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Квазипотенциальные модели релятивистского осциллятора

В рамках квазипотенциального подхода в квантовой теории поля исследованы две точно решаемые одномерные модели релятивистского осциллятора, для которых построены когерентные состояния и найдены динамические группы симметрии.

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Quasipotential Models of the Relativistic Oscillator

Two exactly solvable one-dimensional models of the relativistic harmonic oscillator are investigated in the framework of the quasipotential approach in quantum field theory. The coherent states are constructed and the dynamical symmetry groups are found for both the models.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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The harmonic oscillator, being one of a few exactly solvable problems in the nonrelativistic quantum mechanics, has been extensively used in the various fields of the theoretical physics - statistical mechanics, theory of superconductivity, nuclear physics, and so on. The interest to the harmonic oscillator was revived after the appearance of the quark models, which has made it possible to describe the basic features of hadronic structures. The further development of the quark models has led to the necessity of constructing the relativistic wave functions of compound particles and, in particular, the relativistic harmonic-oscillator models [1-5].

The characteristic feature of the harmonic oscillator is the existence of a class of solutions in the form of coherent states (c.s.). The utilization of the c.s. allows one to use the more simple classical language for the description of quantum phenomena. Originally the c.s. have been introduced for the quantum states with quadratic Hamiltonians, i.e., for the systems, which can be represented in the form of finite or infinite set of harmonic oscillators. The c.s. of quadratic systems are defined as the eigenstates of nonhermitean boson annihilation operators [6] and are the Gaussian wave packets minimizing the coordinate and momentum uncertainty product, the form of which is unchanged in time. The definition of the c.s. for arbitrary quantum systems as the eigenfunctions of the integrals of motion has been suggested in [7].

The c.s. representation turned out to be fruitful also for the investigation of the hadronic interaction at high energies. For instance, in [8] there has been considered the high-energy model, in which the excited states of colliding hadrons have the coherent character, while in /9/ on the basis of the c.s. method the factorization of dual amplitude of semimultiperipheral-type is obtained. The problems arising in the consistent formulation of quantum field theory in the c.s. representation are studied in /10/ .

We investigate two exactly solvable one-dimensional models of the relativistic harmonic oscillator in the framework of the quasipotential approach in quantum field theory /11,12/ . The coherent states are constructed and the dynamical symmetry groups are found for both the models.

The one-dimensional quasipotential equation for the wave function in the p-representation in the case of equal masses is written in the form

$$\left(E_p - E_q\right)\Psi_q(p) = \frac{1}{2\pi}\int \widetilde{V}(p,k;E_q)\Psi_q(k)\,d\Omega_k , \tag{1.1}$$

where $d\Omega_k = \frac{dk}{\sqrt{1+k^2/m^2c^2}}$ is the invariant volume element in the p-representation, $E_q = c\sqrt{m^2c^2+q^2}$ and $\widetilde{V}(p,k;E_q)$ is the quasipotential /12/ . The transition to the relativistic configurational x-representation /13/

$$\Psi_q(x) = \frac{1}{\sqrt{2\pi}}\int d\Omega_p\, \xi(p,x)\Psi_q(p) \tag{1.2}$$

is performed by the expansion in terms of the matrix elements of representations of the Lobachevsky space motion group

$$\xi(p,x) = \left(\frac{E_p - cp}{mc^2}\right)^{-i\frac{mc}{\hbar}x} \equiv e^{i\frac{mc}{\hbar}x\chi_p}, \tag{1.3}$$

where $E_p = mc^2\,\mathrm{ch}\,\chi_p$, $p = mc\,\mathrm{sh}\,\chi_p$ and $\chi_p = \ln\frac{E_p+cp}{mc^2}$ is the rapidity.

Equation (1.1) in the x-representation has the finite difference form:

$$\left(H_0 - E_q\right)\Psi_q(x) = \int_{-\infty}^{\infty} V(x,x';E_q)\Psi_q(x')\,dx', \tag{1.4}$$

$$V(x,x';E_q) = \int \xi(p,x)\widetilde{V}(p,k;E_q)\overset{*}{\xi}(k,x')\,d\Omega_p\,d\Omega_k . \tag{1.5}$$

Here $H_0(x) = mc^2\,\mathrm{ch}\left(\frac{i\hbar}{mc}\frac{d}{dx}\right)$ is the free Hamiltonian, the function (1.3) being the eigenfunction of $H_0(x)$:

$$H_0(x)\,\xi(p,x) = E_p\,\xi(p,x) . \tag{1.6}$$

The free particle momentum operator $\hat{\mathcal{P}}_x = -mc\,\mathrm{sh}\left(\frac{i\hbar}{mc}\frac{d}{dx}\right)$ and Hamiltonian $H_0(x)$ satisfy the following commutation relations with the relativistic coordinate x:

$$\left[\,x, \hat{\mathcal{P}}_x\,\right] = \frac{i\hbar}{mc^2}H_0(x) , \quad \left[\,x, H_0(x)\right] = \frac{i\hbar}{m}\hat{\mathcal{P}}_x . \tag{1.7}$$

In the nonrelativistic limit, i.e., when $c \longrightarrow \infty$, $H_0(x) \longrightarrow mc^2 + \frac{\hat{p}_x^2}{2m}$, $\hat{\mathcal{P}}_x \longrightarrow \hat{p}_x = -i\hbar\frac{d}{dx}$ and the function $\xi(p,x)$ goes over to the one-dimensional plane wave $e^{i\frac{xp_x}{\hbar}}$.

1. In the case of the local quasipotential $\widetilde{V}(p,k;E_q) = \widetilde{V}\left[(p(-)k)^2; E_q\right]$ /13,14/ , equation (1.4) takes the form

$$\left[\,H_0(x) + V(x) - E_q\right]\Psi_q(x) = 0 . \tag{2.1}$$

We consider the model of the relativistic harmonic oscillator, which corresponds to the interaction potential /15/

$$V(x) = \frac{m\omega^2}{2} x^{(2)} \exp\left(\frac{i\hbar}{mc}\frac{d}{dx}\right), \quad x^{(2)} = x\left(x + \frac{i\hbar}{mc}\right). \tag{2.2}$$

With such a choice of the potential $V(x)$ the following commutation relations for the relativistic coordinate X hold:

$$\left[x, H(x)\right] = \frac{i\hbar}{m}\left[\hat{\mathcal{P}}_x - \frac{1}{c}V(x)\right], \quad \left[x, \hat{\mathcal{P}}_x - \frac{1}{c}V(x)\right] = \frac{i\hbar}{mc^2}H(x),$$
$$\left[\left[x, H(x)\right], H(x)\right] = (\hbar\omega)^2 x, \tag{2.3}$$

i.e. in accordance with (1.7) $\hat{\Pi}(x) = \hat{\mathcal{P}}_x - \frac{1}{c}V(x)$ can be called the generalized momentum operator, and the double commutator in (2.3) coincides with the equation of motion of the nonrelativistic oscillator in the Schrödinger representation. In the nonrelativistic limit the potential $V(x) \longrightarrow \frac{m\omega^2}{2}x^2$.

The quasipotential equation (2.1) for this model with the aid of the dimensionless quantity $\tilde{x} = \frac{mc}{\hbar}x$ is written in the form

$$\left[\mathrm{ch}\left(i\frac{d}{d\tilde{x}}\right) + \frac{1}{2}\left(\frac{\hbar\omega}{mc^2}\right)^2 \tilde{x}(\tilde{x}+i)\exp\left(i\frac{d}{d\tilde{x}}\right)\right]\psi_q(\tilde{x}) = \mathrm{ch}\chi_q \psi_q(\tilde{x}). \tag{2.4}$$

The corresponding to (2.4) equation in the p-representation

$$\left[\mathrm{ch}\chi_p - \frac{1}{2}\left(\frac{\hbar\omega}{mc^2}\right)^2 e^{-\chi_p}\left(\frac{d^2}{d\chi_p^2} - \frac{d}{d\chi_p}\right) - \mathrm{ch}\chi_q\right]\psi_q(\chi_p) = 0 \tag{2.5}$$

by changing of the variable $\zeta = \frac{2mc^2}{\hbar\omega}e^{\chi_p}$ ($0 \leq \zeta < \infty$) is reduced to the Whittaker equation

$$\left[\frac{d^2}{d\zeta^2} - \frac{1}{4} + \frac{\lambda}{\zeta} + \frac{1/4 - \mu^2}{\zeta^2}\right]\psi_q(\zeta) = 0 \tag{2.6}$$

with the parameters $\lambda = \frac{mc^2}{\hbar\omega}\mathrm{ch}\chi_q$, $\mu = \sqrt{\frac{1}{4} + \left(\frac{mc^2}{\hbar\omega}\right)^2}$

