIES OF MESONS

The quasiclassical solution to eq. (1.1) is searched in the usual form [9]

$$\Phi_\ell(r) = \exp(\frac{i}{\hbar} g(r)) \tag{2.1}$$

$$g(r) = g_0(r) + \frac{\hbar}{i} g_1(r) + (\frac{\hbar}{i})^2 g_2(r) + \ldots \tag{2.2}$$

With two first terms of the expansion (2.2) the solution is

$$\Phi_\ell(r) = \Phi_\ell^+(r) + \Phi_\ell^-(r) \tag{2.3a}$$

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* For $\ell = 0$ equation (1.1) with the linear potential $V(r) = \sigma r$ has been solved by the Laplace-transform method in [7]. Another approach (based on a somewhat different variant of the Kadyshevsky equation, equation in terms of rapidities [4]) was used for describing the meson spectrum in [8].

$$\Phi_\ell^\pm(r) = \frac{c_\pm\sqrt{Mc}}{\sqrt[4]{X^2-1-\frac{\lambda^2\Lambda^2}{r^2}}} \times \exp\{\frac{i}{\lambda}\int_{r_-}^{r_+} dr\, \ln[X \pm \sqrt{X^2-1-\frac{\lambda^2\Lambda^2}{r^2}}] + \phi\}, \tag{2.3b}$$

where $\Lambda = \ell + 1/2$, $c_\pm$ are normalization constants which in the nonrelativistic limit turn into their nonrelativistic analog by virtue of the factor $\sqrt{Mc}$. Reversal points $r_\pm$ are determined as branch points of the root in eq. (2.3b), i.e., from the equation

$$X^2(r) - 1 - \frac{\lambda^2\Lambda^2}{r^2} = 0 . \tag{2.4}$$

To make our further construction more clear, we consider in detail the solution with $\ell = 0$. Because of the finiteness of the WF at $r = 0$ we represent that solution as follows

$$\Phi_0^\pm(r) = \frac{c_\pm\sqrt{Mc}}{\sqrt{\operatorname{sh}\chi(r)}} \exp\{\frac{i}{\lambda}\int_{r_-}^{r} dr' \chi(r')\}, \tag{2.5}$$

where

$$\chi(r) = \operatorname{Arch} X(r) = \ln[X(r) + \sqrt{X^2(r) - 1}] \tag{2.6}$$

has the meaning of the rapidity of a particle moving in the field $V(r)$. Note that from the geometrical point of view the transition from the nonrelativistic to relativistic theory consists in the change of the Euclidean geometry of three-dimensional momentum space to the Lobachevsky space [11]. In the Lobachevsky momentum space realized on the hyperboloid $p_0^2 - \vec{p}^2 = m^2$ the distance between two points is measured in terms of the rapidity [11]. Therefore, $\chi(r)$ represents a direct relativistic generalization of the nonrelativistic momentum $p(r) = \sqrt{m[E - V(r)]}$ of a particle moving in the potential $V(r)$. This fact is supported by the maintenance of the geometrical meaning of the condition of quantization which in the relativistic case looks as follows

$$\int_{r_-}^{r_+} dr\,\chi(r) = \lambda\pi(n + \frac{1}{2}). \tag{2.7}$$

The reversal points $r_\pm$ are determined by the condition

$$X(r) \equiv \frac{M - V(r)}{2mc^2} = 1 \tag{2.8}$$

The condition of applicability of the WKB method in the relativistic case is given by the inequality

$$\lambda \left| \frac{2\,\mathrm{sh}\,\chi(r) + \mathrm{ch}\,\chi(r)}{\chi(r)\cdot \mathrm{sh}\,\chi(r)} \cdot \frac{d\chi(r)}{dr} \right| \ll 1\,, \tag{2.9}$$

which in the nonrelativistic limit converts to the condition $\left|\frac{d}{dr}\,\lambda_{nonrel}(r)\right| \ll 1$, where $\lambda_{nonrel}(r) = \hbar/\sqrt{m[E - V(r)]}$. In the ultrarelativistic limit $\chi \to \infty$ and (2.9) gives

$$\lambda \left| \frac{d}{dr} \ln \left| \ln \frac{\lambda(r)}{\lambda} \right| \right| \ll 1\,, \tag{2.10}$$

with

$\lambda(r) = \hbar/p(r) = \hbar/mc \cdot \mathrm{sh}\,\chi(r)$.

