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ON THE NOTION OF WORLD LINES AND SCATTERING MATRIX FOR DIRECTLY INTERACTING RELATIVISTIC POINT PARTICLES

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О понятии мировой линии и матрицы рассеяния для взаимодействия релятивистских частиц

Изучается понятие мировой линии в формулировке задачи релятивистского взаимодействия точечных частиц в терминах гамильтоновой динамики со связями. Показано, что в случае нетривиального взаимодействия мировые линии зависят от выбора поверхности равных времен. Однако относительное движение двухчастичной системы и классическая матрица рассеяния не зависят от этого выбора.

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Molotkov V.V., Todorov I.T. **E2 - 12270**

On the Notion of World Lines and Scattering Matrix for Directly Interacting Relativistic Point Particles

The notion of world lines is studied in the constraint Hamiltonian formulation of relativistic point particle dynamics. The particle world lines are shown to depend (in the presence of interaction) on the choice of the equal time hyperline. However, the relative motion of a two-particle system and the (classical) S -matrix are independent of this choice. We infer that particle trajectories should not be regarded as strict observables in the classical theory of relativistic particles.

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Introduction

We study the notion of particle world lines in the relativistic phase space formulation of classical point particles' dynamics developed in /18/ on the basis of Dirac's theory of constraint Hamiltonian systems*) /4-6/.

The N-particle dynamics is determined by a set of N generalized mass-shell equations

$$\varphi_a = \frac{1}{2}(m_a^2 - p_a^2) + \phi_a(x_{12}, \ldots, x_{N-1N}; p_1, \ldots, p_N) = 0, \quad x_{ab} = x_a - x_b, \quad p_a^2 = (p_a^0)^2 - \underline{p}_a^2, \quad a = 1, \ldots, N,$$

which define a 7N dimensional surface $\mathcal{M}$ in the 8N dimensional "large phase space" Γ^N (for the spinless particle case considered here a point in Γ^N is given by N pairs of 4-vectors $\Gamma \ni \gamma = (x_1, p_1, \ldots, x_N, p_N)$). Here ϕ_a are Lorentz invariant functions subject to some conditions recapitulated in Sec. 1B. We mention here the important requirement that φ_a are first class constraints, which means that their Poisson brackets $\{\varphi_a, \varphi_b\}$ vanish on $\mathcal{M}$. The functions φ_a not only determine the generalized N-particle mass shell $\mathcal{M}$ but also generate N vector fields on which

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*) The constraint Hamiltonian approach to relativistic point particle interaction was also adopted (in fact, rediscovered) in /9/. Recent work by F.Rohrlich /15/ which follows a similar path, differs from ours in abandoning the notion of individual particle coordinates and trajectory (a generalized notion of "relative coordinates"- whose sum over all particles is not required to vanish - is used instead). As noted recently by Professor Rohrlich (private communication of October 1978) this difference is not essential: a slight modification of his approach allows one to impose a linear relation among the relative coordinates ξ_a of ref./15/ and hence define single particles' coordinates. A Lagrangian approach to the problem of relativistic point particle interactions which leads to similar constraint equations is being developed in the work of Takabayasi et al. (see /17,8/ and further references cited there).

the restriction $\omega|_{\mathcal{M}}$ of the symplectic form $\omega = \sum_{a=1}^{N} dx_a^\mu \wedge dp_{a\mu}$ on Γ^N is degenerate. The relativistic Hamiltonian is defined as a linear combination of φ_a (with γ-dependent coefficients) that leaves invariant some 3(N−1) dimensional space-like surface in the space of relative coordinates which will be called the "equal time surface". (An example of such a surface is the plane $n x_{ab} = 0$, where $x_{ab} = x_a - x_b$, $a, b = 1, \dots, N$ and n is a time-like vector which may depend on the momenta). The selection of an equal time surface, which excludes the unphysical relative time variables is analogous to specifying a gauge condition and will be also referred to in the sequel in such terms.

In Sec. 2A we introduce a notion of equivalent dynamics which says, essentially, that two sets of constraints $\varphi_a = 0$ and $\overline{\varphi}_a = 0$ are equivalent, if they lead to the same particle world lines (for the same gauge and initial conditions) and to the same realization of the Poincaré group. (Equivalent dynamics corresponds, in general, to different submanifolds $\mathcal{M}$ and $\overline{\mathcal{M}}$ of Γ^N). We prove (in Sec. 2B for the case $N=2$) that only straight world lines (corresponding to a free motion) are independent of the choice of the equal time surface (and hence, independent of the Lagrange multipliers λ_a in the Hamiltonian $H = \sum \lambda_a \varphi_a$). This statement agrees with recent results of Sokolov/16/, obtained in an alternative approach to the description of directly interacting relativistic point particles. It also provides a new interpretation of the so-called "no-interaction theorem" of Currie et al. /2,3,10/ (for a recent discussion see also/7,11/). It is demonstrated in Sec. 2C that in the 2-particle case, for $\phi_1 = \phi_2$, the relative motion (expressed in terms of the variables $x_\perp$ and p :

$$x_\perp = x\Pi \equiv x - \frac{xP}{s} P, \qquad x = x_1 - x_2, \quad P = p_1 + p_2, \quad s = P^2, \tag{0.1}$$

$$p = \mu_2 p_1 - \mu_1 p_2, \qquad \mu_1 + \mu_2 = 1, \quad \mu_1 - \mu_2 = \frac{m_1^2 - m_2^2}{s} \tag{0.2}$$

orthogonal to the total momentum P of the system) is gauge invariant and so is the 2-particle S-matrix.

An appendix includes a synopsis of some basic notions of differential geometry (such as a symplectic form and a vector field) used in the text, as well as a coordinate free formulation of the constraint Hamiltonian approach to relativistic point particle dynamics and of the main theorem (of Sec. 2B).

1. Constraint Hamiltonian Formulation of Relativistic N-Particle Dynamics

For reader's convenience we start with a brief recapitulation of the constraint dynamics approach to the relativistic N-body problem developed in ref./18/ (with a special attention to the case N=2).

A. Poincaré invariant symplectic structure on Γ^N. The free particles' mass shell

In order to avoid unnecessary complications, we shall only deal with <u>spinless</u> point particles in this paper. The general case of (massive) spinning particles is considered in ref./18/.

A manifestly covariant description of relativistic point particles' kinematics requires the use of 4-dimensional coordinates and momenta. We shall consider the space $\mathcal{M}$ of "physical" dynamical variables (including a separate time parameter for each particle) as a 7N-dimensional surface in the 8N dimensional extended phase space Γ^N (which is spanned by pseudoeuclidean particle coordinates $x_1, \dots, x_N$ and momenta $p_1, \dots, p_N$). For example the free particles' mass shell $\mathcal{M}_0$ is given by

$$\mathcal{M}_0 = \{(x_a, p_a) \in \Gamma_a = \mathbb{R}^8,\ a = 1, \dots, N;\ p_a^2 = m_a^2,\ p_a^0 > 0\ (\text{or } p_a^0 = \sqrt{m_a^2 + \vec{p}_a^2})\}.$$

The "large phase space" Γ^N can be regarded as a direct product of single particle spaces: $\Gamma^N = \Gamma_1 \times \dots \times \Gamma_N$. Each carries a natural action of the (connected) Poincaré group $\mathcal{P}$:

$$\mathcal{P} \ni (a, \Lambda): \quad (x, p) \to (\Lambda x + a, \Lambda p)$$

and an $Aut\,\mathcal{P}$ -invariant 1-form

$$d\Theta_a = -p_a dx_a \ (= -p_{a\mu} dx_a^\mu), \quad a = 1, \dots, N. \tag{1.1a}$$