The general solution of equation (2.6) can be represented as

$$\psi_q(\zeta) = c_1 M_{\lambda,\mu}(\zeta) + c_2 M_{\lambda,-\mu}(\zeta), \tag{2.7}$$

where $M_{\lambda,\mu}(\zeta)$ is the Whittaker function, which is expressed through the confluent hypergeometric function as follows:

$$M_{\lambda,\mu}(\zeta) = \zeta^{\mu+\frac{1}{2}} e^{-\frac{\zeta}{2}} \Phi\left(\mu + \frac{1}{2} - \lambda, 2\mu+1; \zeta\right). \tag{2.8}$$

Since the function $\Phi(a,b;\zeta)$ behaves at the origin like a constant and $\frac{1}{2} - \mu = \frac{1}{2} - \sqrt{\frac{1}{4} + \left(\frac{mc^2}{\hbar\omega}\right)^2} < 0$, from (2.8) it follows that the solution (2.7) is regular at the origin only if $C_2 = 0$. The asymptotic behaviour of the confluent hypergeometric function when ζ becomes large is given by

$$\Phi\left(\frac{1}{2} + \mu - \lambda, 2\mu+1; \zeta\right) \simeq \frac{\Gamma(2\mu+1)}{\Gamma(\mu+\frac{1}{2}-\lambda)} \zeta^{-(\lambda+\mu+\frac{1}{2})} e^{\zeta}.$$

Therefore the requirement of regularity of the solution of equation (2.6) when $\zeta \longrightarrow \infty$ leads to the condition

$$\frac{1}{\Gamma(\mu+\frac{1}{2}-\lambda)} = 0, \text{ i.e. } \mu + \frac{1}{2} - \lambda = -n, \quad n = 0,1,2,\ldots, \tag{2.9}$$

which gives the energy quantization rule for the relativistic oscillator under consideration:

$$E_q = mc^2\mathrm{ch}\chi_q = \hbar\omega\lambda = \hbar\omega\left(n + \mu + \frac{1}{2}\right). \tag{2.10}$$

Thus, the wave functions, corresponding to the energy levels (2.10), have in p-representation the form:

$$\psi_q(\zeta;n) = c_n M_{\mu+\frac{1}{2}+n,\mu}(\zeta) = c_n \zeta^{\mu+\frac{1}{2}} e^{-\frac{\zeta}{2}} \Phi(-n, 2\mu+1; \zeta). \tag{2.11}$$

where C_n are the normalization coefficients. The calculation of C_n is simplified if one expresses $\Phi(-n,2\mu+1;\zeta)$ through the generalized Laguerre polynomial:

$$\Phi(-n,2\mu+1;\zeta)=\frac{n!\,\Gamma(2\mu+1)}{\Gamma(2\mu+n+1)}L_n^{2\mu}(\zeta)=\Gamma(2\mu+1)\sum_{m=0}^{n}\frac{C_n^m(-z)^m}{\Gamma(2\mu+m+1)} \qquad (2.12)$$

Now the coefficients C_n are easily calculated since

$$\int_0^\infty y^\alpha L_n^\alpha(y)L_m^\alpha(y)e^{-y}dy=\begin{cases}\dfrac{\Gamma(n+\alpha+1)}{n!}, & n=m\\ 0, & n\neq m\end{cases} \qquad (2.13)$$

Thus, the normalized wave functions in the p-representation are written in the form (2.11), and

$$C_n=\frac{c\,i^n}{\Gamma(2\mu+1)}\left\{\frac{mc\,\Gamma(n+2\mu+1)}{\hbar\, n!}\right\}^{\frac12}. \qquad (2.14)$$

The transition to the wave functions in the x-representation

$$\psi_q(x;n)=\frac{2^\mu\Gamma(2\mu+1)}{\sqrt{\pi}\,c}C_n\left(\frac{\hbar\omega}{mc^2}\right)^{i\tilde{x}}\sum_{m=0}^{n}\frac{(-2)^m C_n^m}{\Gamma(m+2\mu+1)}\Gamma\left(\mu+\tfrac12+m+i\tilde{x}\right)= \qquad (2.15)$$

$$=\frac{2^\mu C_n}{\sqrt{\pi}\,c}\left(\frac{\hbar\omega}{mc^2}\right)^{i\tilde{x}}\Gamma\left(\mu+\tfrac12+i\tilde{x}\right)F\left(-n,\mu+\tfrac12+i\tilde{x};2\mu+1;2\right)$$

is performed with the help of formulae (1.2), (2.11) and the integral representation for the gamma-function

$$\Gamma(\zeta)=\int_0^\infty e^{-t}t^{\zeta-1}dt,\quad \mathrm{Re}\,\zeta>0.$$

When $c\to\infty$ the functions $\psi_q(x;n)$ and $\psi_q(\zeta;n)$ coincide with the corresponding wave functions of the nonrelativistic linear oscillator in the coordinate and momentum representations (see Appendix I).

Coherent states. For constructing the relativistic oscillator c.s., we introduce the variable $\tau=\sqrt{\zeta}=c\sqrt{\frac{2m}{\hbar\omega}}\exp(\frac12\chi)$ ($0\le\tau<\infty$). Then equation (2.6) is rewritten in the form

$$\left[\tau^2-\frac{d^2}{d\tau^2}+\frac{g}{\tau^2}\right]U_q(\tau)=4\lambda U_q(\tau), \qquad (2.16)$$

where $\psi_q(\zeta)=\sqrt{\tau}\,U_q(\tau)$, $g=4\mu^2-\frac14=\frac34+\left(\frac{2mc^2}{\hbar\omega}\right)^2$. The advantage of this form of equation (2.6) is that it allows one to use the analogy with the well-known quantum-mechanical singular linear oscillator /16/ .

We introduce the annihilation and creation operators

$$a=\frac{1}{\sqrt2}\left(\tau+\frac{d}{d\tau}\right),\qquad a^+=\frac{1}{\sqrt2}\left(\tau-\frac{d}{d\tau}\right) \qquad (2.17)$$

acting on the function $U_q(\tau)$ and satisfying the commutation relation $[a,a^+]=1$. Then the total "Hamiltonian", which corresponds to equation (2.6), can be represented as

$$H(\tau)=H^{(0)}(\tau)+\frac{\hbar\omega g}{4\tau^2}=\frac{\hbar\omega}{2}\left(a^+a+\frac12+\frac{g}{2\tau^2}\right). \qquad (2.18)$$

Using the commutation relation $[a,a^+]=1$, it is easy to show that

$$\left[a^n,H^{(0)}(\tau)\right]=\frac{\hbar\omega}{2}n\,a^n,\quad \left[(a^+)^n,H^{(0)}(\tau)\right]=-\frac{\hbar\omega}{2}n\,(a^+)^n, \qquad (2.19)$$

i.e., the operators a and a^+ are formally the integrals of motion of the system, which is described by the "Hamiltonian" $H^{(0)}(\tau)$. However, in this case the operators (2.17) have no direct physical meaning since the parameter $g=\frac34+\left(\frac{2mc^2}{\hbar\omega}\right)^2$,which characterizes the contribution of the singular term to (2.16), never vanishes for any real ω . Therefore, the representation of the Hamiltonian $H(\tau)$ in the form (2.18) and the subsequent use of the operators (2.17) is only a mathematical method, which allows us to construct the integrals of motion for the relativistic oscillator (2.2).

As is known /16/, for the total "Hamiltonian" $H(\tau)$ such invariants are quadratic in the operators (2.17) expressions

$$A(\tau)=\frac{1}{2\Lambda}\left\{a^2-\frac{g}{2\tau^2}\right\}, \qquad A^+(\tau)=-\frac{1}{2\Lambda}\left\{(a^+)^2-\frac{g}{2\tau^2}\right\}, \Lambda=\left(\frac{2mc^2}{\hbar\omega}\right)^{\frac{1}{2}} \tag{2.20}$$

satisfying the following commutation relations:

$$[A,H]=\hbar\omega A, \quad [A^+,H]=-\hbar\omega A^+; \quad [A,A^+]=\frac{1}{mc^2}H. \tag{2.21}$$

The formulae (2.17) and (2.20) define the action of the operators $A(\tau)$ and $A^+(\tau)$ on the function $u_q(\tau)$. Having derived the explicit form of these integrals of motion, we can return to the initial function $\Psi_q(\zeta)=\sqrt{\tau}\,u_q(\tau)$. Since

$$\frac{d}{d\tau}\frac{\varphi(\tau)}{\sqrt{\tau}}=\frac{1}{\sqrt{\tau}}\left\{\frac{d}{d\tau}-\frac{1}{2\tau}\right\}\varphi(\tau),$$

the explicit form of the operators A, A^+ and H in the $\Psi_q(\zeta)$-functions space is obtained from formulae (2.18) and (2.20) by replacing a by $a-\frac{1}{2\sqrt{2}}\cdot\frac{1}{\tau}$ and a^+ by $a^++\frac{1}{2\sqrt{2}}\cdot\frac{1}{\tau}$, i.e.,

$$\begin{aligned}
A(\zeta)\Psi_q(\zeta)&=\frac{i}{\Lambda}\left\{\zeta\frac{d}{d\zeta}+\frac{1}{2}\zeta-\frac{1}{\hbar\omega}H(\zeta)\right\}\Psi_q(\zeta),\\
A^+(\zeta)\Psi_q(\zeta)&=\frac{i}{\Lambda}\left\{\zeta\frac{d}{d\zeta}-\frac{1}{2}\zeta+\frac{1}{\hbar\omega}H(\zeta)\right\}\Psi_q(\zeta),\\
H(\zeta)\Psi_q(\zeta)&=\hbar\omega\zeta\left\{\frac{1}{4}+\left(\frac{mc^2}{\hbar\omega}\right)^2\frac{1}{\zeta^2}-\frac{d^2}{d\zeta^2}\right\}\Psi_q(\zeta),
\end{aligned} \tag{2.22}$$

the same commutation relations (2.21) being satisfied.