For the potential $V(r)$ growing with distance the relativistic WF, as compared to the nonrelativistic WF, has a new characteristic in the behaviour at large r. In region I (see Fig. 1), where $r < r_+$ (r_+ stands for the hight classical reversal point) the WF is given by (2.3), (2.5), and due to oscillations it becomes zero n times. Then, in region II, where $|X(r)| < 1$, like in nonrelativistic quantum mechanics, there the exponential decrease takes place

$$\Phi_0^{II}(r) = \frac{c\cdot\sqrt{Mc}}{\sqrt[4]{1 - X^2}} \exp\left\{ -\frac{1}{\lambda} \int_{r_+}^{r} dr' \arccos X(r') \right\}. \tag{2.11}$$

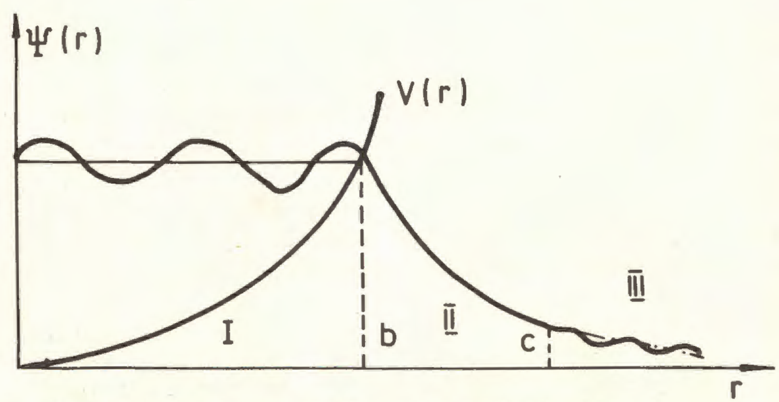

Fig. 1. The behaviour of the relativistic wave function of the two-particle system in the field of confinement potential: I - is the classically admissible region, II - the classically forbidden region, III - the region of pair production.

The difference from nonrelativistic theory occurs in region III which developes from point r_0 defined by the condition of a possible production of a pair of particles:

$$V(r)\big|_{r=r_0} = 2mc^2 + M = 4mc^2 + E_{coupl.}\,. \tag{2.12}$$

In region III, where $r > r_0$, the oscillation

$$\Phi_0^{III}(r) = \frac{c\cdot\sqrt{Mc}}{\sqrt[4]{X^2 - 1}} \cdot \exp\left[-\frac{\pi}{\lambda}(r - r_+)\right] \cdot \exp\left[\frac{i}{\lambda} \int_{r_+}^{r} dr\,\chi(r)\right] \tag{2.13}$$

is superimposed on the exponential decrease (2.11).

For $\ell \neq 0$ we put, like in quantum mechanics, the WF phase $\phi = \pi/4$ that provides the transition of the oscillating WKB solution from the left of the reversal point into the exponentially decreasing solution in classically forbidden region. The condition of quantization for $\ell \neq 0$ in full analogy with nonrelativistic theory, follows from the condition of coincidence of the WF at a point r from the left of the largest reversal point r_+ and WF at a point from the right of the smallest reversal point

$$\exp\left\{-i\frac{\pi}{4} + \frac{i}{\lambda}\int_{r_-}^{r} dr' \ln\left[X + \sqrt{X^2 - 1 - \frac{\lambda^2\Lambda^2}{r'^2}}\right]\right\} + \exp\left\{-\frac{i\pi}{4} + \frac{i}{\lambda}\int_{r_-}^{r} dr' \ln\left[X - \sqrt{X^2 - 1 - \frac{\lambda^2\Lambda^2}{r'^2}}\right]\right\} = \exp\left\{-i\frac{\pi}{4} - \frac{i}{\lambda}\int_{r}^{r_+} dr' \ln\left[X + \sqrt{X^2 - 1 - \frac{\lambda^2\Lambda^2}{r'^2}}\right]\right\} + \exp\left\{-i\frac{\pi}{4} - \frac{i}{\lambda}\int_{r}^{r_+} dr' \ln\left[X - \sqrt{X^2 - 1 - \frac{\lambda^2\Lambda^2}{r'^2}}\right]\right\}. \tag{2.14}$$