(Here Aut $\mathcal{P}$ is the group of automorphisms of the Poincaré group which consists of Poincaré transformations and dilatations $(x, p) \to (\rho x, \frac{1}{\rho} p)$, $\rho > 0$.) Its differential is a symplectic form*) on Γ_a :

$$\omega_a = d\,\Theta_a = dx_a^\mu \wedge dp_{a\mu} \tag{1.1b}$$

(summation over the repeated Minkowski space vector index μ , but not over the particle index a , is to be carried out in the right-hand side of (1.1b)). Thus we can define a (Aut$\mathcal{P}\times \dots \times$ Aut$\mathcal{P}$ -invariant) symplectic form

$$\omega = \sum_{a=1}^{N} \omega_a = d\theta \qquad (\theta = \sum_a \theta_a) \tag{1.1c}$$

on Γ^N which gives rise to the canonical Poisson bracket relations

*) For reader's convenience we have summarized basic definitions and facts concerning symplectic forms and Poisson brackets in Appendix A1.

$$\{p_{a\nu}, x_b^\mu\} = \delta_{ab}\,\delta_\nu^\mu, \quad a, b = 1, \dots, N, \quad \mu, \nu = 0, 1, 2, 3 \tag{1.2}$$

(all other brackets among the basic phase space coordinates vanishing).

We shall assume (see Sec. 1B below) that the surface $\mathcal{M}$ is a Poincaré invariant submanifold of Γ^N. (This is obviously true for the free particle mass shell $\mathcal{M}_0$.) As a consequence $\mathcal{M}$ will inherit the diagonal action of the Poincaré group in Γ^N. Its infinitesimal generators are given by

$$P = p_1 + \dots + p_N, \tag{1.3}$$

$$M = x_1 \wedge p_1 + \dots + x_N \wedge p_N \tag{1.4}$$

where the wedge product $\wedge$ is defined, as usual, by

$$(x \wedge p)_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu . \tag{1.5}$$

For x_a, p_b satisfying the canonical Poisson bracket relations (1.2) P and M satisfy the Poisson bracket relations of the Poincaré Lie algebra.

We shall proceed further with the special case N=2.

If we take as relative momentum (conjugate to the relative coordinate $x = x_1 - x_2$) the variable p (0.2) then the centre of mass coordinate X conjugate to the total momentum P (and completing the set of 4-vectors p, P, x to a canonical quadruplet) should be taken as

$$X = \mu_1 x_1 + \mu_2 x_2 - \frac{\mu_1 - \mu_2}{s}(xP)P . \tag{1.6}$$

(Because of the s-dependence of the weights μ_a (0.2) the nonconventional last term in the right hand-side of (1.6) is required to ensure the vanishing of the Poisson brackets $\{X^\mu, X^\nu\}$ and $\{X^\mu, p_\nu\}$.)

The total angular momentum M (1.4) splits into a centre of mass (orbital) part and a relative (internal) part:

$$M = X_\wedge P + x_\wedge p . \tag{1.4'}$$

On the free particle mass shell $\mathcal{M}_0$ we have

$$-P^2 \approx b^2(s) = \frac{1}{4}\left[s - 2(m_1^2+m_2^2) + \frac{(m_1^2-m_2^2)^2}{s}\right] . \tag{1.7}$$

[The weak equality sign $\approx$ is used for equations that are only valid on the physical subspace $\mathcal{M}$ (which coincides here with $\mathcal{M}_0$) of the large phase space $\Gamma(=\Gamma_2)$]

We note finally that the relative momentum p (0.2) is orthogonal to P on the mass shell:

$$\varphi \equiv \frac{1}{2}(m_1^2 - m_2^2 - p_1^2 + p_2^2) = -pP \approx 0. \tag{1.8}$$

B. The generalized N-particle mass shell

The generalized mass shell $\mathcal{M}$ for N-interacting particles is defined, according to [18], by N constraints of the type

$$\varphi_a = \frac{1}{2}(m_a^2 - p_a^2) + \phi_a \approx 0, \quad a = 1, \dots, N. \tag{1.9}$$

For the purposes of this paper we shall need the following properties of the deviations ϕ_a from the free particle mass shell.

(i) Poincaré invariance: in order to simplify the discussion, we shall make the slightly stronger assumption that ϕ_a are functions of the scalar products of $x_{bc} = x_b - x_c$ and p_b .

(ii) Compatibility: the constraints φ_a are first class,

$$\{\varphi_a , \varphi_b\} \approx 0 . \tag{1.10}$$

(iii) Independence and time-like character: the φ_a's are functionally independent and Eqs.(1.9) can be solved for the particle energies, so that

$$d \equiv \det\left(n \frac{\partial \varphi_a}{\partial p_b}\right) > 0 \qquad (n^2 > 0, \quad n_0 > 0) \tag{1.11}$$

for any choice of the P-dependent time-like vector n; (the sign is chosen to fit the free particle case; if we set for N=2 $n = P$ /for $P \frac{\partial}{\partial p_b}\phi_a = 0$/ then (1.11) is satisfied for $s > |m_1^2 - m_2^2|$).

Conditions (ii) and (iii) along with some regularity properties are partly incorporated in the following mathematical assumption (see Appendix to ref. [5]) which will be also adopted:

(iv) Fibre bundle structure on $\mathcal{M}$: the 7N-dimensional manifold $\mathcal{M}$ is a fibre bundle with N-dimensional fibres spanned by the integral curves of the vector fields X_{φ_a} on which the form $\omega|_{\mathcal{M}}$ is degenerate (see Appendix A.1 as well as the discussion after Eq.(1.19) below). The 6N-dimensional base space Γ_* of $\mathcal{M}$ (whose points γ_* can be identified with the N-dimensional fibres in $\mathcal{M}$) plays the role of the physical phase space.

These requirements are supplemented in ref. [18] by the following additional assumption:

(v) Separability: the free particle mass shell is recovered for large space-time separation between the particles:

$$\lim_{-x^2_{ab\perp}\to\infty} \phi_a = 0 \qquad \text{where} \qquad x_{ab\perp} = x_{ab}\Pi = x_{ab} - P x_{ab}\frac{P}{P^2}$$
$$b = 1, \dots, N$$

(confining potentials - of the type treated in ref. [11] - can also be considered, however the separation of the mass term in Eq. (1.9) becomes then ill defined).

[A regularity requirement also imposed in ref. [18] and designed to exclude strong attractive singularities (leading to falling on a centre) will not be discussed here.]

Now we again proceed to the special case N=2.

The compatibility condition (1.10) assumes a more tractable form in terms of the functions

$$D = \phi_1 - \phi_2 , \qquad \phi = \mu_2 \phi_1 + \mu_1 \phi_2 \tag{1.12}$$

where $\mu_{1,2}$ are defined by (0.2):

$$\mu_{1,2} = \frac{1}{2} \pm \frac{1}{2s} (m_1^2 - m_2^2). \tag{1.13}$$

We have

$$\{ \varphi_1 , \varphi_2 \} = p \frac{\partial D}{\partial x} - P \frac{\partial \phi}{\partial x} + \{ D, \phi \} \approx 0. \tag{1.14}$$

For a given D Eq.(1.14) can be regarded as a first order partial differential equation for ϕ whose solution involves a functional freedom. It was pointed out in ref.[18] that the special solution, for which

$$D = 0 = P \frac{\partial \phi}{\partial x} \tag{1.15}$$

contains enough freedom to accomodate (in its quantized version including spin) the quasipotential equations[12, 14] considered so far. The general Poincaré invariant solution of the second equation (1.15) is a function of five among the six independent scalar products of the vectors x , p , and P (excluding xP):

$$\phi = \phi (x_\perp^2 ; s ; p x_\perp , p^2 ; -pP). \tag{1.16}$$

Other solutions (with $D \neq 0$) will be displayed in Sec. 2A.