It is evident that the solution (2.11) of equation (2.6), which will be denoted by the symbol $|n\rangle$, are the eigenfunctions of the Hamiltonian H:

$$H|n\rangle=E_n|n\rangle, \qquad E_n=\hbar\omega\left(n+\mu+\frac{1}{2}\right). \tag{2.23}$$

Using the representation (2.12) and the recurrence relations for the generalized Laguerre polynomials, one can show that

$$\begin{aligned}
A|n\rangle&=b_n|n-1\rangle, \quad A^+|n\rangle=b_{n+1}|n+1\rangle,\\
|n\rangle&=\beta_n(A^+)^n|0\rangle.
\end{aligned} \tag{2.24}$$

Here we have used the notation $b_n=\sqrt{n(n+2\mu)\frac{\hbar\omega}{2mc^2}}$, $\beta_n=\{b_1b_2\cdots b_n\}^{-1}=\Lambda^n\left\{\frac{\Gamma(2\mu+1)}{n!\,\Gamma(n+2\mu+1)}\right\}^{1/2}$. Since $b_n\cdot\beta_n=\beta_{n-1}$, one can easily check that the functions

$$|\gamma\rangle=C_\gamma\sum_{n=0}^{\infty}\beta_n\gamma^n|n\rangle, \tag{2.25}$$

where C_γ is an arbitrary constant, are the eigenfunctions of the operator A. In fact

$$A|\gamma\rangle=C_\gamma\sum_{n=1}^{\infty}\beta_n\gamma^n b_n|n-1\rangle=C_\gamma\sum_{n=1}^{\infty}\beta_{n-1}\gamma^n|n-1\rangle=\gamma|\gamma\rangle. \tag{2.26}$$

Substituting into (2.25) the expression for the states $|n\rangle$ through the generalized Laguerre polynomials, with the help of the generating function of these polynomials

$$\sum_{n=0}^{\infty}\frac{x^n L_n^{\alpha}(\zeta)}{\Gamma(n+\alpha+1)}=(x\zeta)^{-\frac{\alpha}{2}}e^{x}J_\alpha(2\sqrt{x\zeta}),$$

we obtain the explicit form of the eigenfunctions of the operator A in the p-representation:

$$|\gamma\rangle = c\left\{\frac{mc}{\hbar}\Gamma(2\mu+1)\zeta\right\}^{\frac{1}{2}} C_\gamma \frac{e^{i\Lambda\gamma-\frac{\zeta}{2}}}{(i\gamma\Lambda)^\mu} J_{2\mu}\left(2\sqrt{i\Lambda\gamma\zeta}\right). \tag{2.27}$$

The constant $C_\gamma = (\Lambda|\gamma|)^\mu\left[\Gamma(2\mu+1)I_{2\mu}(2\Lambda|\gamma|)\right]^{-\frac{1}{2}}$ is defined from the normalization condition

$$\int_{-\infty}^{\infty}|\psi(x)|^2 dx = \frac{2\hbar}{mc^3}\int_0^\infty \frac{d\tau}{\tau}|\psi(\tau)|^2 = 1$$

and is calculated by the formula

$$\int_0^\infty d\tau\, J_{2\mu}\left(2e^{\frac{\pi i}{4}}\sqrt{\Lambda\gamma}\,\tau\right)J_{2\mu}\left(2e^{-\frac{\pi i}{4}}\sqrt{\Lambda\bar{\gamma}}\,\tau\right)\tau e^{-\tau^2} = \frac{1}{2}e^{-i\Lambda(\gamma-\bar{\gamma})} I_{2\mu}(2\Lambda|\gamma|).$$

Thus, the normalized eigenfunctions of the operator A have in p-representation the form

$$|\gamma\rangle = c\sqrt{\frac{mc}{\hbar}}N_\gamma\left(-\frac{\bar{\gamma}}{\gamma}\right)^{\frac{\mu}{2}} e^{i\Lambda\gamma-\frac{\zeta}{2}}\sqrt{\zeta}\, J_{2\mu}\left(2\sqrt{i\Lambda\gamma\zeta}\right), \tag{2.28}$$

where $N_\gamma = \left\{I_{2\mu}(2\Lambda|\gamma|)\right\}^{-\frac{1}{2}}$. Insofar as the states $|n\rangle$ coincide when $c \longrightarrow \infty$ with the states of the nonrelativistic linear oscillator in the p-representation and the quantities $b_n \rightarrow \sqrt{n}$, $\beta_n \longrightarrow 1/\sqrt{n!}$, it follows from the relations (2.4) that the operators A and A^+ go over in this limit to the annihilation and creation operators of the nonrelativistic oscillator. Since the normalization constant $C_\gamma \longrightarrow e^{-\frac{1}{2}|\gamma|^2}$ when $c \longrightarrow \infty$ (see Appendix II, formula (A.8)), it also follows from the representation (2.25) that the states $|\gamma\rangle$ coincide in this limit with the nonrelativistic linear oscillator c.s.

The explicit form of the operators A and A^+ in the x-representation can be obtained from (2.22) by means of the transformation (1.2):

$$A(x) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{m\omega}{\hbar}}\,x + \frac{i}{\sqrt{m\hbar\omega}}\Pi(x)\right\}, \quad \overset{+}{A}(x) = \frac{1}{\sqrt{2}}\left\{\sqrt{\frac{m\omega}{\hbar}}\,x - \frac{i}{\sqrt{m\hbar\omega}}\Pi(x)\right\} \tag{2.29}$$

Having used (2.28) and the transformation formula (1.2), in which the substitution of the variable χ_p by $\tau = \sqrt{\zeta} = \Lambda e^{\frac{\chi_p}{2}}$ is made, we get the explicit form of the eigenfunctions $|\gamma\rangle$ of the operator A in the x-representation:

$$|\gamma\rangle_x = \frac{1}{c}\sqrt{\frac{2}{\pi}}\,\Lambda^{-2i\tilde{x}}\int_c^\infty \tau^{2i\tilde{x}-1}|\gamma\rangle_p d\tau = \tag{2.30}$$

$$= \sqrt{\frac{mc}{\pi\hbar}}N_\mu\frac{(2\Lambda|\gamma|)^\mu}{\Gamma(2\mu+1)}e^{-i\Lambda|\gamma|}\,2^{i\tilde{x}}\,\Gamma\left(\mu+\frac{1}{2}+i\tilde{x}\right)\Phi\left(\mu+\frac{1}{2}-i\tilde{x}, 2\mu+1, 2i\Lambda\gamma\right).$$

The integration in (2.30) over the variable τ has been performed with the aid of the following formula for the Bessel function:

$$\Gamma(\mu+1)\int_0^\infty J_\mu(\alpha t)e^{-\gamma^2t^2}t^{\rho-1}dt = \frac{1}{2\gamma^\rho}\Gamma\left(\frac{\mu+\rho}{2}\right)\left(\frac{\alpha}{2\gamma}\right)^\mu e^{-\frac{\alpha^2}{4\gamma^2}}\Phi\left(\frac{\mu-\rho}{2}+1, \mu+1; \frac{\alpha^2}{4\gamma^2}\right), \tag{2.31}$$

$$\mathrm{Re}\,\gamma^2 > 0, \quad \mathrm{Re}(\mu+\rho) > 0.$$

<u>Dynamical symmetry group.</u> The utilization of the operators A and A^+, with the help of which we have constructed c.s. (2.28) and (2.30), makes also easier the problem of searching a dynamical symmetry group of the relativistic oscillator (2.2) /16,17/. In fact, if one introduces

$$M_+ = \Lambda A^+, \quad M_- = -\Lambda A, \quad M_3 = \frac{1}{\hbar\omega}H \tag{2.32}$$

and makes use the commutation relations (2.21), then it can be verified that the operators M_+, M_- and M_3 define the Lie algebra

$$[M_+, M_-] = 2M_3, \quad [M_3, M_\pm] = \pm M_\pm, \tag{2.33}$$

i.e., they are the generators of the group $SU(1,1)$. The direct calculation of the Casimir operator

$$M^2 = M_3^2 + \frac{1}{2}\left(M_+M_- + M_-M_+\right) \tag{2.34}$$

both in the x– and in the p–representation shows that it is equal to

$$M^2 = \left(\frac{mc^2}{\hbar\omega}\right)^2 I = \left(\mu^2 - \frac{1}{4}\right) I \,. \tag{2.35}$$