This relation for $\ell \neq 0$ results in the following condition of quantization

$$\frac{1}{\lambda}\int_{r_-}^{r_+} dr \ln\left[X(r) + \sqrt{X^2(r) - 1 - \frac{\lambda^2\Lambda^2}{r^2}}\right] + \Lambda\,\mathrm{arctg}\frac{r_-}{\lambda\Lambda} + \frac{r_-}{\lambda}\ln\sqrt{1 + \frac{\lambda^2\Lambda^2}{r_-^2}} - \Lambda\,\mathrm{arctg}\frac{r_+}{\lambda\Lambda} - \frac{r_+}{\lambda}\ln\sqrt{1 + \frac{\lambda^2\Lambda^2}{r_+^2}} = \pi(n + 1/2)\,.$$

Before applying to the potentials infinitely rising as $r \to \infty$, let us establish, by an example of the potential given in the relativistic configurational representation

$$V(r) = -\frac{\kappa_s^2}{r} \tag{2.15}$$

what accuracy may be expected from the relativistic WKB method. In ref. [5] it was shown that the exact solution of eq. (1.1) with potential (2.15) determines the condition of quantization

$$\frac{M_n}{2mc^2} = \sqrt{1 - \frac{\kappa_s^4}{4n^2}}\,, \tag{2.16}$$

whereas the calculation of integral (2.7) for potential (2.15) results in the relation

$$\frac{M_n}{2mc^2} = \sqrt{1 - \frac{\kappa_s^4}{4n^2}\left(1 - \frac{x_n}{\pi}\right)}\,; \quad x_n = \arccos\frac{M_n}{2mc^2}\,. \tag{2.17}$$

Comparison of (2.16) and (2.17) shows that in the Coulomb potential (2.15) the accuracy of the relativistic WKB method increases, like in nonrelativistic quantum mechanics, for high lying levels, i.e., when $\arccos\frac{M_n}{2mc^2} \ll \pi$.

For potentials $V(r) = \sigma r^s$, $s > 0$ the condition (2.14) can be simplified by taking the dependence on the centrifugal term out of the integral sign. In the nonrelativistic formalism this transformation, called "modified WKB method" is performed by passing to the variable

$$Z(r) = \int_0^r dr' \sqrt{m[E - V(r')]}$$

(see ref. [10]) in the Schrödinger equation. However, the same result can be achieved (see Appendix) by dividing in the nonrelativistic condition of quantization the integration range into two parts (see Fig. 2). In region I the main contribution is due to the centrifugal term, and the potential is considered as a perturbation while in region II the main contribution comes from the potential, and the centrifugal term is taken as a perturbation.

In the relativistic theory for potentials $V(r) = \sigma r^s$, $s > 0$ the same situation holds: the left reversal point r_- is mainly defined by the centrifugal term

$$r_- \simeq \frac{\lambda\Lambda}{\operatorname{sh}\chi_0}\,; \qquad \chi_0 = \operatorname{Arch}\frac{M}{2mc^2}\,, \tag{2.18}$$

and the right one r_+, by the potential $V(r)$, i.e., like for $\ell = 0$, by the condition