C. Equal time surface, Hamiltonian, gauge transformations

We shall define the time evolution of the system in terms of the constraints (1.9). To this end we have first to select a

family of equal time surfaces. For the sake of simplicity, we restrict our attention to the 2-particle case and take the set of hyperplanes in the space of relative coordinates

$$n x \approx 0 \qquad (x \equiv x_1 - x_2), \tag{1.17}$$

where n is a P -dependent time-like vector such that $\{ n, \varphi_a \} \approx 0$. We shall demand that the Hamiltonian $H^{(n)}$ is a linear combination of the constraints which has zero Poisson brackets with nx on $\mathcal{M}$. This requirement determines $H^{(n)}$ up to a single Lagrange multiplier λ (which will be taken positive in order to fix the direction of the time axis):

$$H_\lambda^{(n)} = \frac{\lambda}{nP} \left\{ \left(n p_2 + n \frac{\partial \phi_2}{\partial p} \right) \varphi_1 + \left(n p_1 - n \frac{\partial \phi_1}{\partial p} \right) \varphi_2 \right\} (\approx 0). \tag{1.18}$$

(Note, that according to (1.11), $n \frac{\partial}{\partial p_1} H_\lambda^{(n)} \approx n \frac{\partial}{\partial p_2} H_\lambda^{(n)} \approx \frac{\lambda}{nP} d \neq 0$.)

A family $\mathcal{T}$ of functions on $\Gamma = \Gamma^2$ is said to be gauge invariant if the time evolution of each $f \in \mathcal{T}$ generated by the Hamiltonian (1.18) for any choice of n and λ , does not lead it out of $\mathcal{T}$. If in addition $\mathcal{T}$ is irreducible in the sense that it does not contain a nontrivial gauge invariant subset, then it will be called a (strict) observable. A special case of an observable is a single function of the dynamical variables which has zero Poisson brackets with the constraints on $\mathcal{M}$. The ten generators (1.3), (1.4) of the Poincaré group provide examples of such simple observables. In general, the observables are given by 2-parameter families of functions $f(\gamma; \sigma_1, \sigma_2)$, $\gamma = (x_1, p_1 ; x_2, p_2)$ defined by the system of partial differential equations

$$\frac{\partial f}{\partial \sigma_1} = \{ f, \varphi_1 \} , \quad \frac{\partial f}{\partial \sigma_2} = \{ f, \varphi_2 \} \tag{1.19a}$$

and the intial condition

$$f(\gamma;0,0)=f(\gamma). \tag{1.19b}$$

We remark that if eqs.(1.19) are globally integrable (for $(\sigma_1,\sigma_2)\in\mathbb{R}^2$) then they define a 2-parameter group, say $G_{(2)}$, of canonical gauge transformations $f(\gamma)\to f(\gamma;\sigma_1,\sigma_2)=f[\gamma(\sigma_1,\sigma_2)]$ ($\gamma\in\mathcal{M}$, $(\sigma_1,\sigma_2)\in\mathbb{R}^2$ implies $\gamma(\sigma_1,\sigma_2)\in\mathcal{M}$). It is a subgroup of an infinite parameter gauge group $\mathcal{G}_{(2)}$ generated by arbitrary linear combinations of φ_1 and φ_2 (with variable coefficients)*). (This infinite parameter group reflects reparametrization invariance of phase space trajectories.) For given $\gamma\in\mathcal{M}$ the groups $\mathcal{G}_{(2)}$ and $G_{(2)}$ have the same 2-dimensional orbit γ_* which is nothing but the fibre in $\mathcal{M}$ containing γ. We shall see in Sec. 2B below that (interacting) particle world lines are only invariant under a subgroup $\mathcal{G}_{(1)}$ of $\mathcal{G}_{(2)}$ generated by multiples of the Hamiltonian.

Note that the Hamiltonian H of ref./18/ is obtained from (1.18) by taking $n=P$, - that is

$$\chi=P_x\approx 0, \tag{1.20}$$

and $\lambda=1$; assuming in addition that $P\frac{\partial\phi}{\partial P}=0$ (as it was done in ref./18/) we find

$$H_1^{(P)}=H=\mu_2\varphi_1+\mu_1\varphi_2=\phi-\tfrac{1}{2}\left(p^2+b^2(s)\right)\ (\approx 0) \tag{1.21}$$

(for $\mu_{1,2}$ given by (0.2)). The Hamiltonians λH where H is given by (1.21) along with the subsidiary condition (1.20) play a privileged role in the 2-particle dynamics because of their manifest covariance and symmetry with respect to particles permutation and we shall often refer to them in what follows.

*) $\mathcal{G}_{(2)}$ can be defined as the subgroup of Diff$(\mathcal{M})$ (the group of all diffeomorphisms of $\mathcal{M}$) which leaves each fibre γ_* invariant.

Note, finally, that the physical phase space Γ_* can be parametrized in the gauge (1.17) by the points of the surface of initial conditions

$$n\bar{x}_1\approx t_0\approx n\bar{x}_2\ (\subset\mathcal{M}),\quad \bar{x}_a=x_a-\frac{\mu_1-\mu_2}{s}(xP)P, \tag{1.22}$$

where the time parameter t_0 has by definition, zero Poisson brackets with all dynamical variables.

2. Space-Time Trajectories and Scattering Matrix

A. The notion of world line. Equivalent dynamics

For a Hamiltonian of type (1.18) choosing the time variable as

$$t=nX\ (\approx n\bar{x}_1\approx n\bar{x}_2) \tag{2.1}$$

(or as some monotonously increasing function of t) amounts to fixing the Lagrange multiplier λ. Indeed, in order to ensure the consistency of (2.1) with the time evolution, we should require

$$\frac{d}{dt}\left(t-nX\right)=1-\{nX,\ H_\lambda^{(n)}\}\approx 0. \tag{2.2a}$$

Taking into account that $\frac{\partial}{\partial p}=\frac{\partial}{\partial p_1}-\frac{\partial}{\partial p_2}$ and, hence,

$$\{n\bar{x}_1,\ H_\lambda^{(n)}\}\approx\{n\bar{x}_2,\ H_\lambda^{(n)}\}\approx\{nX,H_\lambda^{(n)}\}\approx\frac{\lambda d}{nP} \tag{2.2b}$$

where d is the determinant (1.11), we find

$$\lambda=\frac{nP}{d}\qquad\left(d=\det\begin{pmatrix} n\frac{\partial H}{\partial P} & n\frac{\partial H}{\partial p}\\ n\frac{\partial\varphi}{\partial P} & n\frac{\partial\varphi}{\partial p}\end{pmatrix},\ H=\mu_2\varphi_1+\mu_1\varphi_2,\ \varphi=\varphi_1-\varphi_2\right). \tag{2.2c}$$

(If we set $\bar{t}=f(nX)$ where $f'>0$ then we get $\bar{\lambda}=\frac{\lambda}{f'(nX)}$.) Given a pair of points (ξ_1,ξ_2) on the plane (1.22) and initial velocities $\dot{x}_1 = v_1$, $\dot{x}_2 = v_2$ at $t = t_0$ (where the dot indicates differentiation with respect to the variable t), we have a unique pair of trajectories $(x_1(t), x_2(t))$ satisfying the initial conditions

$$x_a(t_0) = \xi_a , \quad \dot{x}_a(t_0) = v_a \quad / \text{or } \bar{x}_a(t_0) = \bar{\xi}_a = \xi_a - \frac{\mu_1 - \mu_2}{s}(\xi P)P, \ \xi = \xi_1 - \xi_2, \text{etc.} /$$

$a = 1, 2.$

These trajectories are independent of λ (since a change of λ amounts to a rescaling of the time parameter on each world line), but depend, in general, on n .

