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Обобщенные неравенства Баргманна

Выведено широкое семейство неравенств для $(\psi,-\Delta\psi)$ в $L^2(R^n)$, которое содержит неравенства Баргманна как частный случай.

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Generalized Bargmann Inequalities

A wide set of inequalities is derived for $(\psi,-\Delta\psi)$ in $L^2(R^n)$ which contains the Bargmann inequalities as a special case.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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Some years ago Bargmann proved validity of a set of inequalities /1/ which includes the conventional uncertainty principle relations in $L^2(R^n)$, $n \geq 2$, as well as for example the known relation

$$\int_{R^3} |\nabla \psi(x)|^2 d^3x \geq \frac{1}{4} \int_{R^3} r^{-2} |\psi(x)|^2 d^3x\,, \quad \psi \in C_0^\infty(R^3). \qquad (*)$$

Inequalities of this set can be applied in at least two directions. The relation (*) is useful for self-adjointness proofs (cf. /2/, Sec.X.2) and also its generalizations can serve to the same purpose /3/. On the other hand, some of the Bargmann inequalities give, e.g., the (exact) lower bounds for energies of orbital series of the hydrogen-like atom in n dimensions, $n \geq 2$ /1/. Another inequality from the considered set analogous to (*) can be also used for energy estimates /4/.

Recently Sachrajda, Weldon and Blankenbecler (SWB) have derived formally inequalities which generalize the Bargmann-type inequalities in the one-dimensional case /5/; the purpose of their paper was again to give some lower bounds for the Schrödinger operators. We shall use here their idea to prove validity of a set of inequalities which will contain the Bargmann inequalities as well as the exact version of the SWB-result as special cases. Applications of these inequalities will be discussed elsewhere.

Let us first introduce notation. We shall deal with the Hilbert space $\mathcal{H}^n = L^2(R^n)$, elements of R^n and their distances from origin will be denoted by $x = (x_1, \dots, x_n)$ and r, respectively. The operators $\nabla = (\nabla_1, \dots, \nabla_n)$ and $\Delta = \nabla^2$ are defined conventionally (/2/, Sec. IX.7; /8/, chap.7):

$$(\nabla_j \psi)(x) = i\check{(k_j \hat{\psi}(k))}(x) = i(F_n^{-1}(k_j F_n \psi))(x), \qquad (1a)$$

$$(-\Delta\psi)(x) = \check{(k^2 \hat{\psi}(k))}(x) = (F_n^{-1}(k^2 F_n \psi))(x) \qquad (1b)$$

with the appropriate domains; here $k = (k_1, \dots, k_n)$ $k^2 = \sum_{j=1}^{n} k_j^2$, and F_n denotes the n-dimensional Fourier-Plancherel operator.

PROPOSITION 1: Let $\psi \in D(-\Delta)$ and assume g to be a real function on R^n, $g\psi \neq 0$, g, ∇g Lebeague measurable and such that $\|g\phi\| < \infty$, $\|g r^{-1} x \nabla \phi\| < \infty$, $\|(\nabla g r^{-1} x)\phi\| < \infty$, $\|g r^{-1} x_j \phi\| < \infty$, $j = 1, \dots, n$, and $\|\nabla g r^{-1} x \phi\| < \infty$ for $\phi \in \{\psi\} \cup \mathcal{S}(R^n)$. Then

$$(\psi, -\Delta\psi) \geq \frac{1}{4} \|g\psi\|^{-2} (\psi, (r^{-1} x \nabla g + (n-1) r^{-1} g)\psi)^2. \qquad (2)$$

Proof: Since $|r^{-1} x_j| \leq 1$ and $\psi \in D(-\Delta)$, we have

$$(\psi, -\Delta\psi) = \|\nabla\psi\|^2 \equiv \sum_{j=1}^{n} \|\nabla_j \psi\|^2 \geq \sum_{j=1}^{n} \|r^{-1} x_j \nabla_j \psi\|^2 \equiv \|r^{-1} x \nabla \psi\|^2. \qquad (3)$$

The last expression can be estimated by the Schwartz inequality:

$$\|r^{-1} x \nabla \psi\|^2 \geq \frac{(r^{-1} x \nabla \psi, g\psi)^2}{\|g\psi\|^2} \geq \left[\frac{\mathrm{Re}(r^{-1} x \nabla \psi, g\psi)}{\|g\psi\|}\right]^2. \qquad (4)$$

The operator of multiplication by g (denoted also as g) is self-adjoint on $D(g) = \{\psi \in \mathcal{H}^n : g\psi \in \mathcal{H}^n\}$, analogously $r^{-1} x_j$ are Hermitean. Further each ∇_j is skew-symmetric on $D(-\Delta)$, thus

$$2\mathrm{Re}(r^{-1} x \nabla \psi, g\psi) = (g r^{-1} x \nabla \psi, \psi) - (\nabla r^{-1} x g \psi, \psi). \qquad (5)$$

For any $\phi \in \mathcal{S}(R^n)$ we have $\nabla r^{-1} x g \phi = (\nabla r^{-1} x g)\phi + r^{-1} x g \nabla \phi$. Since $\mathcal{S}(R^n)$ is dense in $\mathcal{H}^n$ and all ∇_j's are closed, there exists a sequence $\{\phi_k\} \subset \mathcal{S}(R^n)$ to any $\psi \in D(-\Delta)$, $\phi_k \to \psi$, $\nabla \phi_k \to \nabla \psi$. The multiplication operators $r^{-1} x_j g$ and $\nabla r^{-1} x_j g$ are self-adjoint, thus also closed. Due to the assumptions $\{\nabla_j \psi\} \cup \mathcal{S}(R^n) \subset D(r^{-1} x_j g)$, $j = 1, \dots, n$ and $\{\psi\} \cup \mathcal{S}(R^n) \subset D(\nabla r^{-1} x g)$ so that $r^{-1} x g \nabla \phi_k \to r^{-1} x g \nabla \psi$, $(\nabla r^{-1} x g)\phi_k \to (\nabla r^{-1} x g)\psi$. The same argument applied to the operators $r^{-1} x_j g$ gives $r^{-1} x_j g \phi_k \to r^{-1} x_j g \psi$; using once more the closedness of ∇_j's we obtain $\nabla r^{-1} x g \phi_k \to \nabla r^{-1} x g \psi$, and therefore

$$\nabla r^{-1} x g \psi = (\nabla r^{-1} x g)\psi + r^{-1} x g \nabla \psi. \qquad (6)$$

Now the relations (3)-(6) together with $\nabla r^{-1} x g = r^{-1} x \nabla g + (n-1) r^{-1} g$ prove the desired result. ■

REMARKS: 1) The inequality (2) will be denoted as B(n;g) or B(n;g,0). Especially for $n \geq 2$, $g(x) = r^{\mu+1}$, $\mu \geq -2$, we obtain the inequalities formally identical with (C_μ) of /1/. Assumptions of the proved proposition are, however, more restrictive than those of /1/; that is the price we pay for the more general g. Later we shall give weaker assumptions related to a special class of the functions g (Corollary 1 to Proposition 4).

2) The inequalities under consideration can be presented also in the form $\|\nabla\psi\|^2 \geq \dots$, then $\psi \in D(-\Delta)$ may be replaced by $\psi \in D(\nabla) \equiv \bigcap_{j=1}^{n} D(\nabla_j)$. The form (2) is, however, more suitable for estimating the kinetic energy terms of the Schrödinger operators.

3) The condition $\|\nabla g r^{-1} x \phi\| < \infty$ is a shorthand for $g r^{-1} x \phi_j \in D(\nabla_j)$, $j = 1, \dots, n$; in general it has to be verified with the help of the definition (1a).