For the eigenvalues of the invariant operator M^2 the notation $S(S+1)$ is usually employed and a representation of the group SU(1,1) is characterized by a number S. From (2.35) it follows that in this case S can take two values: $S_1 = -\left(\mu + \frac{1}{2}\right)$ and $S_2 = \mu - \frac{1}{2}$. Since $\mu = \sqrt{\frac{1}{4} + \left(\frac{mc^2}{\hbar\omega}\right)^2} \geq \frac{1}{2}$, then $S_2 \geq 0$ and has to be discarded as the corresponding representation is nonunitary. The first value $S_1 = -\left(\mu + \frac{1}{2}\right)$ determines the representation $D^+\left(-\mu - \frac{1}{2}\right)$, which is characterized by the fact, that the eigenvalues of the operator $M_3 = \frac{1}{\hbar\omega} H$ are bounded below and equal to $-S_1 + n = \mu + \frac{1}{2} + n$, $n = 0, 1, 2, \ldots$

Thus, we obtain correct spectrum of the operator $H = \hbar\omega M_3$ and as in the nonrelativistic case the dynamical symmetry group of the relativistic linear oscillator (2.2) is the group SU(1,1). The functions $|n\rangle$, being defined by (2.11) and (2.15), are the basis functions of the infinite-dimensional irreducible unitary representation $D^+\left(-\mu - \frac{1}{2}\right)$ of the group SU(1,1) in p– and x–spaces respectively.

II. In constructing the second model the key role is played by the variable $k_p = 2mc\,\mathrm{sh}\frac{\chi_p}{2}$, which has the clear

geometrical meaning x) and coincides with nonrelativistic momentum when $c \to \infty$. The energy of motion $e_p = E_p - mc^2$ in terms of k_p has the nonrelativistic form $e_p = \frac{k_p^2}{2m}$. Therefore, it is natural to postulate the following equation for the linear oscillator in the momentum representation:

$$\left(\frac{k_p^2}{2m} - \frac{m\omega^2\hbar^2}{2}\frac{d^2}{dk_p^2} - \frac{k_q^2}{2m}\right)\psi_q(k_p) = 0 \,. \tag{3.1}$$

Thus, in the second model the quasipotential is the differential operator

$$V(k_p) = -\frac{m\omega^2\hbar^2}{2}\frac{d^2}{dk_p^2} = -\frac{\hbar^2\omega^2}{2mc^2}\frac{1}{\mathrm{ch}^2\frac{\chi_p}{2}}\left(\frac{d^2}{d\chi_p^2} - \frac{1}{2}\mathrm{th}\frac{\chi_p}{2}\frac{d}{d\chi_p}\right). \tag{3.2}$$

In this case we can use the fact, that the solutions of (3.1) are well known and expressed through the Hermite polynomials

$$\psi_n(k_p) = \frac{C_0}{\sqrt{2^n n!}} e^{-\frac{k_p^2}{2m\hbar\omega}} H_n\left(\frac{k_p}{\sqrt{m\hbar\omega}}\right) \tag{3.3}$$

$$e_n = \hbar\omega\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \ldots, \quad C_0 = \left(\frac{\hbar}{\pi m\omega}\right)^{\frac{1}{4}},$$

while the functions $\psi_n(k_p)$ satisfy the following orthogonality and completeness conditions:

$$\int_{-\infty}^{\infty} dk_p\, \psi_m^*(k_p)\psi_n(k_p) = \delta_{mn} \tag{3.4}$$

$$\sum_{n=0}^{\infty} \psi_n^*(k_p)\psi_n(k_q) = \delta(k_p - k_q) \,. \tag{3.5}$$

---

x) For the details see, for instance, /18/.

In the nonrelativistic limit the quasipotential (3.2), equation (3.1) and its solutions (3.3) go into the quasipotential, Schrödinger equation and its solutions for the nonrelativistic linear oscillator, respectively.

The quasipotential (3.2) in the x-representation is written as

$$V\left(x,\frac{d}{dx}\right)=\frac{\hbar^2\omega^2}{2mc^2}\left(ch\frac{i}{2}\frac{d}{d\tilde{x}}\right)^{-2}\left(\tilde{x}+\frac{i}{2}th\frac{i}{2}\frac{d}{d\tilde{x}}\right)\tilde{x}, \qquad (3.6)$$

where, by definition,

$$\left(ch\,i\alpha\frac{d}{dx}\right)^{-\kappa}=-i\lim_{\varepsilon\to 0}\int_0^\infty d\tau\, e^{-\tau\varepsilon+i\tau\left(ch\,i\alpha\frac{d}{dx}\right)^{\kappa}}, \quad \kappa=1,2,\dots \qquad (3.7)$$

For instance, the action of the operator (3.7) on an exponent is given by

$$\left(ch\,i\alpha\frac{d}{dx}\right)^{-\kappa}e^{\pm i x \chi}=\left(ch\,\alpha\chi\right)^{-\kappa}e^{\pm i x\chi}.$$

The finite difference analogue of equation (3.1) has in x-representation the form

$$\left[H_0-E_q+\frac{\hbar^2\omega^2}{2mc^2}\left(ch\frac{i}{2}\frac{d}{d\tilde{x}}\right)^{-2}\left(\tilde{x}+\frac{i}{2}th\frac{i}{2}\frac{d}{d\tilde{x}}\right)\tilde{x}\right]\psi_q(\tilde{x})=0, \qquad (3.8)$$

or, after multiplying from the left by $\left(ch\frac{i}{2}\frac{d}{d\tilde{x}}\right)^3$,

$$\left\{\left(ch\frac{i}{2}\frac{d}{d\tilde{x}}\right)^3(H_0-E_q)+\frac{\hbar^2\omega^2}{2mc^2}\left[\left(\tilde{x}^2-\frac{1}{2}\right)ch\frac{i}{2}\frac{d}{d\tilde{x}}+\frac{3}{2}i\tilde{x}\,sh\frac{i}{2}\frac{d}{d\tilde{x}}\right]\right\}\psi_q(\tilde{x})=0. \qquad (3.9)$$

Equation (3.1) in the x-representation can also be written in the integral form

$$\left(H_0-E_q\right)\psi_q(x)+\int_{-\infty}^{\infty}V(x,x')\psi_q(x')dx'=0. \qquad (3.10)$$

According to (1.5) and (3.2), the following integral representation for the quasipotential $V(x,x')$ is valid:

$$V(x,x')=\frac{\hbar^2\omega^2}{4\pi c}\int_{-\infty}^{\infty}e^{i\tilde{x}\chi}\frac{1}{ch^2\frac{\chi}{2}}\left(\frac{d^2}{d\chi^2}-\frac{1}{2}th\frac{\chi}{2}\frac{d}{d\chi}\right)e^{-i\tilde{x}'\chi}d\chi. \qquad (3.11)$$

Hence we obtain, after performing the integration, that the function $V(x,x')$ is nonlocal and is equal to

$$V(x,x')=\frac{\hbar\omega^2}{c}\frac{\check{x}\check{x}'(\check{x}-\check{x}')}{sh\,\pi(\check{x}-\check{x}')}. \qquad (3.12)$$

In the Appendix III it is shown that in the nonrelativistic region the function (3.12) takes the local form:

$$V(x,x')\xrightarrow[c\to\infty]{}\frac{m\omega^2}{2}x^2\delta(x-x'). \qquad (3.13)$$

The comparison of two different forms (3.9) and (3.10) of equation (3.1) in the x-representation leads to

$$\left(ch\frac{i}{2}\frac{d}{d\tilde{x}}\right)^3V(x,x')=\frac{\hbar^2\omega^2}{2mc^2}\left[\left(\tilde{x}^2-\frac{1}{2}\right)ch\frac{i}{2}\frac{d}{d\tilde{x}}+\frac{3}{2}i\tilde{x}\,sh\frac{i}{2}\frac{d}{d\tilde{x}}\right]\delta(\tilde{x}-\tilde{x}'), \qquad (3.14)$$

i.e., the quasipotential (3.12) satisfies the inhomogeneous difference equation.

It is necessary to note, that since the formal action of the difference differentiation operators entering (3.14) is accompanied by the extension to the complex X-plane, the action of these operators on δ-function of the real argument is defined, as usually, by the corresponding representation in the form of infinite series. For instance,

$$e^{i\frac{d}{dx}}\delta(x-x')=\sum_{n=0}^{\infty}\frac{i^n}{n!}\delta^{(n)}(x-x'). \qquad (3.15)$$

Now we find the wave functions (3.3) in the configurational representation. According to (1.2) for the ground state wave function

$$\psi_0(x)=\frac{c_0 mc}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{\infty}d\chi\, \exp\left(i\tilde{x}\chi-\frac{k_0^2}{2m\hbar\omega}\right)=\frac{c_0 mc}{\sqrt{2\pi\hbar}}\int\limits_{-\infty}^{\infty}d\chi\, \exp\left(i\tilde{x}\chi-\Lambda^2 \mathrm{sh}^2\frac{\chi}{2}\right) \qquad (3.16)$$

with the help of the integral representation for the Macdonald function

$$K_\nu(z)=\frac{1}{2}\int\limits_{-\infty}^{\infty}e^{-z\,\mathrm{ch}\,t+\nu t}\,dt \qquad (3.17)$$

we get

$$\psi_0(x)=\sqrt{\frac{2}{\pi\hbar}}\,c_0 mc\, e^{\frac{\Lambda^2}{2}}K_{i\tilde{x}}\left(\frac{\Lambda^2}{2}\right). \qquad (3.18)$$

The excited state wave functions ($n=1,2,3,\ldots$)

$$\psi_n(x)=\frac{c_0 mc}{\sqrt{2^{n+1}\pi\hbar\, n!}}\int\limits_{-\infty}^{\infty}d\chi\, e^{i\tilde{x}\chi-\Lambda^2\mathrm{sh}^2\frac{\chi}{2}}\,H_n\left(\sqrt{2}\Lambda\,\mathrm{sh}\frac{\chi}{2}\right) \qquad (3.19)$$

can be written in the form

$$\psi_n(x)=\frac{(-1)^n}{\sqrt{2^n n!}}H_n\left(\sqrt{2}\Lambda\,\mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}\right)\psi_0(x), \qquad (3.20)$$

if one takes into account the formula

$$\left(\mathrm{sh}\, i\alpha\frac{d}{d\tilde{x}}\right)^n e^{i\tilde{x}\chi}=(-1)^n(\mathrm{sh}\,\alpha\chi)^n e^{i\tilde{x}\chi}. \qquad (3.21)$$

Then using the explicit form of the Hermite polynomials

$$H_n(x)=n!\sum_{m=0}^{[\frac{n}{2}]}\frac{(-1)^m(2x)^{n-2m}}{m!(n-2m)!}, \quad \left[\frac{n}{2}\right]=\begin{cases}\frac{n}{2} & \text{for even } n\\ \frac{n-1}{2} & \text{for odd } n\end{cases} \qquad (3.22)$$

and the integral representation (3.16) for the ground state wave function $\psi_0(x)$, we obtain that

$$\psi_n(x)=\frac{(-1)^n}{2^{\frac{n-1}{2}}}\sqrt{\frac{n!}{\pi\hbar}}\,c_0 mc\, e^{\frac{\Lambda^2}{2}}\sum_{k=0}^{[\frac{n}{2}]}\frac{(-1)^k(2^{\frac{3}{2}}\Lambda)^{n-2k}}{k!(n-2k)!}\sum_{s=0}^{n-2k}(-1)^s C^s_{n-2k}K_{i\tilde{x}+k+s-\frac{n}{2}}\left(\frac{\Lambda^2}{2}\right) \qquad (3.23)$$

From the orthogonality condition (3.4) for the wave functions in the p-representation it follows, that in the x-representation they satisfy the following "nonlocal" orthogonality condition

$$\int\limits_{-\infty}^{\infty}dx\,\psi_n^*(x)\,\mathrm{ch}\frac{i\hbar}{2mc}\frac{d}{dx}\psi_m(x)=\delta_{nm} \qquad (3.24)$$

or

$$\frac{1}{2}\int\limits_{-\infty}^{\infty}dx\,\psi_n^*(x)\left[\psi_m\left(x-\frac{i\hbar}{2mc}\right)+\psi_m\left(x+\frac{i\hbar}{2mc}\right)\right]=\delta_{nm}. \qquad (3.25)$$

The nonunitarity of the orthogonality condition for the functions $\psi_n(x)$ is bound up with the fact that the quasipotential (3.2) is nonhermitian with respect to the scalar product, defined by the volume element $d\Omega_p=mc\,d\chi_p$.

Let us also write the completeness condition (3.5) in the x-representation, which has the form

$$\sum_{n=0}^{\infty}\psi_n^*(x)\,\mathrm{ch}\left(\frac{i\hbar}{2mc}\frac{d}{dx'}\right)\psi_n(x')=\delta(x-x'), \qquad (3.26)$$

or

$$\frac{1}{2}\sum_{n=0}^{\infty}\psi_n^*(x)\left[\psi_n\left(x'+\frac{i\hbar}{2mc}\right)+\psi_n\left(x'-\frac{i\hbar}{2mc}\right)\right]=\delta(x-x') \qquad (3.27)$$

It is easy to verify, that $\psi_n(x)$ have the correct nonrelativistic limit:

$$\psi_n(x)\longrightarrow\frac{c_0'}{\sqrt{2^n n!}}e^{-\frac{m\omega}{2\hbar}x^2}H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right),\quad c_0'=\left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{4}}. \qquad (3.28)$$

Indeed, the nonrelativistic limit of the ground state wave function (3.18) is obtained with the aid of the asymptotic representation for the function

$$K_{i\nu}(x) \simeq \frac{\sqrt{\pi}}{\sqrt[4]{x^2-\nu^2}} \exp\left(-\sqrt{x^2-\nu^2} - \nu\cdot \arcsin\frac{\nu}{x}\right), \qquad (3.29)$$

which is valid in the case $x > \nu > 0$ and $\nu \to \infty$. To find the nonrelativistic limit of the excited state wave functions, it is more suitable to start from the representation (3.21). Taking into account that when $c \to \infty$ the relation $\sqrt{2}\Lambda \mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}} \to i\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}$ does hold, we get

$$\psi_n(x) \longrightarrow \frac{(-1)^n C_0^{n\prime}}{\sqrt{2^n n!}} H_n\left(i\sqrt{\frac{\hbar}{m\omega}}\frac{d}{dx}\right) e^{-\frac{m\omega}{2\hbar}x^2} \qquad (3.30)$$

Now using (3.22) and the formula

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} \qquad (3.31)$$

for the Hermite polynomials, we obtain

$$\psi_n(x) \longrightarrow c_0^{\prime n} i^n \sqrt{n!}\, e^{-\frac{m\omega}{2\hbar}x^2} \sum_{k=0}^{[\frac{n}{2}]} \frac{H_{n-2k}\left(x\sqrt{\frac{m\omega}{2\hbar}}\right)}{2^k\cdot k!\,(n-2k)!} . \qquad (3.32)$$

Since for the Hermite polynomials the following relation is valid (for a proof see Appendix IV)

$$2^{\frac{n}{2}} n! \sum_{k=0}^{[\frac{n}{2}]} \frac{H_{n-2k}\left(x\sqrt{\frac{m\omega}{2\hbar}}\right)}{2^k\cdot k!(n-2k)!} = H_n\left(x\sqrt{\frac{m\omega}{\hbar}}\right) \qquad (3.33)$$

from (3.32) it follows (3.28).

Coherent states. For constructing the c.s. of the model under consideration (3.2), we introduce the annihilation and creation operators

$$a = \sqrt{\frac{m\hbar\omega}{2}}\left(\frac{k_p}{m\hbar\omega} + \frac{d}{dk_p}\right), \quad a^+ = \sqrt{\frac{m\hbar\omega}{2}}\left(\frac{k_p}{m\hbar\omega} - \frac{d}{dk_p}\right), \qquad (3.34)$$

satisfying the commutation relation $[a, a^+] = 1$. Let us express them through the variable χ :

$$a = \Lambda \mathrm{sh}\frac{\chi}{2} + \frac{1}{\Lambda \mathrm{ch}\frac{\chi}{2}}\frac{d}{d\chi}, \quad a^+ = \Lambda \mathrm{sh}\frac{\chi}{2} - \frac{1}{\Lambda \mathrm{ch}\frac{\chi}{2}}\frac{d}{d\chi} . \qquad (3.35)$$

It is evident that when $c \to \infty$ they go into the annihilation and creation operators of the nonrelativistic linear oscillator, respectively.

We find the explicit form of the operators (3.35) in the x-representation. Here, as in the case of the quasipotential (3.2), it is possible to write two forms: The first

$$a_x = -\left[\Lambda \mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}} + \frac{i}{\Lambda}\left(\mathrm{ch}\frac{i}{2}\frac{d}{d\tilde{x}}\right)^{-1}\tilde{x}\right], \quad a_x^+ = -\left[\Lambda \mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}} - \frac{i}{\Lambda}\left(\mathrm{ch}\frac{i}{2}\frac{d}{d\tilde{x}}\right)^{-1}\tilde{x}\right], \qquad (3.36)$$

and the second

$$a_x F(x) = -\Lambda \mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}F(x) - \frac{i}{\Lambda}\int_{-\infty}^{\infty}\frac{\tilde{x}' F(x')\,d\tilde{x}'}{\mathrm{ch}\,\pi(\tilde{x}-\tilde{x}')},$$

$$a_x^+ F(x) = -\Lambda \mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}F(x) + \frac{i}{\Lambda}\int_{-\infty}^{\infty}\frac{\tilde{x}' F(x')\,d\tilde{x}'}{\mathrm{ch}\,\pi(\tilde{x}-\tilde{x}')} . \qquad (3.37)$$

The annihilation and creation operators (3.36) and (3.37) in the x-representation satisfy the commutation relation

$$[a_x, a_x^+] = 1 .$$

For the first form (3.36) this is easily verified and for the second one is proven in the Appendix V.

The transition from one form (3.36) to another (3.37) is accomplished with the aid of the relation x)

x) The action of the operator $\mathrm{ch}\frac{i}{2}\frac{d}{d\tilde{x}}$ on the function $1/\mathrm{ch}\,\pi\tilde{x}$ is defined as a limit of $\mathrm{ch}\left\{\frac{1}{2}(i-\varepsilon)\frac{d}{d\tilde{x}}\right\}\frac{1}{\mathrm{ch}\,\pi\tilde{x}}$ when $\varepsilon \to +0$

$$
\mathrm{ch}\frac{i}{2}\frac{d}{dx}\frac{1}{\mathrm{ch}\,\pi(x-x')} = \frac{1}{2\pi}\,\mathrm{ch}\frac{i}{2}\frac{d}{dx}\int_{-\infty}^{\infty}\frac{e^{i(x-x')\chi}\,d\chi}{\mathrm{ch}\frac{\chi}{2}} = \delta(x-x') \;. \tag{3.38}
$$

We note that in the nonrelativistic limit

$$
a_x \longmapsto -i\sqrt{\frac{\hbar}{m\omega}}\left(\frac{d}{dx}+\frac{m\omega}{\hbar}x\right), \qquad a_x^{+} \longrightarrow -i\sqrt{\frac{\hbar}{m\omega}}\left(\frac{d}{dx}-\frac{m\omega}{\hbar}x\right) . \tag{3.39}
$$

For the representation (3.38) it is also evident and for the representation (3.39) it is shown in the Appendix III.

Having defined the operators a and a^{+}, the c.s. are construoted in the standard way – as the eigenvalues of the operator a :

$$
a|\alpha\rangle = \alpha|\alpha\rangle , \tag{3.40}
$$

where α is an arbitrary complex number. The state $|\alpha\rangle$ is connected with the ground state $\psi_0 \equiv |0\rangle$ of the considered relativistic oscillator by the Weyl unitary operator $D(\alpha) = \exp(\alpha a^{+} - \alpha^{*}a)$ /6/ , i.e.,

$$
|\alpha\rangle = D(\alpha)|0\rangle \; . \tag{3.41}
$$

From here the representation also follows, which aotually is the expansion of $|\alpha\rangle$ in terms of the oscillator states,

$$
|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|n\rangle , \tag{3.42}
$$

where the Dirac notation $\psi_n \equiv |n\rangle$ for the state vector is used.

The explicit form of the c.s. in the p- and x-representations can be obtained with the help of the generating function of the Hermite polynomials

$$
\sum_{n=0}^{\infty}\frac{z^n}{n!}H_n(x) = e^{2xz-z^2} . \tag{3.43}
$$

In the p-representation they have the form

$$
\langle \kappa_p|\alpha\rangle \equiv \psi(\chi,\alpha) = c_0\exp\left\{-\frac{1}{2}\left(\alpha^2+|\alpha|^2\right)-\frac{\kappa_p^2}{2m\hbar\omega}+\alpha\sqrt{\frac{2}{m\hbar\omega}}\,\kappa_p\right\} , \tag{3.44}
$$

while in the x-representation they are given by

$$
\langle x|\alpha\rangle \equiv \psi(x,\alpha) = \exp\left\{-\frac{1}{2}\left(\alpha^2+|\alpha|^2\right)-2\alpha\Lambda\,\mathrm{sh}\frac{i}{2}\frac{d}{d\check{x}}\right\}\psi_0(x) . \tag{3.45}
$$

Calculating in the last formula the action of the finite-difference operator $\mathrm{sh}\frac{i}{2}\frac{d}{d\check{x}}$ on $\psi_0(x)$, we get

$$
\psi(x,\alpha) = \sqrt{\frac{2}{\pi}}\,c_0 mc\, e^{\frac{1}{2}(\Lambda^2-\alpha^2-|\alpha|^2)}\sum_{n=0}^{\infty}\frac{(-\alpha\Lambda)^n}{n!}\sum_{k=0}^{n}(-1)^k C_n^k\, K_{i\check{x}+k-\frac{n}{2}}\left(\frac{\Lambda^2}{2}\right) \tag{3.46}
$$

It is easy to verify that the c.s. in the x-representation are the eigenstates of the annihilation operator a_x both in the form (3.36) and in the form (3.37), i.e.,

$$
a_x\psi(x;\alpha) = \alpha\,\psi(x,\alpha) . \tag{3.47}
$$

On the other hand, if we make use of formula (3.37), we obtain the following integral relation for the c.s. in the x-representation:

$$
\int_{-\infty}^{\infty}\frac{\check{x}'\,\psi(x';\alpha)\,d\check{x}'}{\mathrm{ch}\,\pi(\check{x}-\check{x}')} = i\Lambda\left(\Lambda\,\mathrm{sh}\frac{i}{2}\frac{d}{d\check{x}}+\alpha\right)\psi(x,\alpha) . \tag{3.48}
$$

In the particular case when $\alpha = 0$ from (3.48) we get for the ground state $\psi_0(x)$:

$$\int_{-\infty}^{\infty}\frac{\tilde{x}'\psi_0(x')d\tilde{x}'}{\mathrm{ch}\,\pi(\tilde{x}-\tilde{x}')} = i\Lambda^2\,\mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}\psi_0(x). \qquad (3.49)$$

With the aid of (3.20) this relation can be rewritten in the form

$$\int_{-\infty}^{\infty}\frac{\tilde{x}'K_{i\tilde{x}'}(\frac{\Lambda^2}{2})d\tilde{x}'}{\mathrm{ch}\,\pi(\tilde{x}-\tilde{x}')} = i\Lambda^2\,\mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}K_{i\tilde{x}}\left(\frac{\Lambda^2}{2}\right) = \frac{i\Lambda^2}{2}\left[K_{i\tilde{x}-\frac{i}{2}}\left(\frac{\Lambda^2}{2}\right) - K_{i\tilde{x}+\frac{i}{2}}\left(\frac{\Lambda^2}{2}\right)\right]. \qquad (3.50)$$

Formula (3.50) is easily reduced to the Kontorowich-Lebedev transformation, which is often used in solving the boundary value problems:

$$g(t) = \int_0^\infty f(\tau)K_{i\tau}(t)d\tau, \qquad (3.51)$$

$$f(\tau) = \frac{2}{\pi^2}\tau\,\mathrm{sh}\,\pi\tau\int_0^\infty g(t)K_{i\tau}(t)\frac{dt}{t}.$$

As it follows from (3.50) and (3.51) in our case

$$f(\tilde{x}') = \frac{\tilde{x}'\,\mathrm{sh}\,\pi\tilde{x}'}{\mathrm{ch}\,2\pi\tilde{x} + \mathrm{ch}\,2\pi\tilde{x}'}, \quad g\left(\frac{\Lambda^2}{2}\right) = \frac{i\Lambda^2}{4\,\mathrm{sh}\,\pi\tilde{x}}\,\mathrm{sh}\frac{i}{2}\frac{d}{d\tilde{x}}K_{i\tilde{x}}\left(\frac{\Lambda^2}{2}\right). \qquad (3.52)$$

The c.s. (3.44) and (3.45) form an overcomplete system of functions, but they are nonorthogonal. The scalar product and the completeness condition for the c.s. in the p-representation are written in the following form:

$$\int_{-\infty}^{\infty}\psi^*(\chi_p;\alpha)\psi(\chi_p;\beta)dk_p = e^{\alpha^*\beta - \frac{1}{2}(|\alpha|^2+|\beta|^2)},$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}d^2\alpha\,\psi^*(\chi_p;\alpha)\psi(\chi_p';\alpha) = \frac{\delta(\chi_p-\chi_p')}{mc\,\mathrm{ch}\frac{\chi_p}{2}} = \delta(k_p-k_p'), \quad d^2\alpha = d\alpha_1 d\alpha_2 \qquad (3.53)$$

In the x-representation the scalar product and the completeness condition for the c.s., as well as the orthogonality (3.24) and completeness (3.26) conditions for the wave functions (3.21), have the nonlocal form:

$$\int_{-\infty}^{\infty}dx\,\psi^*(x;\alpha)\,\mathrm{ch}\frac{i\hbar}{2mc}\frac{d}{dx}\psi(x;\beta) = e^{\alpha^*\beta - \frac{1}{2}(|\alpha|^2+|\beta|^2)} \qquad (3.54)$$

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\psi^*(x;\alpha)\,\mathrm{ch}\frac{i\hbar}{2mc}\frac{d}{dx'}\psi(x';\alpha)d^2\alpha = \delta(x-x').$$

It is clear that the functions (3.44) and (3.45) in the p- and x-representations respectively, when $c\to\infty$ go into the c.s. of the nonrelativistic linear oscillator

$$\psi(\chi;\alpha)\xrightarrow[c\to\infty]{}\tilde{\psi}(p;\tilde{\alpha}) = \frac{1}{(\pi m\hbar\omega)^{1/4}}\exp\left\{-\frac{1}{2}(\tilde{\alpha}^2+|\tilde{\alpha}|^2) - \frac{p^2}{2m\hbar\omega} + \tilde{\alpha}\sqrt{\frac{2}{m\hbar\omega}}\,p\right\}, \qquad (3.55)$$

$$\psi(x;\alpha)\xrightarrow[c\to\infty]{}\tilde{\psi}(x;\tilde{\alpha}) = \left(\frac{m\omega}{2\hbar}\right)^{\frac{1}{4}}\exp\left\{\frac{1}{2}(\tilde{\alpha}^2-|\alpha|^2) - \frac{m\omega}{2\hbar}x^2 + i\sqrt{\frac{2m\omega}{\hbar}}\,\tilde{\alpha}x\right\}, \qquad (3.56)$$

where $\tilde{\alpha}$ is given by

$$\tilde{\alpha} = \lim_{c\to\infty}\alpha = \frac{\overline{p}}{\sqrt{2m\hbar\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}\,\overline{x}. \qquad (3.57)$$

The dynamical symmetry group for the second model (3.2) of relativistic oscillator, generating its energy spectrum, is also the group SU(1,1). This follows from the fact that the total Hamiltonian for this model is represented in the form

$$H = \hbar\omega\left(a^+a + \frac{1}{2} + \frac{mc^2}{\hbar\omega}\right) \qquad (3.58)$$

both in the p- and x-representations, the corresponding expressions for the annihilation a and creaction a^+ operators being defined by formulae (3.34) and (3.36). The generators of the group SU(1,1), satisfying the commutation relations (2.29), are expressed through the operators a and a^+ in the standard way /16,17/ :

$$M_+ = \frac{1}{2}(a^+)^2, \quad M_- = -\frac{1}{2}a^2, \quad M_3 = \frac{1}{2}\left(aa^+ + \frac{1}{2}\right) = \frac{H - mc^2}{2\hbar\omega}. \qquad (3.59)$$

Calculating by formula (2.30) the invariant Casimir operator, we obtain that $M^2 = s(s+1) = -\frac{3}{16}$. Consequently, the representations of the group SU(1,1) are characterized by two negative numbers $s_1 = -\frac{1}{4}$ and $s_2 = -\frac{3}{4}$, to which there correspond the irreducible unitary representations $D^+(-\frac{1}{4})$ and $D^+(-\frac{3}{4})$. The eigenvalues of the operator M_3 are bounded below and equal to

1) in the case of $D^+(-\frac{1}{4})$ representation: $-s_1 + n = \frac{1}{4} + n$, i.e. $H = mc^2 + \hbar\omega(2n + \frac{1}{2})$;
2) in the case of $D^+(-\frac{3}{4})$ representation : $-s_2 + n = \frac{3}{4} + n$, i.e $H = mc^2 + \hbar\omega(2n + 1 + \frac{1}{2})$.

The functions $|n\rangle$, which are defined by formulae (3.3) and (3.23), for $n = 2k$ $(k = 0,1,2,\ldots)$ form the basis of the irreducible unitary representation $D^+(-\frac{1}{4})$ whereas for $n = 2k + 1$ $(k = 0,1,2,\ldots)$ of the irreducible unitary representation $D^+(-\frac{3}{4})$ of the group SU(1,1) in p- and x-spaces, respectively.

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APPENDIX

I. The direct calculation of the nonrelativistic limit of the wave functions, defined by relations (2.10) and (2.11), is a rather complicated problem. Therefore, we will proceed as follows. Making use of the representation (2.20) and the recurrence formula for the generalized Laguerre polynomials

$$(2n + 1 + \alpha - x)L_n^{\alpha}(x) = (n+1)L_{n+1}^{\alpha}(x) + (n+\alpha)L_{n-1}^{\alpha}(x)$$

it is easy to show that

$$\sqrt{(n+1)(n+1+2\mu)}\,\psi_q(\zeta;n+1) = i(2n+1+2\mu-\zeta)\psi_q(\zeta;n) + \sqrt{n(n+2\mu)}\,\psi_q(\zeta;n-1). \qquad (A.I)$$

From (1.3) it follows that $\chi_p \simeq \frac{p}{mc}$ as $c \to \infty$, i.e., the variable $\zeta = \frac{2mc^2}{\hbar\omega}e^{\chi_p} \simeq \frac{2mc^2}{\hbar\omega}\left\{1 + \frac{p}{mc} + \cdots\right\}$. Since the parameter $\mu = \sqrt{\frac{1}{4} + \left(\frac{mc^2}{\hbar\omega}\right)^2} \simeq \frac{mc^2}{\hbar\omega}$, the recurrence relation (A.1) when $c \to \infty$ takes the form

$$\sqrt{n+1}\,\psi_q^{(\infty)}(\zeta;n+1) = \sqrt{n}\,\psi_q^{(\infty)}(\zeta;n-1) - i\sqrt{2}\,\eta\,\psi_q^{(\infty)}(\zeta;n), \qquad (A.2)$$

where the dimensionless quantity η is equal to $\frac{p}{\sqrt{m\hbar\omega}}$. The solution of (A.2) is defined up to an arbitrary function of η and is expressed through the Hermite polynomial

$$\psi_q^{(\infty)}(\zeta;n) = \frac{(-i)^n\,\varkappa(\eta)H_n(\eta)}{\sqrt{2^n\,n!}}. \qquad (A.3)$$

Since $H_0(\eta) = 1$, from (A.3) it follows that the function $\varkappa(\eta)$ describes the asymptotic behaviour of the ground state $\psi_q(\zeta;0)$ when $c \to \infty$. Therefore, with the help of the representation

$$\Gamma(z) \simeq \sqrt{\frac{2\pi}{z}}\left(\frac{z}{e}\right)^z\left\{1 + \frac{1}{12z} + \cdots\right\}$$

for the gamma function at large values of $|z|$, from (2.10) we obtain

$$\varkappa(\eta) = \Psi_q^{(\infty)}(\xi;0) = \frac{mc^2}{\sqrt{\hbar}} \frac{e^{-\frac{\eta^2}{2}}}{(\pi m\hbar\omega)^{1/4}}$$

Thus, when $c \longrightarrow \infty$ the functions $\Psi_q(\xi;n)$ really go into the wave functions $a_n(p)$ of the nonrelativistic linear oscillator in the p-representation:

$$\Psi_q^{(\infty)}(\xi;n) = \frac{mc^2}{\sqrt{\hbar}}\, a_n(p) = \frac{mc^2}{\sqrt{\hbar}} \frac{e^{-\frac{\eta^2}{2}}}{(\pi m\hbar\omega)^{1/4}} \frac{(-i)^n H_n(\eta)}{\sqrt{2^n \cdot n!}} . \qquad (A.4)$$

In the same way the nonrelativistic limit is obtained in the x-representaion:

$$\Psi_q^{(\infty)}(x;n) = \Psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac14} \frac{H_n(\xi)\,e^{-\frac{\xi^2}{2}}}{\sqrt{2^n \cdot n!}}, \qquad \xi = \sqrt{\frac{m\omega}{\hbar}}\, x . \qquad (A.5)$$

II. To find the nonrelativistic limit of the normalization constant $C_\gamma = (\Lambda|\gamma|)^{\mu}\{\Gamma(2\mu+1)\,I_{2\mu}(2\Lambda|\gamma|)\}^{-1/2}$, it is necessary to know the asymptotic behaviour of the modified Bessel function of the first kind

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m+\nu}}{m!\,\Gamma(m+\nu+1)} = \frac{z^\nu}{2^\nu \Gamma(\nu)} \sum_{m=0}^{\infty} \frac{\left(\frac{z}{2}\right)^{2m}}{m!\,\nu(\nu+1)\cdots(\nu+m)} \qquad (A.\ 6\ )$$

in the case when both the argument $z = 2\Lambda|\gamma| = 2c\sqrt{\frac{2m}{\hbar\omega}}\,|\gamma|$ and the order $\nu = 2\mu \simeq \frac{2mc^2}{\hbar\omega} = \Lambda^2$ tend to infinity. Since the ratio of the argument squared to the order remains fixed, from (A.6) it follows that

$$I_{\Lambda^2}(2\Lambda x) = \frac{(\Lambda x)^{\Lambda^2}}{\Gamma(\Lambda^2)} \sum_{m=0}^{\infty} \frac{(\Lambda x)^{2m}}{m!\,\Lambda^2(\Lambda^2+1)\cdots(\Lambda^2+m)} = \frac{(\Lambda x)^{\Lambda^2}}{\Gamma(\Lambda^2)} \sum_{m=0}^{\infty} \frac{x^{2m}}{m!\,\Lambda^2(1+\frac{1}{\Lambda^2})\cdots(1+\frac{m}{\Lambda^2})} \simeq$$

$$\underset{\Lambda\to\infty}{\simeq} \frac{(\Lambda x)^{\Lambda^2}}{\Gamma(\Lambda^2+1)} \sum_{m=0}^{\infty} \frac{x^{2m}}{m!} = \frac{(\Lambda x)^{\Lambda^2}}{\Gamma(\Lambda^2+1)}\, e^{x^2} . \qquad (A.7)$$

Therefore, when $c \longrightarrow \infty$ the normalization constant tends to

$$C_\gamma \simeq (\Lambda|\gamma|)^{\frac{\Lambda^2}{2}} \{\Gamma(\Lambda^2+1)\, I_{\Lambda^2}(2\Lambda|\gamma|)\}^{-\frac12} \simeq e^{-\frac{|\gamma|^2}{2}} . \qquad (A.8)$$

III. Now we will show that the nonlocal quasipotential $V(x,x')$ (3.12), as well as the annihilation and creation operators (3.37), have the correct nonrelativistic limit. For that it is sufficient to verify that the following formulae are valid:

$$\lim_{\alpha\to\infty} \frac{2\alpha^2 x}{\operatorname{sh}\pi\alpha x} = \delta(x) , \qquad (A.9)$$

$$\lim_{\alpha\to\infty} \frac{\alpha}{\operatorname{ch}\pi\alpha x} = \delta(x) . \qquad (A.10)$$

In fact, firstly

$$\int_{-\infty}^{\infty} \frac{2\alpha^2 x}{\operatorname{sh}\pi\alpha x}\, dx = \int_{-\infty}^{\infty} \frac{\alpha\, dx}{\operatorname{ch}\pi\alpha x} = 1 . \qquad (A.11)$$

Besides, for an arbitrary infinitely-differentiable function

$$F(x) = \sum_{n=0}^{\infty} F^{(n)}(0)\, \frac{x^n}{n!} \qquad (A.12)$$

the relations do hold

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} F(x) \frac{2\alpha^2 x}{\operatorname{sh}\pi\alpha x}\, dx = \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \frac{\alpha F(x)}{\operatorname{ch}\pi\alpha x}\, dx = F(0) \qquad (A.13)$$

since for any $n \geqslant 1$

$$\lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \frac{2\alpha^2 x^{2n+1}\, dx}{\operatorname{sh}\pi\alpha x} = \lim_{\alpha\to\infty} \int_{-\infty}^{\infty} \frac{\alpha x^{2n}\, dx}{\operatorname{ch}\pi\alpha x} = 0 . \qquad (\ A.14)$$

Thus, when $\alpha \longrightarrow \infty$ the functions $\frac{2\alpha^2 x}{\operatorname{sh}\pi\alpha x}$ and $\frac{\alpha}{\operatorname{ch}\pi\alpha x}$ really coincide with the function $\delta(x)$.

IV. Here we will justify the summation formula for the Hermite polynomials, which has been used in the text

$$n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(2\sqrt{\alpha})^{n-2k}}{k!(n-2k)!}H_{n-2k}(x\sqrt{\alpha})=(4\alpha-1)^{\frac{n}{2}}H_n\left(\frac{2\alpha x}{\sqrt{4\alpha-1}}\right). \qquad (A.15)$$

To this end let us consider the expression

$$Z_n(x)=H_n\left(i\frac{d}{dx}\right)e^{-\alpha x^2} \qquad (A.16)$$

and construct the generating function for Z_n with the help of formula (3.43)

$$I(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}Z_n(x)=e^{-t^2+2it\frac{d}{dx}}\cdot e^{-\alpha x^2}.$$

Then, after some transformation

$$I(t)=\exp\left\{\frac{\alpha x^2}{4\alpha-1}-(4\alpha-1)\left(\frac{2\alpha x}{4\alpha-1}+it\right)^2\right\}.$$

Consequently

$$Z_n(x)=\frac{d^n}{dt^n}I(t)\Big|_{t=0}=i^n(4\alpha-1)^{\frac{n}{2}}e^{\frac{\alpha x^2}{4\alpha-1}}\frac{d^n}{d\eta^n}e^{-\eta^2}\Big|_{\eta=\frac{2\alpha x}{\sqrt{4\alpha-1}}} \qquad (A.17)$$

and taking now into account (3.31), from (A.17) we obtain

$$Z_n(x)=(-i)^n(4\alpha-1)^{\frac{n}{2}}H_n\left(\frac{2\alpha x}{\sqrt{4\alpha-1}}\right)e^{-\alpha x^2}. \qquad (A.18)$$

On the other hand, with the aid of formula (3.22) and the definition (A.16) of $Z_n(x)$, we have also

$$Z_n(x)=(-i)^n n!\sum_{k=0}^{[\frac{n}{2}]}\frac{(2\sqrt{\alpha})^{n-2k}}{k!(n-2k)!}H_{n-2k}(x\sqrt{\alpha})e^{-\alpha x^2}. \qquad (A.19)$$

The comparison of (A.18) and (A.19) gives us the sought formula (A.19). We note that when $\alpha\to\frac{1}{4}$ formula (A.15) goes into

$$n!\sum_{k=0}^{[\frac{n}{2}]}\frac{H_{n-2k}(\frac{x}{2})}{k!(n-2k)!}=x^n.$$

V. In this paragraph we show that the operators a_x and a_x^+ in the form (3.37) satisfy the standard commutation relation

$$[a_x,a_x^+]=1. \qquad (A.20)$$

The direct use of formulae (3.37) leads to

$$[a_x,a_x^+]F(x)=2i\int_{-\infty}^{\infty}\frac{\tilde{x}'d\tilde{x}'}{\operatorname{ch}\pi(\tilde{x}-\tilde{x}')}\operatorname{sh}\frac{i}{2}\frac{d}{d\tilde{x}'}F(x')-2i\operatorname{sh}\frac{i}{2}\frac{d}{d\tilde{x}}\int_{-\infty}^{\infty}\frac{\tilde{x}'F(x')d\tilde{x}'}{\operatorname{ch}\pi(\tilde{x}-\tilde{x}')} \qquad (A.21)$$

Let us prove that the right-hand side of (A.21) is equal to $F(x)$. To this end we substitute $F(x')$ in (A.21) by its Fourier-transform

$$F(x)=\frac{mc}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{i\tilde{x}\chi}F(\chi)d\chi. \qquad (A.22)$$

Then the integration over $\tilde{x}'$ gives for the commutator of the operators a_x and a_x^+ the following expression:

$$[a_x,a_x^+]F(x)=\sqrt{\frac{2}{\pi}}mc\int_{-\infty}^{\infty}\frac{e^{i\tilde{x}\chi}}{\operatorname{ch}\frac{\chi}{2}}d\chi\int_{-\infty}^{\infty}F(\chi')\left(\operatorname{sh}\frac{\chi}{2}-\operatorname{sh}\frac{\chi'}{2}\right)\frac{\partial}{\partial\chi'}\delta(\chi-\chi')d\chi'. \qquad (A.23)$$

Performing now the integration in (A.23) over χ' with the help of formula

$$\int_{-\infty}^{\infty}F(x')\frac{\partial}{\partial x'}\delta(x-x')dx'=-\frac{dF(x)}{dx} \qquad (A.24)$$

and taking again into account (A.22), we come to the conclusion that the right-hand side of (A.23) is really equal to $F(x)$

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