Two pairs of constraints $\varphi_1 \approx 0 \approx \varphi_2$ and $\bar{\varphi}_1 \approx 0 \approx \bar{\varphi}_2$ shall be considered as physically equivalent if for any fixed choice of the time-like vector n they lead to the same world lines for the same initial conditions, and if in addition they give rise to the same realization of the Poincaré group (the latter means that there is a one-to-one correspondence $\gamma \to \bar{\gamma}$ of $\mathcal{M}$ onto $\bar{\mathcal{M}}$ which leaves the world lines and the Poincaré group generators (1.3), (1.4) invariant).*)

Note that this notion of physical equivalence singles out the Minkowski space trajectory along with the Poincaré group generators as a more fundamental object than the phase space picture. The notion of particle momentum (for fixed coordinates) is not determined by the canonical Poisson bracket relations. Indeed the transformations

$$x_a \to \bar{x}_a = x_a , \quad p_a \to \bar{p}_a = p_a + \partial_a F(\{\tfrac{1}{2}x_{bc}^2\}) \tag{2.3}$$

$$a, b, c = 1, \dots, N, \qquad x_{bc} = x_b - x_c ,$$

are easily verified to be canonical (for any choice of the smooth function F). Moreover, they leave the coordinates unaltered and

*) We shall assume in addition that for large (space like) separations, $-x_\perp^2 \to \infty$, the particle momenta p_a and $\bar{p}_a$ tend to the same (time independent) limit. This implies the vanishing of F in Eq. (2.3) for large (negative) arguments.

because of the Poincaré invariance of F , the generators of the Poincaré group do not change either:

$$\sum_a \bar{p}_a = \sum_a p_a \ (= P) , \quad \sum_a \bar{x}_a \wedge \bar{p}_a = \sum_a x_a \wedge p_a \ (= M). \tag{2.4}$$

In fact, it is not difficult to prove that locally the transformations of type (2.3) are the most general ones with all these properties. In the 2-particle case the second equation (2.3) can be rewritten in terms of the single relative coordinate $x = x_{12}$ as follows *):

$$\bar{p}_1 = p_1 + x\, B(\tfrac{1}{2}x^2), \qquad \bar{p}_2 = p_2 - x\, B(\tfrac{1}{2}x^2) \tag{2.5}$$

(where $B(u) = \frac{dF}{du}$). Such transformations leave the x -space trajectories invariant and, therefore, relate physically equivalent theories (in the above terminology) to each other.

We can use the freedom in the choice of B in (2.5) (or F in (2.3)) to select a standard representative of the constraints (1.9). One way to do that is to assume that for the privileged gauge condition (1.20) and Hamiltonian (1.21) the relative velocity $\dot{x}$ vanishes weakly for $p = 0$. We have

$$\dot{x} = p - \Pi\frac{\partial\phi}{\partial p} + \frac{1}{s}\Big[\Big(P\frac{\partial D}{\partial p}\Big)\frac{\partial\phi}{\partial p} - \Big(P\frac{\partial\phi}{\partial p}\Big)\frac{\partial D}{\partial p}\Big]. \tag{2.6}$$

Hence, our standardization condition is

$$\Big\{\Pi\frac{\partial\phi}{\partial p} - \frac{1}{s}\Big[\Big(P\frac{\partial\phi}{\partial p}\Big)\frac{\partial D}{\partial p} - \Big(P\frac{\partial D}{\partial p}\Big)\frac{\partial\phi}{\partial p}\Big]\Big\}\Big|_{p=0=\chi} \approx 0. \tag{2.7}$$

(Note that the left-hand side of (2.7) has the form $A(s, x^2)\,x = -B(\tfrac{1}{2}x^2)\,x$, since the equation $H \approx 0$ for $p = 0 = \chi$ allows to express s as a function of x^2 .)

*) The fact that the functions $\bar{\varphi}_{1,2} = \frac{1}{2}[m_{1,2}^2 - (p_{1,2} \pm x\, B(\tfrac{1}{2}x^2))^2]$ are in involution was first noticed in ref. /19/.

B. Gauge dependence of interacting particles' world lines

We shall demonstrate in this section that the notion of gauge invariance (introduced in Sec. 1C) is too restrictive to accomodate space-time particle trajectories in the presence of a nontrivial interaction. More precisely, we shall establish the following negative result.

Theorem. Consider for each point γ of the generalized 2-particle mass-shell $\mathcal{M}$,

$$\gamma = (x_1, p_1\,;\, x_2, p_2) \in \mathcal{M}, \tag{2.8}$$

the 2-dimensional fibre $\gamma_* = \{$set of $\gamma(\sigma_1, \sigma_2)$, $(\sigma_1,\sigma_2) \in \mathbb{R}^2$, such that $\frac{\partial \gamma}{\partial \sigma_a} = \{\gamma, \varphi_a\}$; $\gamma(0,0) = \gamma\}$ through γ. The projections $T_a = \pi_a \gamma_*$, $a = 1,2$ of this fibre into the Minkowski space of each particle,

$$T_a = \{ x_a(\sigma_1, \sigma_2) \in M_a\,;\, (\sigma_1, \sigma_2) \in \mathbb{R}^2 \} \quad a = 1,2 \tag{2.9}$$

are one-dimensional, if and only if the trajectories T_a are straight lines.

Remark. In a less technical language the theorem says that a 2-particle system has gauge invariant world lines (in Minkowski space) only if the motion is free. Indeed, if the projections were 2-dimensional we would need a (gauge dependent) subsidiary condition to define the 1-dimensional world line of each particle.

Proof. In one direction the theorem is trivial. If the constraints are given by

$$\begin{aligned} \varphi_1^{fr} &\equiv \tfrac{1}{2}\left[m_1^2 - \left(p_1 + x\, B(\tfrac{1}{2}x^2)\right)^2 \right] \approx 0 \\ \varphi_2^{fr} &\equiv \tfrac{1}{2}\left[m_2^2 - \left(p_2 - x\, B(\tfrac{1}{2}x^2)\right)^2 \right] \approx 0 \end{aligned} \tag{2.10}$$

(cf. Sec. 2A), then obviously

$$\frac{\partial x_1}{\partial \sigma_2} = \{x_1, \varphi_2^{fr}\} = 0 = \frac{\partial x_2}{\partial \sigma_1} \quad \left(= \{x_2, \varphi_1^{fr}\}\right), \tag{2.11}$$

and hence, the projections $T_a = \pi_a \gamma_*$ of the fibre γ_* are one dimensional.

The converse statement is both more interesting and more difficult to establish: given that

$$\dim T_a = 1, \quad a = 1, 2, \tag{2.12}$$

where T_a is the manifold (2.9) to prove that the constraints φ_a can be chosen in the form (2.10). We shall proceed in two steps. First, we shall see, that the assumption of the theorem leads to Eq.(2.11) for some linear combinations of the original constraints. Second, we shall show that the general solution of (2.11) satisfying conditions (i)-(iv) of Sec. 1B is given by (2.10). These two steps form the content of the following two lemmas.