The main difficulty of the presented proof was to establish the relation (6). It can be done in a simpler way if the action of ∇ is explicitly known. Such a situation occurs if $n = 1$; then $D(\nabla)$ consists of all functions $\psi \in L^2(R)$ which are absolutely continuous in any finite interval of R and the derivatives $\psi' \in L^2(R)$. This domain is conventionally denoted as $AC[R]$, action of ∇ on it being $\nabla\psi = \psi'$. Analogously $D(-\Delta) = AC^2[R] \equiv \{\psi \in AC[R] : \psi' \in AC[R]\}$ and $-\Delta\psi = -\psi''$. With these prerequisites we can prove the exact version of the SWB-inequalities:

PROPOSITION 2: Let $\psi \in AC^2[R]$ and $g: R \to R$ Lebesgue measurable such that g' exists a.e. in R, $0 < \|g\psi\| < \infty$, $g\psi \in AC[R]$ and $\|g'\psi\| < \infty$. Then

$$(\psi, -\psi'') \geq \frac{1}{4} \|g\psi\|^{-2} (\psi, g'\psi)^2 \tag{7}$$

Proof: In the same way as above we obtain $(\psi, -\psi'') \geq$ $\geq \|g\psi\|^{-2} (\mathrm{Re}(\psi', g\psi))^2$ and

$$2\,\mathrm{Re}(\psi', g\psi) = (g\psi', \psi) - ((g\psi)', \psi); \tag{8}$$

the last equality needs $g\psi' \in L^2(R)$ besides the assumptions explicitly stated. Both the functions g, ψ are differentiable a.e. in R (ψ' is even continuous), thus the same holds for g and $(g\psi)' = g'\psi + g\psi'$, further $g\psi'$ as the difference of L^2-vectors belongs to $L^2(R)$. Substitution of the last equality into (8) gives (7). ∎

Bargmann has shown that for $g(x) = r^{\mu+1}$ the inequalities stronger than (2) hold on eigenspaces of the "angular momentum". We shall deduce an analogous result for spherically symmetric g's. Let us start with some auxiliary statements borrowed from /1/. The harmonic polynomials Z_ℓ are defined by the relations

$$x\nabla Z_\ell = \ell Z_\ell, \quad \Delta Z_\ell = 0; \tag{9}$$

they are related to standard spherical harmonics $Y_\ell = Z_\ell \restriction S_n^1$, S_n^r being a sphere in R^n of radius r, by $Z_\ell(x) = r^\ell Y_\ell(r^{-1}x)$. Further $\{Y_{\ell t}\}_{t=1}^{s_\ell}$ will denote some ortonormal basis among the spherical harmonics with given ℓ:

$$(Y_{\ell t}, Y_{\ell t'})_S \equiv \int_{S_n^1} \overline{Y_{\ell t}(x)} Y_{\ell t'}(x) d\sigma_n(x) = \delta_{tt'},$$

$t, t' = 1, \dots, s_\ell$, where $d\sigma_n^r$ is the Euclidean measure on S_n^r, $d\sigma_n \equiv d\sigma_n^1$ and $Z_{\ell t}(x) = r^\ell Y_{\ell t}(r^{-1}x)$. Finally, let $(.,.)_\ell$ be the inner product in $L^2(R_+, r^{\beta_\ell} dr)$, $\beta_\ell = n-1+2\ell$.

PROPOSITION 3: (a) The following decomposition is valid for $n \geq 2$

$$\mathcal{H}^n = \sum_{\ell=0}^{\infty} \oplus \mathcal{H}_\ell^n, \quad \mathcal{H}_\ell^n = L^2(R_+, r^{\beta_\ell} dr) \otimes E_\ell L^2(S_n^1, d\sigma_n), \tag{10}$$

where the projection E_ℓ refers to the subspace of $L^2(S_n^1, d\sigma_n)$ spanned by $\{Y_{\ell t}\}_{t=1}^{s_\ell}$; analogously each $\mathcal{H}_\ell$ can be decomposed into a direct sum of orthogonal subspaces $\mathcal{H}_{\ell t}^n = E_{\ell t} \mathcal{H}_\ell^n = \{\psi : \psi(x) = f(r) Z_{\ell t}(x),\ \|f\|_\ell < \infty\}$.
(b) $\psi \in \mathcal{H}_{\ell t}^n$, $\psi(x) = f(r) Z_{\ell t}(x)$, belongs to $D(\nabla)$ if and only if f is absolutely continuous on R_+ and $\|f'\|_\ell < \infty$; then it holds

$$\|\nabla\psi\|^2 = \|f'\|_\ell^2 . \tag{11}$$

(c) If $\psi \in D(\nabla)$, the same is true for each $E_{\ell t}\psi$ and $\|\nabla\psi\|^2 = \sum_{\ell, t} \|\nabla E_{\ell t}\psi\|^2$.

Let us further consider a real-valued function g on R^n, the values of which depend on r only: $g(x) = \tilde{g}(r)$, $\tilde{g}$ being a function on R_+. We shall assume $\tilde{g}$ to be absolutely continuous in (any finite interval of) R_+ so that $\tilde{g}'$ exists a.e. in R_+. One can easily verify that for each $\psi \in \mathcal{H}_{\ell t}^n$, $\psi(x) = f(r) Z_{\ell t}(x)$, the following relations hold

$$\|g\psi\| = \|\tilde{g}f\|_\ell, \quad \|r^{-1} g\psi\| = \|r^{-1}\tilde{g}f\|_\ell,$$
$$\|(\nabla g)\psi\|^2 \equiv \sum_{j=1}^{n} \|(\nabla_j g)\psi\|^2 = \|\tilde{g}' f\|_\ell^2 . \tag{12}$$

PROPOSITION 4: Assume $\tilde{g}$ to be absolutely continuous in R_+, then for any $\psi \in \mathcal{H}_\ell^n \cap D(\nabla)$, $n \geq 2$, such that $0 < \|g\psi\| < \infty$,

$||r^{-1} g\psi|| < \infty$ and $||(\nabla g)\psi||^2 < \infty$ the inequality

$$||\nabla\psi||^2 \geq \frac{1}{4}||g\psi||^{-2}(\psi,(g'+\beta_\ell r^{-1} g)\psi)^2 \qquad (13)$$

(denoted as $B(n; g, \ell)$) holds, where $g' \equiv r^{-1} x\nabla g$, $g'(x) = \tilde{g}'(r)$.

Proof: (a) Let us first take $\psi \in \mathcal{H}^n_{\ell t}$, $\psi(x) = f(r)Z_{\ell t}(x)$. With the help of (11), (12) and the Schwartz inequality we obtain

$$||\nabla\psi||^2||g\psi||^2 = ||f'||^2_\ell \, ||\tilde{g}f||^2_\ell \geq (\mathrm{Re}(f',\tilde{g}f)_\ell)^2; \qquad (14)$$

all these experessions make sense due to our assumptions. The functions $f, \tilde{g}$ are absolutely continuous, thus also $f\tilde{g}r^{\beta_\ell}$ is absolutely continuous and $I_a^\beta = \int_a^\beta \overline{f'(r)}\tilde{g}(r)f(r)r^{\beta_\ell}\,dr$ may be integrated by parts:

$$I_a^\beta = [\tilde{g}(r)|f(r)|^2 r^{\beta_\ell}]_a^\beta - \int_a^\beta \overline{f(r)}\tilde{g}(r)f'(r)r^{\beta_\ell}\,dr - \int_a^\beta (\tilde{g}'(r)+\beta_\ell r^{-1}\tilde{g}(r))|f(r)|^2 r^{\beta_\ell}\,dr.$$

According to the assumptions all three integrals here are absolutely convergent for $a \to 0$, $\beta \to \infty$ so that there exist finite limits $c_0 = \lim_{r\to 0}\tilde{g}(r)|f(r)|^2 r^{\beta_\ell}$, $c_\infty = \lim_{r\to\infty}\tilde{g}(r)|f(r)|^2 r^{\beta_\ell}$. On the other hand, $||r^{-1}\tilde{g}f||_\ell < \infty$ means that the integral $\int_0^\infty \tilde{g}(r)|f(r)|^2 r^{\beta_\ell - 1}\,dr$ dr is absolutely convergent what is impossible unless $c_0 = c_\infty = 0$. Using further the fact that $\tilde{g}$ is real-valued we obtain

$$\mathrm{Re}(f',\tilde{g}f)_\ell = \frac{1}{2}I_0^\infty + \frac{1}{2}\int_0^\infty f'(r)\tilde{g}(r)\overline{f(r)}r^{\beta_\ell}\,dr = -\frac{1}{2}(f,(\tilde{g}'+\beta_\ell r^{-1}\tilde{g})f)_\ell,$$

i.e.,

$$\mathrm{Re}(f',\tilde{g}f)_\ell = -\frac{1}{2}(\psi,(g'+\beta_\ell r^{-1} g)\psi). \qquad (15)$$

Substituting this equation into (14), we get (13) for the considered ψ. (b) Further we take $\psi \in \mathcal{H}^n_\ell$, $\psi = \sum_{t=1}^{s_\ell}\psi_{\ell t}$, $\psi_{\ell t}(x) = f_{\ell t}(r)Z_{\ell t}(x)$. The operator of multiplication by g is of the form $\tilde{g}\otimes I_S$, i.e., $g\psi_{\ell t}$, $g\psi_{\ell t'}$ are orthogonal for $t \neq t'$ and $||g\psi|| < \infty$ implies $||g\psi_{\ell t}|| < \infty$, $t = 1, \dots, s_\ell$. Analogously we obtain $||r^{-1}g\psi_{\ell t}|| < \infty$, $||(\nabla g)\psi_{\ell t}|| < \infty$ and with the help of Proposition 3(c) also $\psi_{\ell t} \in D(\nabla)$ for $t = 1,\dots,s_\ell$; thus the relation (13) may be used for each $\psi_{\ell t}$. For $G_\ell = g' + \beta_\ell r^{-1} g$ it holds $G_\ell\psi_{\ell t} \in \mathcal{H}^n_{\ell t}$, and therefore

$$\frac{1}{2}(\psi, G_\ell\psi) = \frac{1}{2}\sum_{t=1}^{s_\ell}(\psi_{\ell t}, G_\ell\psi_{\ell t}) \leq \sum_{t=1}^{s_\ell}[||\nabla\psi_{\ell t}||^2||g\psi_{\ell t}||^2]^{1/2}.$$

Finally, the Hölder inequality gives the desired result

$$\frac{1}{2}(\psi, G_\ell\psi) \leq [\sum_{t=1}^{s_\ell}||\nabla\psi_{\ell t}||^2]^{1/2}[\sum_{t=1}^{s_\ell}||g\psi_{\ell t}||^2]^{1/2} = (||\nabla\psi||^2||g\psi||^2)^{1/2}. \qquad \blacksquare$$

REMARKS: 1) The inequalities $(B(n; r^{\mu+1}, \ell))$, $\mu \geq -2$, $n \geq 2$, coincide with $(C'_{\mu,\ell})$ of Bargmann. The more detailed analysis given in /1/ shows that in this case $\psi \in \mathcal{H}^n_\ell \cap D(\nabla)$ and $||g\psi|| < \infty$ is sufficient, or even only $\psi \in \mathcal{H}^n_\ell \cap D(\nabla)$ if $-2 \leq \mu \leq -1$ (except for $g\psi \neq 0$).
2) It holds $\beta_\ell = n - 1 + 2\ell$, thus the inequalities (13) are stronger than (2) if $(\psi, r^{-1}g\psi) \geq 0$.

COROLLARY 1: Let the assumptions of the Proposition 4 be valid with replacement of $\mathcal{H}^n_\ell$ by $\mathcal{H}^n$ and $D(\nabla)$ by $D(-\Delta)$. If $(\psi, r^{-1}g\psi) \geq 0$, then the inequality (2) holds.

The proof is essentaially the same as the part (b) of the preceding proof, the only thing to add is the inequality

$$\frac{1}{2}(\psi, G_0\psi) = \frac{1}{2}\sum_{\ell,t} (\psi_{\ell t}, G_0\psi_{\ell t}) \leq \frac{1}{2}\sum_{\ell,t} (\psi_{\ell t}, G_\ell\psi_{\ell t}).$$

■

One can obtain also a result analogous to Proposition 2 for $\mathcal{H} = L^2(R_+)$:

COROLLARY 2: Let g: $R_+ \to R$ be absolutely continuous, $\psi \in AC^2[R_+]$ and $0 < ||g\psi|| < \infty$, $||g'\psi|| < \infty$, then the inequality of the form (7) holds.

The proof reproduces essentially the part (a) of the preceding proof with $\beta_\ell = 0$. ■

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