$$X(r_+) = 1\,. \tag{2.19}$$

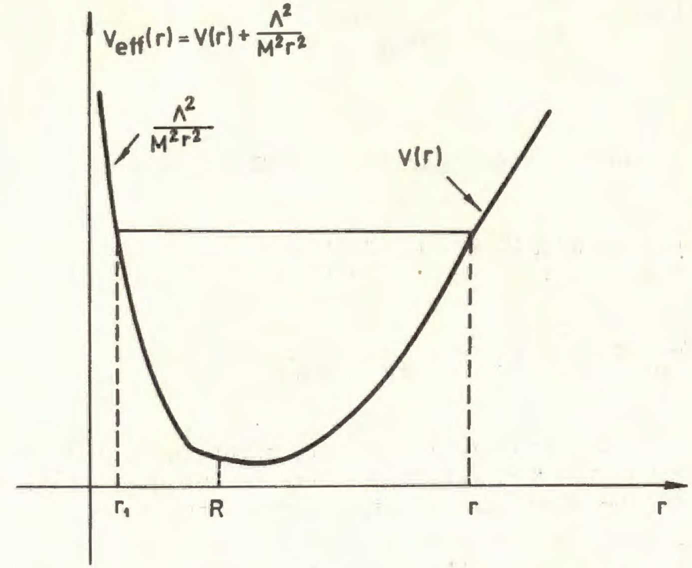

Fig. 2. The shape of the effective potential in the radial quasiclassical relativistic equation.

Therefore we make here the same splitting of the integral (2.14) into two parts at a point R lying in the classically admissible region of motion (the value of R is taken to be large in comparison with $r_- = \lambda\Lambda/\operatorname{sh}\chi_0$)

$$\frac{1}{\lambda}\int_{r_-}^{r_+} dr \ln\left[X(r) + \sqrt{X^2(r) - 1 - \frac{\lambda^2\Lambda^2}{r^2}}\right] = I_1 + I_2\,. \tag{2.20}$$

Assuming the contribution of the interaction potential to be small in this region, we obtain for the term I_1 by expanding in powers of $1/R^2$ the following expression

$$I_1 = \frac{1}{\lambda}\int_{r_-}^{R} dr \ln\left[X + \sqrt{X^2 - 1 - \frac{\lambda^2\Lambda^2}{r^2}}\right] \approx \frac{R}{\lambda}\chi_0 +$$

$$+ \frac{\lambda\Lambda^2}{R}\cdot\frac{e^{-\chi_0}}{2\operatorname{sh}\chi_0} - \frac{\Lambda\pi}{2} +$$

$$+\Lambda\,[\,\arcsin\frac{\mathrm{sh}\,\chi_0}{\mathrm{ch}\,\chi_0}-\frac{\ln(\mathrm{ch}\,\chi_0)}{\mathrm{sh}\,\chi_0}\,]. \tag{2.21}$$

For I_2 expansing the logarithm in $1/R^2$ we get

$$I_2=\frac{1}{\lambda}\int_R^{r_+}dr\ln[X+\sqrt{X^2-1-\frac{\lambda^2\Lambda^2}{r^2}}\,]\approx$$
$$\approx\frac{1}{\lambda}\int_0^{r_+}dr\,\chi(r)-\chi_0\frac{R}{\lambda}-\frac{\lambda\Lambda^2}{2R}\cdot\frac{e^{-\chi_0}}{2\,\mathrm{sh}\chi_0}, \tag{2.22}$$

where r_+ is the classical reversal point for $\ell=0$ defined by the condition $X(r_+)=1$. The condition of quantization (2.14) in the same approximations takes the form

$$\frac{1}{\lambda}\int_{r_-}^{r_+}dr\ln[X(r)+\sqrt{X^2(r)-1-\frac{\lambda^2\Lambda^2}{r^2}}\,]+$$
$$+\Lambda[\frac{\ln(\mathrm{ch}\chi_0)}{\mathrm{sh}\chi_0}+\arcsin\frac{1}{\mathrm{ch}\chi_0}]-\frac{\Lambda\pi}{2}=\pi(n+\frac{1}{2}). \tag{2.23}$$

Inserting (2.20) defined by (2.21) and (2.22) in (2.23) leads to the condition of quantization

$$\frac{1}{\lambda}\int_{r_-}^{r_+}dr\,\chi(r)=\pi(n+\frac{\Lambda}{2}+\frac{1}{2}), \tag{2.24}$$

whence for potentials $V(r)=\sigma r^s$ $(s>0)$ we derive the following condition of quantization

$$\sqrt{\frac{\pi}{2}}\,(\frac{2mc^2}{\sigma})^{1/s}\cdot[\mathrm{sh}\chi_n]^{1/2+1/s}\cdot\Gamma(1+\frac{1}{s})P_{-1/2}^{-1/2-1/s}(\mathrm{ch}\chi_n)=$$
$$=\lambda\pi(n+\ell/2+3/4), \tag{2.25}$$

where $P_\mu^\nu(\mathrm{ch}\chi)$ is the Legendre function.

For the oscillator interaction $s=2$ we get

$$2\sqrt{\mathrm{ch}\chi_n+1}\,[\,K(\mathrm{th}\frac{\chi_n}{2})-E(\mathrm{th}\frac{\chi_n}{2})]=$$
$$=\sqrt{\frac{\sigma}{2mc^2}}\,\lambda\pi(n+\frac{\ell}{2}+\frac{3}{4}), \tag{2.26}$$

where K and E are complete elliptic integrals.

For the linear potential $V(r)=\sigma r$ the condition of quantization is of the simplest form

$$\chi_n\mathrm{ch}\chi_n-\mathrm{sh}\chi_n=\frac{\sigma}{2mc^2}\lambda\pi(n+\frac{\ell}{2}+\frac{3}{4}). \tag{2.27}$$

Expressions (2.24) to (2.27) determine the explicit dependence of orbital momentum ℓ on the resonance energy and, thus, the relativistic Regge trajectories of mesons composed of two quarks.

In [2] it was shown that equation (1.1) is relativistic invariant due to the invariant nature of the modulus of the "relativistic relative coordinate" and the invariance of eigenvalues of the square of the orbital momentum of the relative motion, introduced in this representation in ref.[2]

3. CALCULATION OF THE LEPTON WIDTHS OF MESON DECAYS

It is commonly assumed that the lepton (ℓ) width of decays of mesons (μ) in state $\ell=0$ is given, by analogy with the positronium, by the formula (see refs.[12,13])

$$\Gamma(\mu\to e^+e^-)=16\pi\cdot e_q^2\ \alpha^2\cdot\frac{|\Psi_{\ell=0}(0)|^2}{M^2}, \tag{3.1}$$

where e_q^2 is the quark charge squared in the three-colour model. The quasiclassical wave function $\Psi_{WKB}^{\ell=0}(r)$ finite at the origin is of the form

$$\Psi_{\ell=0}(r)=\frac{\Phi_0(r)}{r}=\frac{c_0\sqrt{Mc}}{\sqrt{\mathrm{sh}\,\chi(r)}}\cdot\frac{1}{\sqrt{4\pi}}\cdot\frac{1}{r}\cdot\sin[\frac{1}{\lambda}\int_{r_-}^{r}dr'\chi(r')]. \tag{3.2}$$

The normalization coefficient c_0 in (3.2) can easily be calculated by using the fact that in the range of applicability of quasiclassical theory the argument of sin in (3.2) is a rapidly oscillating function. Therefore, like in the nonrelativistic theory, the sin squared in the normalization integral can be replaced by its average, 1/2:

$$4\pi\int_0^\infty |\Phi_{\ell=0}(r)|^2\, dr = \frac{|c_0|^2}{2}\cdot 4\pi\cdot\int_{r_-}^{r_+}\frac{dr}{\sqrt{X^2-1}} = 1 .$$

On the other hand, differentiating (2.7) with respect to principal quantum number we get

$$|c_0|^2 = [2\pi\int_{r_-}^{r_+}\frac{dr}{\mathrm{sh}\,\chi(r)}]^{-1} = \frac{1}{4\pi^2}\cdot\frac{1}{\hbar c}\cdot\frac{dM}{dn} . \qquad (3.3)$$

Consider first the quark confinement potentials without singularities at the origin: $V(r) = \sigma r^s$, $s>0$. Evidently, in this case the solution of eq. (1.1) with large n should turn into solutions of the free equation /4/ at small distances

$$\Psi_{\ell=0}^{WKB}(r)|_{r\to 0} \to \Psi_{\ell=0}^{free}(r) = \frac{\Phi_0^{free}(r)}{r} =$$

$$= a_0\frac{\sin\frac{r}{\lambda}\chi_n}{\frac{r}{\lambda}\cdot \mathrm{sh}\,\chi_n}; \qquad M_n = 2mc^2\cdot \mathrm{ch}\chi_n . \qquad (3.4)$$

Comparing (3.2) with (3.4) at $r=0$ results in the following relation between coefficients c_0 and a_0

$$a_0 = \frac{c_0}{\lambda}\sqrt{\mathrm{sh}\chi_n} . \qquad (3.5)$$

With (3.3) and (3.5) we arrive at the relativistic expression

$$|\Psi_{\ell=0}(0)|^2 = |\frac{\Phi_0(r)}{r}|^2|_{r=0} = |\frac{\chi_n}{\mathrm{sh}\,\chi_n}\cdot a_0|^2 =$$

$$= \frac{\lambda^{-2}}{4\pi^2\cdot\hbar c}\cdot\frac{\chi_n^2}{\mathrm{sh}\,\chi_n}\cdot\frac{dM}{dn} = \frac{\lambda^{-3}}{4\pi^2}\cdot\frac{2}{3}\cdot\frac{d(\chi_n)^3}{dn} . \qquad (3.6)$$

For the linear potential, (3.6) and (2.27) produce the relation

$$|\Psi_{\ell=0}^{rel.}(0)|^2 = \frac{1}{\hbar^2}\cdot\frac{\sigma m}{4\pi}\cdot\frac{\chi_n}{\mathrm{sh}\chi_n} , \qquad (3.7)$$

which differs from the nonrelativistic relation

$$|\Psi_{\ell=0}^{nonrel.}(0)|^2 = \frac{1}{\hbar^2}\cdot\frac{\sigma m}{4\pi}$$

by the relativistic factor $\chi_n/\mathrm{sh}\chi_n$ only*. The factor $\chi/\mathrm{sh}\chi$ does not contribute in the nonrelativistic limit ($\chi/\mathrm{sh}\chi\to 1$ as $c\to\infty$) and serves as a measure of the contribution of relativistic effects /14/.

As is shown in papers on the quark model /13/ correct widths of meson decays results from using the combined potentials

$$V(r) = -\frac{\kappa_s^2}{r} + \sigma r^s , \quad s>0 . \qquad (3.8)$$

In the field of such a confinement potential (of the "funnel" type) there may exist the energy levels $M<2mc^2$. Let us find for this case the decay widths.

Let the energy value in the potential (3.8) for a given level be determined from the condition of quantization. Then for a given fixed energy $M<2mc^2$ the wave function around the origin is known to be defined by the Coulomb part of the potential /5/:

$$\Psi_{Coulomb,\,\ell=0}^{M_n<2mc^2}(r) = c_{Coul.}\cdot e^{-\chi_n r\lambda^{-1}}\cdot e^{-i\chi_n}\cdot e^{i\chi_n\frac{\kappa_s^2}{2\sin\chi_n}}$$

$$\times {}_2F_1(1-ir\lambda^{-1},\ 1-\frac{\kappa_s^2}{2\sin\chi_n};\ 2;\ 1-e^{2i\chi_n}), \qquad (3.9)$$

$$\chi_n = \arccos\frac{M_n}{2mc^2},$$

whence it follows that

$$|\Psi_{Coul.}^{M_n<2mc^2}(0)|^2 = |c_{Coul.}|^2 \times$$

$$\times|{}_2F_1(1-\frac{\kappa_s^2}{2\sin\chi_n},\ 1;2;1-e^{2i\chi_n})|^2 . \qquad (3.10)$$

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* The analogous factor appeared also at the width in ref. /8/.

The normalization constant, $c_{\mathrm{Coul.}}$ will be calculated by comparing the exact WF (3.9) and quasiclassical solution at r large enough but still providing the dominant role of the Coulomb part (3.8). As can be easily verified, in quantum mechanics the correct value of the normalization constant for the Coulomb WF follows from comparing the exact solution with energy fixed with the WKB solution taken in the classically forbidden region*.

Comparing the asymptotic of the exact solution of (1.1) with the Coulomb potential (2.15)

$$\Psi^{M_n<2mc^2}_{\mathrm{Coul.},\,\ell=0}(r) \simeq \frac{c_{\mathrm{Coul.}}}{\Gamma\left(1+\frac{\alpha}{2\sin x}\right)} \cdot \frac{1}{2\sin x} \cdot \frac{1}{r/\lambda} \cdot \exp\left[-x\frac{r}{\lambda} - \frac{\alpha}{2\sin x}\ln\left(\frac{2r}{\lambda}\cdot\sin x\right)\right] \tag{3.11}$$

with the WKB solution in the classically forbidden region

$$\Psi^{M_n<2mc^2}_{\mathrm{WKB},\,\ell=0} = \frac{1}{\sqrt{4\pi}} \cdot \frac{1}{r} \cdot \frac{c_{\mathrm{WKB}}}{2\sqrt{Mc}} \cdot \frac{1}{\sqrt[4]{1-X^2(r)}} \times \exp\left[-\lambda\int_{r_+}^{r} dr' \operatorname{arc\,cos}\{X(r')\}\right] \tag{3.12}$$

at large r, we arrive at the relation between constants

$$c^2_{\mathrm{Coul.}} = c^2_{\mathrm{WKB}} \cdot \frac{Zm\alpha_s}{4h^3}, \tag{3.13}$$

which exactly coincides with the analogous relation in quantum mechanics /15/.

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* Asymptotics of both these solutions at large r have no oscillations and coincide in form. This approach gives the same result as a known method based on the comparison of the exact solution of the Schrödinger equation obtained neglecting the energy value (that provides its oscillatory nature) with the WKB solution in the classically admissible region /15/.

As a result combining formulae (3.1), (3.3), (3.10) and (3.13) we obtain for the width of the decay of meson μ with mass M and in state with $\ell=0$ into the lepton-antilepton pair the following expression

$$\Gamma_\ell(\mu\to e^+e^-) = 16\pi\cdot e_q^2\cdot\alpha^2\cdot\frac{Zm\alpha_s}{16\pi^2\hbar^3}\cdot\frac{1}{M^2}\cdot\frac{dM}{dn}\cdot \times\left|{}_2F_1\left(1-\frac{\kappa_s^2}{2\sin\chi_n}, 1; 2; 1-e^{2i\chi_n}\right)\right|^2 \tag{3.14}$$

$$\chi_n = \arccos\frac{M_n}{2mc^2}; \quad \alpha_s = \frac{\kappa_s^2}{\hbar c}.$$

It is interesting to note that in the case of positronium, i.e., for the pure Coulomb interaction, $\kappa_s = e$, $m = m_e$ the expression (3.14) differs from its nonrelativistic analog only by the hypergeometrical function which due to the condition of quantization (2.16) $\kappa_s^2/2\sin\chi_n = n$ is a polynomial and for first values of n has the form

$$\left|{}_2F_1\left(1-\frac{\kappa_s^2}{2\sin\chi_n}, 1; 2; -2ie^{i\chi_n}\sin\chi_n\right)\right|^2 = \begin{cases} 1, & n=1 \\ 1+\frac{3\alpha^2}{16}, & n=2. \end{cases} \tag{3.15}$$

From (3.15) it is clear that for $n=1$ (3.14) coincides in form with the nonrelativistic expression, and the hypergeometrical function contributes to higher states $n>1$ giving corrections of order α^2 and negligible corrections of higher orders in α^{2n}.

CONCLUSION

We have found the relativistic analog of the modified WKB method of quantum mechanics applicable for solving the relativistic two-particle quasipotential equation in the relativistic configurational representation. Simple relativistic formulae are derived for calculating the energy levels and Regge trajectories of the two-particle system of, what is the same, the mass spectrum of mesons considered as a quark-antiquark system.

A combination of the relativistic WKB condition of normalization with the known Coulomb solution of the relativistic quasipotential equation allowed us to obtain formulae for the lepton widths of meson decays.

Experimental data on the mass spectrum and Regge trajectories of ρ , ω mesons family have been analysed on the basis of formulae (2.24) and (2.25) in ref.[16]. It was found that the Regge trajectories of ρ meson in the interval of experimentally measured masses are linear with high accuracy. This fact can be explained by the essentially relativistic nature of the system of two light quarks that is reflected in a considerable difference of factor $\chi_n / \mathrm{sh}\chi_n$ from unity.

Regge trajectories of the family of J/Ψ mesons are plotted in Fig.3 which displays a tendency of approaching the linear dependence for trajectories of higher radial excitations Ψ'' and Ψ''' for which the relativistic factor $v^2/c^2 \geq 0.4$[13] and $\chi/\mathrm{sh}\chi$ noticeably differs from zero. Results of the description of the family of γ meson within this model by formula (2.25) and $s=1$ were presented earlier in ref.[17].

However, it should be noted that a more detailed comparison with experiment and discussion of the spectrum is to be carried out after including the dependence on quark spins. We intend to solve this task on the basis of the three-dimensional covariant spin approach developed in refs.[6,18].

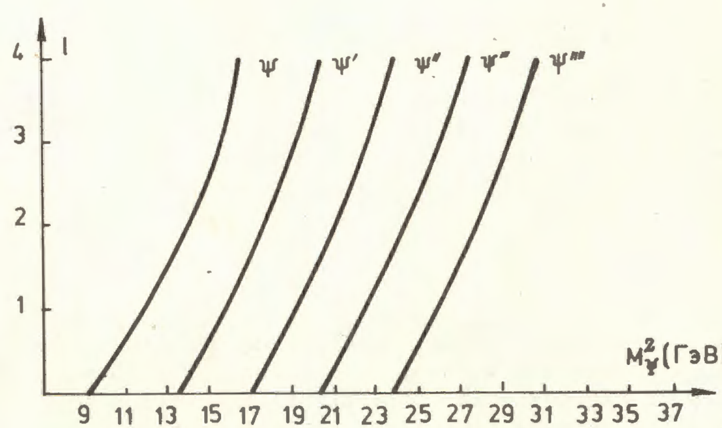

Fig. 3. The Regge trajectories of J/Ψ mesons calculated by formula (2.27).

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APPENDIX A.

MODIFIED CONDITION OF QUANTIZATION IN NONRELATIVISTIC QUANTUM MECHANICS

Following the procedure described in Sec. 2 we split the range in nonrelativistic integral

$$I = \int_{r_-}^{r_+} dr \sqrt{k^2 - V(r) - \frac{\Lambda^2}{r^2}} = \pi(n+1/2) \qquad (A.1)$$

into the (first) range from r_- to arbitrary (large) R, where the most contribution comes from the centrifugal term Λ^2/r^2 and the (second) range from that R to r_+, where the term Λ^2/r^2 is small compared to $V(r)$

$$I = I_1 + I_2 = k\int_{r_-}^{R} \frac{dr}{r}\sqrt{r^2 - \frac{\Lambda^2}{r^2}} + \int_{R}^{r_+} dr\, [\sqrt{k^2 - V(r)} - \frac{\Lambda^2}{2r^2\sqrt{k^2 - V(r)}}]. \qquad (A.2)$$

The first integral being limited to the terms of lowest orders in r_-/R is:

$$I_1 = k\{R - \frac{1}{2}\frac{r_-^2}{R} - r_-(\frac{\pi}{2} - \frac{r_-}{2})\}$$

and I_2 for $R/r_+ \ll 1$

$$I_2 = \int_0^{r_+} dr\sqrt{k^2 - V(r)} - kR - \frac{\Lambda^2}{2R\cdot k}$$

that results in the modified condition of quantization

$$\int_0^{r_+} dr\sqrt{k^2 - V(r)} = \pi(n + \frac{\Lambda}{2} + 1/2) \qquad (A.3)$$

coinciding with that obtained in ref.[10].

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