Lemma 1. If assumption (2.12) is satisfied, then we can find in the neighbourhood of each point γ of $\mathcal{M}$ two independent linear combinations of the original constraints

$$\overline{\varphi}_a = C_{a1} \varphi_1 + C_{a2} \varphi_2\,, \quad a = 1, 2 \tag{2.13}$$

that satisfy (2.11) or, equivalently,

$$\frac{\partial \overline{\varphi}_1}{\partial p_2} = 0 = \frac{\partial \overline{\varphi}_2}{\partial p_1}\,. \tag{2.14}$$

Proof of Lemma 1. Let σ_1 and σ_2 be the proper-time parameters on the world lines T_1 and T_2. Assumption (2.12) implies that one can choose in the neighbourhood of each point $\gamma \in \mathcal{M}$ σ_1 and σ_2 as local coordinates in the fibre (smoothly depending on the fibre); then

$$\{x_b^\mu, \varphi_a\} = B_{ab} \frac{d x_b^\mu}{d\sigma_b}\,, \quad a, b = 1, 2. \tag{2.15}$$

where B_{ab} may depend on the point γ of $\mathcal{M}$ but not on the index μ (there is no summation in the right-hand side). It follows from (1.11) that

$$|B| \equiv \det B_{ab} \quad (= B_{11} B_{22} - B_{12} B_{21}) \neq 0. \tag{2.16}$$

Setting

$$(C_{ab}) = (B^{-1}{}_{ab}) \quad \text{or} \quad C_{11} = \frac{B_{22}}{|B|}, \; C_{22} = \frac{B_{11}}{|B|}, \; C_{12} = -\frac{B_{12}}{|B|}, \; C_{21} = -\frac{B_{21}}{|B|}, \tag{2.17}$$

we obtain

$$\{ x_1, \overline{\varphi}_a \} = \frac{dx_1}{d\sigma_1} \delta_{a1}, \qquad \{ x_2, \overline{\varphi}_a \} = \frac{dx_2}{d\sigma_2} \delta_{a2}. \tag{2.18}$$

One can construct such a continuation of C_{ab} (and B_{ab}) off $\mathcal{M}$ for which Eq. (2.18) becomes strong, and hence Eq. (2.14) is also valid (in the strong sense).

Note. This part of the proof readily extends to the N-particle case. Assuming the validity of (2.15), (2.16) for $a, b = 1, \ldots, N$ and setting

$$\overline{\varphi}_a = \sum_{b=1}^{N} (B^{-1})_{ab} \varphi_b , \tag{2.19}$$

we obtain the counterpart of (2.18) for $a = 1, \ldots, N$. If we wish to use (2.15) and (2.19) as global relations on $\mathcal{M}$ we have to assume that $\mathcal{M}$ is a globally trivial fibre bundle (so that it is diffeomorphic to $\Gamma_{*} \times \mathbb{R}^N$). Such an assumption, however, is not needed for the validity of our theorem (cf. Appendix A).

Lemma 2. The constraints $\overline{\varphi}_a \approx 0$, satisfying (1.10) (2.14) can be replaced by equivalent constraints of type (2.10) (which describe the same manifold $\mathcal{M}$).

Proof of Lemma 2. Poincaré invariance, along with (2.14) tells us that each $\overline{\varphi}_a$ depends on 3 scalar variables *)

*) Again the global validity of this statement is not clear, but it is actually only needed locally as explained in Appendix A.

which will be chosen as

$$s_a = \tfrac{1}{2} p_a^2, \quad u_a = x p_a, \quad a = 1, 2, \quad v = \tfrac{1}{2} x^2 . \tag{2.20}$$

Eqs. (2.18) imply that $\frac{\partial \overline{\varphi}_a}{\partial s_a} \neq 0$, $a = 1, 2$. Indeed, assuming that at least one of these derivatives vanishes, we can easily show that the compatibility condition (1.10) has no non-trivial solution satisfying (2.15), (2.16). Physically, this should be expected, since the 4-velocities $\frac{dx_a}{d\sigma_a}$, $a = 1, 2$ have to be timelike vectors while x $\left(= \frac{\partial u_a}{\partial p_a}\right)$ is space like (as a consequence of (1.17) for any choice of n). Setting $\varphi_a = -\left(\frac{\partial \overline{\varphi}_a}{\partial s_a}\right)^{-1} \overline{\varphi}_a$, we can write

$$\varphi_a = F_a(u_a, v) - s_a, \quad a = 1, 2. \tag{2.21}$$

Using the Poisson bracket relations

$$\begin{gathered} \{s_1, u_2\} = \{u_1, s_2\} = p_1 p_2, \quad \{s_1, v\} = u_1, \; \{v, s_2\} = u_2, \\ \{u_1, v\} = 2v = \{v, u_2\}, \quad \{u_1, u_2\} = u_1 + u_2, \end{gathered} \tag{2.22}$$

we obtain

$$\begin{aligned} 0 = \{\varphi_1, \varphi_2\} = & -\left(\frac{\partial F_1}{\partial u_1} + \frac{\partial F_2}{\partial u_2}\right) p_1 p_2 - \frac{\partial F_2}{\partial v} u_1 - \frac{\partial F_1}{\partial v} u_2 + \\ & + \frac{\partial F_1}{\partial u_1} \frac{\partial F_2}{\partial u_2} (u_1 + u_2) + 2v \left(\frac{\partial F_1}{\partial u_1} \frac{\partial F_2}{\partial v} + \frac{\partial F_1}{\partial v} \frac{\partial F_2}{\partial u_2} \right). \end{aligned} \tag{2.23}$$

Since the variable $p_1 p_2$ only appears in the first term, its coefficient should vanish so that $\frac{\partial F_1}{\partial u_1} = -\frac{\partial F_2}{\partial u_2}$; here the left-hand side is independent of u_2 while the right-hand side is independent of u_1, hence

$$F_1 = C_1(v) - u_1 B(v), \quad F_2 = C_2(v) + u_2 B(v). \tag{2.24}$$

Inserting in (2.23), we find

$$-C_1' u_2 - C_2' u_1 - (u_1 + u_2) B^2 + 2 v B [C_1' - C_2' - B' (u_1 + u_2)] = 0,$$

which leads to

$$C_1' = C_2' = - B^2 - 2 v B B'$$

and we can set

$$C_1 = \tfrac{1}{2} m_1^2 - v B^2, \qquad C_2 = \tfrac{1}{2} m_2^2 - v B^2 \tag{2.25}$$

in accord with (2.10).

This completes the proof of Lemma 2 and hence of our theorem.

We conjecture that the theorem is true for any $N \geq 2$. (The validity of this conjecture was verified for N=3.)

The negative result thus established is the counterpart of the no-interaction theorem of Currie, Jordan, Sudarshan and Leutwyler /2,3,7,10/ in the constraint formulation of relativistic classical dynamics.

C. Gauge invariance of the relative motion and of the scattering matrix

The result of the preceding section looks at first sight rather distressing. In the presence of a non-trivial interaction, particle world lines can only be defined if we fix an initial data surface (of type (1.17)) and then they depend on the choice of the vector n. It turns out, however, that (at least for constraints satisfying (1.15)) the gauge dependence is confined to the time evolution of the centre of mass variable and can be taken explicitly into account. The physically relevant relative

coordinate trajectory and the 2-particle scattering matrix are gauge invariant.

In order to see this, we first write the Hamiltonian that leaves the gauge condition (1.17) invariant, in the form

$$\begin{aligned} H^{(n)} &= \frac{1}{nP} \left\{ \left(n p_2 + n \frac{\partial \phi}{\partial p}\right) \varphi_1 + \left(n p_1 - n \frac{\partial \phi}{\partial p}\right) \varphi_2 \right\} \\ &= H + \frac{1}{nP} \left(n \frac{\partial \phi}{\partial p} - n p\right) \varphi \quad (\approx 0), \end{aligned} \tag{2.26}$$

where H is given by (1.21). (It is obtained from the expression (1.18) for $\phi_1 = \phi_2 = \phi$ and $\lambda = 1$. We observe that

$$\{x_\perp, \varphi\} = 0 = \{p, \varphi\}, \tag{2.27}$$

so that the time evolution of $x_\perp$ and p is indeed independent of the choice of n:

$$\{x_\perp, H^{(n)}\} \approx \{x_\perp, H\}, \qquad \{p, H^{(n)}\} \approx \{p, H\}.$$

The gauge dependence of the centre of mass variable (1.6) can be found explicitly by solving the equation

$$\frac{\partial X}{\partial \sigma} = \{X, \varphi\} = p, \qquad \frac{\partial^2 X}{\partial \sigma^2} = 0. \tag{2.28}$$

If we denote the variable conjugate to H by τ, so that

$$\frac{\partial X}{\partial \tau} = \{X, H\}, \tag{2.29}$$

then the σ-dependence of $X(\tau, \sigma)$ is given by

$$X(\tau, \sigma) = X(\tau, 0) + p(\tau)\sigma. \tag{2.30}$$

The evolution of X with respect to the time parameter $\tau^{(n)}$ conjugate to the Hamiltonian $H^{(n)}$ (2.26) can then be expressed in terms of $X(\tau, \sigma)$:

$$X[\tau^{(n)}] = X\left(\tau^{(n)}, \frac{1}{nP}\left(n \frac{\partial \phi}{\partial p} - n p\right) \tau^{(n)}\right). \tag{2.31}$$

Similarly, for the individual particle coordinates we have

$$x_1[\tau^{(n)}] = x_1(\tau^{(n)}, 0) + \frac{1}{nP}\left(n\frac{\partial\phi}{\partial p} - np\right) p_1(\tau^{(n)})\,\tau^{(n)},$$
$$x_2[\tau^{(n)}] = x_2(\tau^{(n)}, 0) - \frac{1}{nP}\left(n\frac{\partial\phi}{\partial p} - np\right) p_2(\tau^{(n)})\,\tau^{(n)}. \tag{2.32}$$

Assume now, that we have an elastic scattering problem, for which the following limits exist:

$$\lim_{\tau\to\pm\infty} p_a(\tau) = p_a^{\pm} = \lim_{\tau\to\pm\infty} \frac{1}{\tau} x_a(\tau, 0) \tag{2.33a}$$

where

$$p_1^+ + p_2^+ = p_1^- + p_2^- = P; \tag{2.33b}$$

$$\lim_{\tau\to\pm\infty} \left[x_\perp(\tau) - (p_1^{\pm} - p_2^{\pm})\Pi\tau \right] = a^{\pm}; \tag{2.34}$$

then for $\tau^{(n)} \to \pm\infty$ we have

$$\frac{dx_a^{\pm}}{d\tau^{(n)}} = \left[1 + (-1)^{a-1} \frac{1}{nP}\left(n\frac{\partial\phi}{\partial p} - np\right)\right] p_a^{\pm}, \quad a = 1, 2.$$

Obviously, the corresponding 4-velocities $u_a^{\pm}$ (normalized by $(u_a^{\pm})^2 = 1$) are independent of n. Since the scattering matrix transforms (by definition) the vectors u_1^-, u_2^-, a^- into u_1^+, u_2^+, a^+ (all of which are gauge invariant), it is gauge invariant as well.

3. Summary and discussion

The results of this paper may be summarized in the following simplified manner.

Define the generalized mass shell $\mathcal{M}$ and the equal time surface by

$$\mathcal{M}: \begin{array}{l} H = -\frac{1}{2}(b^2(s) + p^2) + \phi(x_\perp, p, s) \approx 0, \\ \varphi = \frac{1}{2}(m_1^2 - m_2^2 - p_1^2 + p_2^2) = -pP \approx 0; \end{array} \tag{3.1}$$

$$\chi^{(n)} = nx \approx 0 \qquad (n^2 > 0), \tag{3.2}$$

where we use the kinematical notation of Sec. 1A. Every linear combination of the constraints (3.1) (with variable coefficients) which has zero Poisson bracket with $\chi^{(n)}$ is proportional to the "Hamiltonian"

$$H^{(n)} = H + \frac{1}{nP}\left(n\frac{\partial\phi}{\partial p} - np\right)\varphi \quad (\approx 0). \tag{3.3}$$

The time evolution generated by this Hamiltonian leads to individual particle trajectories [given by (2.32)] which depend explicitly on the time-like vector n. In the free particle case the time $\tau^{(n)}$ dependence of x_a still includes the vector n, however the (straight) world lines of the two particles do not; this can be made manifest by introducing the proper time variables

$$\tau_1 = \left(1 - \frac{np}{nP}\right)\tau^{(n)}, \quad \tau_2 = \left(1 + \frac{np}{nP}\right)\tau^{(n)}, \tag{3.4}$$

in terms of which we can write

$$x_a(\tau_a) - x_a(0) = p_a\tau_a, \qquad a = 1, 2. \tag{3.5}$$

The theorem proven in Sec. 2B asserts that the free motion is the only one for which particle trajectories are "gauge" (i.e., n-) independent.

In the general case, the relative coordinate still has a non-invariant evolution law

$$x[\tau^{(n)}] = x_1[\tau^{(n)}] - x_2[\tau^{(n)}] = x(\tau^{(n)}, 0) + \frac{1}{nP}\left(n\frac{\partial\phi}{\partial p} - np\right) P\tau^{(n)} \tag{3.6}$$

but the n -dependence is only present in the term proportional to P and hence disappears in the orthogonal relative variable $x_\perp$ (0.1). Thus the relative motion is gauge independent. An elementary analysis carried out in Sec. 2C shows that the same is true for the 2-particle scattering matrix.

It is only in the centre of mass motion (which is commonly regarded as physically uninteresting) that the gauge dependence of particle trajectories (in the presence of interaction) manifests itself. This result seems to indicate that centre of mass motion should not be regarded as a strict observable. It should be noted however, that such a conclusion would imply non-observability of some relative variables in the N-particle case (for $N \geq 3$). We would like to point out that the constraint dynamical approach to the case $N \geq 3$ is not yet fully understood *).

It is a pleasure to thank F.Rohrlich and S.N.Sokolov as well as the participants of the quantum field theory seminar of the Steklov Mathematical Institute for useful discussions.

*) Note that in the recent work by H.Crater/1/ on the 3-particle case the compatibility condition $\{\varphi_a, \varphi_b\} = 0$ is not satisfied. Concerning the complications inherent to the many particle case see /13/ and /16/.

APPENDIX A.

A general coordinate free formulation of results

The objective of this appendix is two-fold. First, in Sec. A1, we summarize for the benefit of the physicist reader some of the terminology concerned with symplectic forms and vector fields. Then, we present a general and precise differential geometric formulation of the results of the present paper, including a coordinate free recapitulation of the relevant part of the phase space approach to relativistic point particle dynamics of ref./18/.

A1. Symplectic forms, vector fields, Poisson brackets - a synopsis*)

Let Γ be a differentiable manifold with local coordinates $\gamma = (\gamma^\alpha)$. A 2-form ω on Γ is given in local coordinates by

$$\omega = \frac{1}{2}\omega_{\alpha\beta}\, d\gamma^\alpha \wedge d\gamma^\beta \qquad (1 \leq \alpha, \beta \leq \dim \Gamma) \tag{A.1}$$

where $\omega_{\alpha\beta}(\gamma)$ is a skew symmetric tensor and $d\gamma^\alpha \wedge d\gamma^\beta = -d\gamma^\beta \wedge d\gamma^\alpha$. A vector field $X = X(\gamma)$ on Γ is given in local coordinates by an expression of the form

$$X = X^\alpha \frac{\partial}{\partial \gamma^\alpha} \tag{A.2}$$

which determines (for each point γ) a contravariant vector $X^\alpha(\gamma)$. To each vector field X and 2-form ω we make correspond a 1-form

$$X\omega = X^\alpha \omega_{\alpha\beta}\, d\gamma^\beta \tag{A.3}$$

which is obtained formally from (A.1), (A.2) if we consider $\partial/\partial\gamma^\alpha$ as a differentiation with respect to $d\gamma^\alpha$ (regarded as independent variable) keeping in mind the antisymmetry of the wedge product. Similarly, for any 1-form $\theta = \theta_\alpha d\gamma^\alpha$ and vector field X

*) In writing this synopsis the authors were influenced by the lucid exposition of the Appendix to ref. /5/.

we make correspond a scalar function $X\theta = X^{\alpha}\theta_{\alpha} \equiv \theta(X)$.
For a pair of vector fields X and Y we shall write

$$\omega(X,Y) \equiv X(Y\omega) = Y^{\alpha}\omega_{\alpha\beta}X^{\beta}\,; \tag{A.4}$$

clearly, $\omega(X,Y) = -\omega(Y,X)$. The form ω is called non-degenerate if the equality $\omega(X,Y)=0$ for any choice of X and fixed Y implies $Y=0$. For the non-degeneracy of a 2-form ω on Γ it is necessary that Γ is even dimensional: $\dim\Gamma = 2n$; a necessary and sufficient condition for ω to be non-degenerate is then provided by the requirement that the 2n-form ω^n does not vanish. In the special case of the form (1.1b), we have $\omega_\alpha^4 \neq 0$, since it is the volume form on Γ_α. The form ω is called closed if

$$d\omega = \tfrac{1}{2}\,\partial_\rho\,\omega_{\alpha\beta}\,d\gamma^\rho \wedge d\gamma^\alpha \wedge d\gamma^\beta = 0 \qquad \left(\partial_\rho = \frac{\partial}{\partial\gamma^\rho}\right); \tag{A.5}$$

that is equivalent to the Jacobi condition $\partial_\rho\omega_{\alpha\beta} + \partial_\alpha\omega_{\beta\rho} + \partial_\beta\omega_{\rho\alpha} = 0$.
A closed, non-degenerate 2-form ω is called symplectic.

To each (smooth) function $f(\gamma)$ and symplectic form ω on Γ we make correspond a vector field X_f defined by

$$\omega(X_f, Y) = -(X_f\,\omega)(Y) = df(Y) = Y^{\alpha}\partial_\alpha f. \tag{A.6}$$

For each pair of functions f and g we define the Poisson bracket

$$\{f,g\} = \omega(X_f, X_g) = df(X_g) = -dg(X_f). \tag{A.7}$$

It is easily verified that in local coordinates

$$X_f = \{\gamma^\alpha, f\}\frac{\partial}{\partial\gamma^\alpha} = \omega^{\alpha\beta}\frac{\partial f}{\partial\gamma^\beta}\frac{\partial}{\partial\gamma^\alpha} \tag{A.8a}$$

$$\{f,g\} = \frac{\partial f}{\partial\gamma^\alpha}\,\omega^{\alpha\beta}\,\frac{\partial g}{\partial\gamma^\beta}, \tag{A.8b}$$

where $\omega^{\alpha\beta}$ is the inverse matrix of $\omega_{\alpha\beta}$ (which is also skew-symmetric):

$$\omega^{\alpha\sigma}\omega_{\sigma\beta} = \delta^{\alpha}_{\ \beta}. \tag{A.8c}$$

Eq. (1.2) is then derived by noticing that

$$\{\gamma^\alpha, \gamma^\beta\} = \omega^{\alpha\beta}. \tag{A.9}$$

Note that according to (1.1b) $\omega = \sum_{a=1}^{N}\omega_a$, where ω_a is the canonical symplectic form on the cotangent vector bundle *)

$$\Gamma_a = T^* M_a^4 \tag{A.10}$$

on Minkowski space

$$\omega_a = d\theta_a = dx_a^\mu \wedge dp_{a\mu}. \tag{A.11}$$

*) The tangent bundle TM over a differentiable manifold M is a vector bundle with base M and fibre the tangent vector space $T_x M$ at each point x (regarding M as the configuration space of a dynamical system we can identify $T_x M$ with the velocity space at x). The cotangent bundle T^*M is defined in a similar fashion by replacing $T_x M$ by its dual $T_x^* M$ (which is the space of linear forms on $T_x M$; in the above mechanical picture the vectors in $T_x^* M$ play the role of momenta).

A2. General constrained formulation of relativistic Hamiltonian dynamics

We define the extended N-particle phase space as the direct product

$$\Gamma^N = \Gamma_1 \times \dots \times \Gamma_N \tag{A.12}$$

of the ("large") single particle phase spaces (A.10). It is equipped with the symplectic form

$$\omega = \sum_{a=1}^{N} \omega_a \tag{A.13}$$

(where ω_a is given by (A.11)), and with the related Poisson bracket structure (1.2). We have a natural action of the Poincaré group $\mathcal{P}$ in Γ^N with generators (1.3), (1.4):

$$(a,\Lambda):\ (x_1,p_1;\dots;x_N,p_N) \to (\Lambda x_1+a,\ \Lambda p_1;\dots;\Lambda x_N+a,\ \Lambda p_N) \tag{A.14}$$

(that is the diagonal action of $\mathcal{P}$ in Γ^N). We say that a point $\gamma \in \Gamma^N$ is $\mathcal{P}$-regular if the orbit $\mathcal{P}\gamma$ has a maximal dimension (7 for N=1, 10 for $N \geq 2$).

The N-particle relativistic Hamiltonian dynamics is specified by giving a submanifold $\mathcal{M} \subset \Gamma^N$ with the following properties.

(1) $\mathcal{M}$ is a connected Poincaré invariant submanifold of Γ of codimension N (in other words, $\dim \mathcal{M} = 7N$). The $\mathcal{P}$-regular points of $\mathcal{M}$ form a dense open subset of $\mathcal{M}$.

(2) The set $Ker(\omega|_{\mathcal{M}})$ of all tangent vectors on $\mathcal{M}$, on which the restriction $\omega|_{\mathcal{M}}$ of the 2-form (A.13) is degenerate, is an N-dimensional integrable vector subbundle of TM, such that the foliation

$$\mathcal{M} \to \Gamma_* = \mathcal{M}/Ker(\omega|_{\mathcal{M}}) \tag{A.15}$$

is a locally trivial fibre bundle (cf. Appendix to ref. [5]). (Condition (2) ensures, in particular, the compatibility condition (ii) of Sec. 1B for the constraints $\varphi_a = 0$ which define $\mathcal{M}$ in the neighbourhood of each point).

(3) Let $\pi : \mathcal{M} \to M^N = M_1 \times \dots \times M_N$ be the projection of $\mathcal{M}$ on the configuration space (so that $\pi(x_1,p_1;\dots;x_N,p_N) = (x_1,\dots,x_N)$). Denote by $d\pi$ the corresponding tangent map (mapping vectors of $T_\gamma\mathcal{M}$ on vectors of $T_x M^N$ for $x = \pi(\gamma)$). Then we demand that (a) the rank of the map $d\pi(Ker(\omega|_{\mathcal{M}})_\gamma)$ is N and, moreover, (b) $d\pi(Ker(\omega|_{\mathcal{M}})_\gamma)$ is spanned by N time-like vectors. (This is the counterpart of condition (iii) of Sec. 1B.)

(4) $\mathcal{M}$ is maximal in the sense that it is either closed, or, if it is not, then its closure $\bar{\mathcal{M}}$ contains a non-empty set S of points on which some of the conditions (1)-(3) is violated. (In particular, S should contain all singular points of $\bar{\mathcal{M}}$ and all points in which the rank of the map $d\pi(Ker(\omega|_{\mathcal{M}})_\gamma)$ of condition (3) is smaller than N, but no other boundary.) Moreover, we demand that $\mathcal{M} = \bar{\mathcal{M}} \setminus S$.

A manifold $\mathcal{M}$ with the above properties is called the generalized (N-particle) mass shell. The 6N dimensional factor space Γ_* (A.15) is called the (proper) phase space of the system. The form $\omega|_{\mathcal{M}}$ gives rise to a (non-degenerate) symplectic form ω_* on Γ_*. Indeed it follows from (A.15) that the values $\omega(X,Y)$ of the 2-form ω for $X, Y \in T_\gamma\mathcal{M}$ do not change when X and Y vary in the corresponding equivalence classes so that $\omega|_{\mathcal{M}}$ does indeed define a 2-form ω_* on Γ_*. The non-degeneracy of ω_* follows from the definition of $Ker(\omega|_{\mathcal{M}})$. Since $\mathcal{M}$ and ω are both $\mathcal{P}$-invariant, Γ_* inherits the action of the Poincaré group and ω_* is invariant with respect to this action.

We end up this section with a few comments.

1) Conditions (1)-(4) under which a submanifold $\mathcal{M} \subset \Gamma^N$ specifies a relativistic N-particle dynamics are a generalization of assumptions (i)-(iv) of Sec. 1B. (The main difference is that we do not demand that $\mathcal{M}$ is defined globally in terms of N equations of the type (1.9). On the other hand if assumption (1) is satisfied and $\gamma \in \mathcal{M}$ is $\mathcal{P}$ -regular then in some sufficiently small neighbourhood of γ in Γ^N there exist N Poincaré invariant functions $\varphi_1, \dots \varphi_N$ such that $\mathcal{M}$ is given by the system of equations $\varphi_a = 0,\ a = 1, \dots, N$ in this neighbourhood.)

These conditions, however, do not exhaust the physical desiderata, listed in ref./18/. Most important, the separability requirement (v) of Sec. 1B is not included in (1)-(4), since it is not used in the present paper.

2) Following the pattern of Sec. 2A we can say that two submanifolds $\mathcal{M}$ and $\mathcal{M}'$ of Γ^N, satisfying (1)-(4) define physically equivalent dynamics if there exists a one to one canonical diffeomorphism of a neighbourhood of $\mathcal{M}$ on a neighbourhood of $\mathcal{M}'$ which maps $\mathcal{M}$ onto $\mathcal{M}'$, preserves the form ω and the generators of the Poincaré group, and commutes with the projection π on configuration space (or, in other words, preserves the particle coordinates x_a).

3) The somewhat complicated requirement (4) cannot be replaced by the simpler condition that $\mathcal{M}$ is closed if we want to incorporate zero mass particles.

4) The condition that $\mathcal{M}$ contains a dense open set of $\mathcal{P}$-regular points cannot be replaced by the stronger condition that $\mathcal{M}$ consists entirely of such regular points, since this is

not the case even for the free particles' mass shell (the singular set corresponding to collinear or complanar momenta and relative coordinates). It can be demonstrated that the $\mathcal{P}$-regular points form a dense open set in Γ^N and that there intersection with $\mathcal{M}$ is dense and open in $\mathcal{M}$ for $N \neq 2$.

A3. Non-invariance of particle world lines

Let us now assume the existence of invariant world lines. This means that the projection $\pi_a(\gamma_*)$ of a fibre $\gamma_* \subset \mathcal{M}$ (or, equivalently, of a point $\gamma_* \in \Gamma_*$) on the a-th copy of Minkowski space, M_a is a one-dimensional submanifold in M_a. Our objective is to find (under this assumption) a canonical form for the local equations of the 14-dimensional Poincaré invariant surface $\mathcal{M} \subset \Gamma = \Gamma^2$.

Part of the discussion is general and will be carried out in the N-particle case. For every $\gamma \in \mathcal{M}$ there exist an (open) neighbourhood $U \subset \Gamma^N$ of γ and N smooth functions $\varphi_1, \dots, \varphi_N$ defined in U, such that

$$\mathcal{M} \cap U = \mathcal{M}_1 \cap \mathcal{M}_2 \cap \dots \cap \mathcal{M}_N \,, \tag{A.16a}$$

where

$$\mathcal{M}_a = \{\gamma \in U;\ \varphi_a(\gamma) = 0\} \quad a = 1, \dots, N. \tag{A.16b}$$

Moreover, as a consequence of (2),

$$\{\varphi_a, \varphi_b\}\big|_{\mathcal{M} \cap U} = 0, \quad a, b = 1, \dots, N. \tag{A.17}$$

Proposition. For every $\gamma \in \mathcal{M}$ there exist a neighbourhood $U \subset \Gamma$ of γ and N functions $\varphi_a \in C^\infty(U)$, $a = 1, \dots, N$ satisfying (A.16), (A.17) and such that

$$\varphi_a \big|_U = \varphi_a(x_{12}, \dots, x_{N-1\,N}; p_a) \qquad x_{bc} = x_b - x_c,$$

$$\frac{\partial \varphi_a}{\partial p_a}\Big|_U \neq 0, \quad a = 1, \dots, N. \qquad \text{(A.18)}$$

The proof is essentially that of Lemma 1 (Sec. 2B).

Corollary. If, in addition, the point $\gamma \in \mathcal{M}$ is $\mathcal{P}$-regular then the neighbourhood U and the functions φ_a can be chosen in such a way that φ_a only depend on the scalar products of their arguments:

$$\varphi_a = \varphi_a(x_{12}^2, \dots, x_{N-1\,N}^2; x_{12} p_a, \dots, x_{N-1\,N} p_a; p_a^2), \quad a = 1, \dots, N \qquad \text{(A.19)}$$

We now come to the 2-particle case.

Theorem. In a relativistic two-particle theory with invariant world lines if the point $\gamma \in \mathcal{M}$ is $\mathcal{P}$-regular, then there exist a neighbourhood $U \subset \Gamma (= \Gamma^2)$ of γ and a (smooth) function $B(\frac{1}{2}x^2)$ in U such that the constraints φ_1 and φ_2 have the form (2.10). Moreover, the particle world lines are (globally) straight timelike lines.

The proof of the first statement is actually given in Sec. 2B. The fact that the world lines are straight lines globally is a consequence of the corresponding local statement and of the smoothness of world lines. