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Part I

Афанасьев Г.Н., Асанов Р.А. **E2 - 12390**

О задаче двух тел в специальной теории относительности. Часть I

Метод Пуанкаре в данной работе применяется к рассмотрению задачи двух тел в рамках специальной теории относительности. Теория содержит две произвольные функции от инвариантов группы Пуанкаре. При подходящем выборе этих функций оказывается возможным описать три решающих опыта теории относительности. Аналогичные результаты получаются в полностью ковариантном двухчастичном формализме. С помощью разложения по обратным степеням скорости света получены приближенные лоренц-ковариантные уравнения, которые не содержат эффекта запаздывания.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Afanasiev G.N., Asanov R.A. **E2 - 12390**

On the Special Relativistic Two-Body Problem. Part I

The method of Poincare is applied to the consideration of the two-body problem within the Special Relativity. The formulation of the theory contains two arbitrary functions of the Lorentz invariants. A specific choice of these functions leads to the correct description of the three crucial experiments of the General Relativity. The expansion on the inverse powers of the light velocity being performed, the approximate Lorentz covariant two-body equations without retardation effects are obtained.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. At present only few attempts are made to solve the two-body problem in the framework of the Special Relativity (SR). With some reservations these may be divided as follows:

a) theories in which the two-particle interaction propagates with the finite constant velocity equal to that of light [1-6];

b) relativistic Action of a Distance theories [7-9];

c) relativistic theories in which the velocity of the two-body interaction is not fixed [10-12];

d) relativistic approaches, corresponding to the quasipotential equation approach in quantum field theory [13-15];

e) theory of world minimal surfaces [16].

It is our aim to apply these theories mentioned in a) to the calculation of the concrete physical effects.

2. The early attempts to create the Lorentz-covariant theory of gravity are due to H.Poincare [1] and H.Minkowski [2]. The equations of motion suggested by H.Poincare are:

$$\frac{d^2x_{1i}}{d\tau_1^2} = x_i\frac{f_1}{B^3} + f_2\cdot\frac{1}{c}\frac{dx_{1i}}{d\tau_1} - \frac{1}{c}\frac{dx_{2i}}{d\tau_2}\left(f_2 + \frac{A\cdot f_1}{B^3}\right)\cdot\frac{1}{C},$$
$$c\frac{d^2t_1}{d\tau_1^2} = r\cdot\frac{f_1}{B^3} + f_2\cdot\frac{dt_1}{d\tau_1} - \frac{dt_2}{d\tau_2}\left(f_2 + \frac{A\cdot f_1}{B^3}\right)\frac{1}{C}. \qquad (1)$$

The following notation was used in (1). x_{ki} and t_k are the i-th cartesian coordinate and the time of k-th particle; τ_k and s_k are the proper time and the invariant interval of the k-th particle:

$$ds_k = c\cdot d\tau_k = \sqrt{c^2\cdot dt_k^2 - (d\vec{x}_k)^2}, \quad x_i = x_{1i} - x_{2i},$$
$$r^2 = \Sigma x_i^2, \quad t_1 = t_2 + r/c,$$

c is the velocity of light. A,B,C are the following Lorentz-invariant combinations of the relative coordinates and the particle velocities $\vec{v}_k = \frac{d\vec{x}_k}{dt_k}$:

$$A = \frac{1}{\sqrt{1-\beta_1^2}}\left(r - \frac{1}{c}\Sigma \vec{x}\vec{v}_1\right),\ B = \frac{1}{\sqrt{1-\beta_2^2}}\left(r - \frac{1}{c}\Sigma \vec{x}\vec{v}_2\right),\ C = \frac{1-\vec{v}_1\vec{v}_2/c^2}{\sqrt{1-\beta_1^2}\sqrt{1-\beta_2^2}}.$$

All the quantities referring to the first particle are taken at the time $t_1 = t$, while those for the second one at the time $t_2 = t_1 - \frac{r}{c}$. The functions f_1 and f_2 entering in (1) are arbitrary functions of the Lorentz invariants A,B,C which have the following asymptotic behaviour as c goes to infinity:

$$f_2 \sim O\left(\frac{1}{c}\right),$$

$$f_1 \sim \gamma + O\left(\frac{1}{c^2}\right).$$

Equations (1) are made of in a such way as to reproduce the well-known Newtonian gravitational equations (up to the order $1/c^2$) in the limit $c \to \infty$. Note the difference of expression (1) from the one given by H.Poincare /1/ who for the sake of simplicity set $f_2 = 0$. Even more special cases were considered by H.Minkowski /2/ and in ref. /3/. Using the analogy with electrodynamics as a guideline, they defined the two-body forces with the aid of the Lienard-Wichert potentials /17/. In this case $f_2 = 0$, $f_1 = \mathrm{const}$. Clearly, equations (1) may be easily generalized for arbitrary potential dependence upon the interparticle distance.

3. Leaving the discussion of the approximate solution methods of (1) and of the known exact particular solutions for the latter consider here the potential limit $(M_2 \to \infty)$. In this case particle 2 undergoes uniform straight line motion. The Lorentz transformation to the rest frame of particle 2 being made, the following equations for the motion of particle 1 are easily obtained:

$$\frac{d^2x_i}{d\tau_1^2} = x_i \frac{f_1}{r^3} + f_2 \cdot \frac{1}{c} \frac{dx_i}{d\tau_1},$$
$$\frac{d^2t}{d\tau_1^2} = \frac{1}{c^2}\frac{f_1}{r^3}\Sigma(xv_1) + \frac{f_2}{\sqrt{1-\beta_1^2}}\frac{v_1^2}{c^3}. \qquad (3)$$

Change the proper time in (3) to the coordinate time t_1

$$\frac{d^2x_i}{dt_1^2} = (1-\beta_1^2)\left\{x_i \cdot \frac{f_1}{r^3} + \frac{1}{c}\frac{dx_i}{dt_1}\left[f_2 \cdot \sqrt{1-\beta_1^2} - \frac{f_1}{r^3}\frac{(xv_1)}{c}\right]\right\}. \qquad (4)$$

Now compare equations (4) with those for the test body motion in General Relativity (GR) in Schwarzschild metric:

$$\frac{d^2x_i}{dt_1^2} = \frac{dx_i}{dt_1} \cdot \mu \cdot (xv_1) - x_i\left[\frac{1}{2r}\frac{d\mu}{dr}(xv_1)^2 + \mu v_1^2 + \frac{mc^2}{r^3}\right] \cdot \left(1 - \frac{2m}{r}\right). \qquad (5)$$

Here $\mu = \frac{2m}{r^3}\frac{1}{1-2m/r}$, $m = \frac{GM}{c^2}$ *.

The equations of motion (4) and (5) are exactly the same if the following choice of the functions f_1 and f_2 is made:

$$f_1 = -\frac{1-2m/r}{1-\beta_1^2}\left[\frac{r^2}{2}\frac{d\mu}{dr}(xv_1)^2 + \mu v_1^2 \cdot r^3 + mc^2\right],$$
$$\frac{1}{c} f_2 \sqrt{1-\beta_1^2} = \frac{f_1}{r}\frac{(xv_1)}{c} + \frac{\mu \cdot (xv_1)}{1-\beta_1^2}. \qquad (6)$$

4. The same coincidence with GR in the potential limit could be also obtained in the framework of the manifest covariant 2-body formalism /7-9/. In this formalism only those parts of particle trajectories interact which have the same time in a given Lorentz frame. The equations of motion are:

$$w_{1\nu} = (x_\nu - y_i v_{1\nu}) \cdot f + (v_{2\nu} - y_4 \cdot v_{1\nu}) \cdot g.$$

$$w_{2\nu} = -(x_\nu - y_2 v_{2\nu}) \cdot F + (v_{1\nu} - y_4 \cdot v_{2\nu}) \cdot G.$$

Here $x_\mu = x_{1\mu}(t) - x_{2\mu}(t)$, $v_{i\nu}$ and $w_{i\nu}$ are the four-velocity and acceleration of the i-th particle; $y_1 = (v_1 \cdot x)$, $y_2 = (v_2 \cdot x)$, $y_3 = (x \cdot x)$, $y_4 = (v_1 \cdot v_2)$ f, g, F, G are the functions of invariants y_1, y_2, y_3, y_4.

The condition of the Lorentz-covariance of the preceding equations leads to the system of four nonlinear differential equations for the functions f, g, F, G /9/. In the potential limit one of the particles (say, 2) moves with a constant velocity. Then $G = F = 0$ and the mentioned above system of equations reduces to the following linear one:

$$D \cdot f = 0, \quad D \cdot g + f = 0,$$

where D is the differential operator:

$$D = y_4 \frac{\partial}{\partial y_1} + \frac{\partial}{\partial y_2} + 2y_2 \frac{\partial}{\partial y_3}.$$

These equations are easily solved. The result is: f is an arbitrary function of two invariant variables $y_2^2 - y_3$ and $y_1 - y_4 y_2$, g is equal to

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* $M = M_2$ is the mass of the second particle.

$$g = - f \cdot \frac{y_1}{y_4} + g_1 ,$$

where g_1 is also an arbitrary function of the same invariant variables.

From this it is evident that these arbitrary functions can be chosen as to obtain the equation identical to that of test particle motion in GR.

5. The identity of (4) and (5) means that numerical values of those effects which may be calculated without recourse to the concrete form of the interval are the same both within the treated Lorentz-covariant theory and the GR. Fortunately, this is true for the three decisive experiments supporting the GR. We demonstrate this without too much details.

First, calculate the angular momenta integrals. It follows from (4) or (5):

$$\frac{x_i \dot{x}_j - x_j \dot{x}_i}{1 - 2m/r} = L_{ij} = \text{const.} \tag{7}$$

The dot means the coordinate time derivation. The energy integral is equal to:

$$\frac{(\dot{x} \cdot \dot{x})}{(1-2m/r)^2} + \frac{2m}{r^3} \frac{(x \cdot \dot{x})^2}{(1-2m/r)^3} - \frac{2mc^2}{r(1-2m/r)} = \epsilon. \tag{8}$$

It follows from (7) that the test particle orbit always lies in a plane, which may be chosen as (XY) one. Then (7) and (8) take a simpler form

$$\frac{r^2 \dot{\phi}}{1 - 2m/r} = L,$$

$$\frac{\dot{r}^2 + r^2 \dot{\phi}^2}{(1-2m/r)^2} + \frac{2m}{r} \frac{\dot{r}^2}{(1-2m/r)^3} - \frac{2mc^2}{1-2m/r} \frac{1}{r} = \epsilon. \tag{9}$$

Excluding time from (9) one easily obtains the orbital equation for motion of the test particle:

$$u_\phi^2 + (1 - 2mu) u^2 = \frac{2mu}{L^2}(c^2 - \epsilon) + \frac{\epsilon}{L^2} , \tag{10}$$

$$(u = \frac{1}{r}).$$

Differentiate (10) with respect to ϕ:

$$u_{\phi\phi} + u = 3mu^2 + \frac{mu}{L^2}(c^2 - \epsilon). \tag{11}$$

Equation (11) does not differ from that of describing the test particle motion in GR. Because of this the advance of planetary perihelia is the same in both cases. Further, as for the photon the value of ϵ (= energy per the unit of mass) is equal to c^2, one has:

$$u_{\phi\phi} + u = 3mu^2$$

that coincides with the light propagation equation in GR.

So, the bending of light is also the same. Regarding the half of the first two terms in the energy integral (8) as the kinetic energy contribution (this follows from non-relativistic limit) and equating it to the photon energy $h\nu$, one obtains the correct value of the red shift:

$$\nu \sim \frac{GM}{r}.$$

Some precaution is needed, however. Equations (9) form a complete system. So it is impossible to add to (9) something like that

$$\dot{r}^2 + r^2 \dot{\phi}^2 = c^2 \tag{12}$$

as it is demanded by Special Relativity for the photon velocity. Obviously, Eqs. (9) and (12) are not consistent. The relation (12) means that light does not interact with gravity. The fact that the absence of such an interaction leads to the numerous paradoxes and is not consistent with the energy conservation was recognized by A.Einstein as early as in 1911 [18]. So, for the light we do not impose condition (12). Instead of that we consider the motion of photons and test bodies on the same footing and demand velocity of light to be equal to c at the infinity (or in the absence of gravity)*. Then relation (8) fixes energy constant ϵ equal to c^2 for photons. The aforesaid is applied of course only to eq. (9), obtained from general eq. (4) with a very specific choice of the functions f_1 and f_2 (which aimed to reproduce the exact motion equations of the GR). It seems that different choices of f_1 and f_2 which reproduce the experimental situation for slow motions and do not disagree with (12) for photons are also possible.

Elementary calculations show that a simplified electrodynamic version of the two-particle forces suggested in [2,3] gives in the potential limit wrong value for the precession

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*This suggests that photon mass is very small, but finite.

of Mercury (equal to minus one sixth of the experimental value).

A few attempts are known to simulate the motion equations in GR by using the specific forces in the framework of SR. It was shown in refs.[19] that if the exact coincidence in the framework of equations takes place, the force in SR dynamics is the polynom of the fourth degree relative to the velocities:

$$\frac{d^2x^\alpha}{d\tau^2} = F^\alpha = \Omega^\alpha_{\lambda\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\rho}{d\tau} - \frac{dx^\alpha}{d\tau}\Omega_{\lambda\sigma\rho}\frac{dx^\lambda}{d\tau}\frac{dx^\sigma}{d\tau}\frac{dx^\rho}{d\tau}. \qquad (3a)$$

In the treated case the quantities $\Omega^\alpha_{\lambda\rho}$ are the differences of the Christoffel symbols $\Gamma^\alpha_{\lambda\rho}$ for the flat (i.e., Lorentz) and curved (i.e., Schwarzschild) spaces. Prove that equations (3a) and (3) are equivalent. In the cartesian coordinates all the symbols $\Gamma^\alpha_{\lambda\rho}$ for the flat space are zeroes. Then, eq. (3a) is the motion equation in Schwarzschild metrics, which coincides with eq. (5) if the parameter τ is chosen as the coordinate time. This means that the triple sum in the right-hand side of Eq. (3a) is converted to the first term of the right-hand side of Eq. (5). So, Eqs. (3a) and (5) are the same. As Eq. (5) follows from (4) (if the prescription (6) for the functions f_1 and f_2 is made), so Eq. (3a) is equivalent to (4) and (3).

6. The system of equations (1) which defines the motion of the first particle should be completed with the equations for the second one. The choice for the difference $t_1 - t_2 (= +\frac{r}{c})$ made in (1) corresponds to the retarded action of particle 2 on particle 1. An opposite choice $(t_1 - t_2 = -\frac{r}{c})$ leads to the advanced action (the action from particle 2 reaches particle 1 before it leaves particle 2). One may well use instead of retarded case (1) the half sum of the retarded and advanced interactions. Then, for the electrodynamic case mentioned above it is possible to find the Lorentz-invariant Lagrangian and to recover the integrals of energy, angular and linear momenta [20]. The particular exact solutions of the Special Relativistic two-body problem are known for this case [4]. The particles move along the concentric circular orbits with the constant angular velocity. Exact solutions corresponding to the similar concentric motion may be found for the short range potentials too [5]. If the interaction of 1 with 2 is retarded and 2 with 1 is advanced, then in addition to the concentric motion one may recover the exact solutions corresponding to the straight line relativistic two-particle motion [8].

The exact solutions are unknown if both interactions (1-2 and 2-1) are retarded. For the slow motions it is possible to carry out the expansion of equations (1) on the inverse powers of $\frac{1}{c}$. Keeping the terms up to the order $\frac{1}{c^2}$ one can easily obtains:

$$\frac{d^2x_{1i}}{dt^2} = -\frac{GM_2}{r^3}\left[1 + \frac{1}{c^2}\phi_1 - \frac{v_1^2}{c^2} - \frac{3}{2}\frac{(\vec{r}\vec{w}_2)}{c^2} - \frac{3}{2}\left(\frac{\vec{r}\vec{v}_2}{rc}\right)^2\right]x_i + \frac{1}{c^2}(v_{1i} - v_{2i})\left[\phi_2 + \frac{GM_2}{r^3}(\vec{r}\vec{v}_1)\right], \qquad (4)$$

where we set:

$$f_1 = -GM_2\left(1 + \frac{1}{c^2}\phi_1\right)$$

$$f_2 = \frac{1}{c}\phi_2.$$

Note, that all the quantities entering into Eq. (14) (i.e., coordinate, velocities, acceleration for both particles) are taken at the same time t. The motion of the second particle satisfies the same equation (with the replacement of indices $(1 \to 2)$). Eqs. (14) are the system of the ordinary differential equations which may be solved with the usual means if the initial coordinates and velocities of the particles are known.

If the motion is not slow, but the masses of particles are essentially different, then one may use the method suggested in [21] for solving equations (1). Exactly, in the first approximation the motion of the greater mass M_2 is supposed to be the uniform and straight-lined. For the given motion of M_2 equations (1) are the ordinary differential equations. Solve them and recover the motion M_1. So the motion of M_1 is known. Inserting it in the equations for M_2 one again obtains the equation for M_2 but with the corrections of an order of M_1/M_2. This procedure may be continued further and if it is convergent (no proof is known) then one has a definite answer.

In both approximations the knowledge of the initial positions and velocities of the particle is sufficient for recovering the future particles story. But for the exact system of equations (1) (plus those for the second particle) it seems impossible to formulate a well-defined initial value problem: Due to the finite propagation of the interaction one must either specify initial conditions on the finite part of the particle trajectories or specify at a given point not only the initial coordinates and velocities but all higher derivatives too [22]. So, the approximate solutions mentioned above fill a very narrow gap in a total variety of exact solutions. A possible way to overcome these difficulties, which is greatly appreciated is to formulate a suitable heuristic principle [16], which permirs one to restrict the mentioned above variety of solutions. However, the relation of this principle to an experiment is unclear.

In ref. [23] a general form of the approximately invariant relativistic 2-body Lagrangian was found. By comparing eq. (14) with that obtained from approximately invariant relativistic Lagrangian [23], one easily restores Lagrangian, corresponding to Eq. (14). This in turn gives motion integrals corresponding to an energy, angular and linear momenta.

7. The approach adopted here is rather phenomenological. In fact, unknown functions may be approximately fixed by comparing either with an experiment or with GR. This contrasts to the fundamentality of the Einstein approach in GR, where the only information which is needed is the distribution of matter. But until now there is no satisfactory solution of the 2-body problem in GR. On the other hand, the Special Relativity suggests the interesting possibilities, which were mentioned at the beginning of this paper.

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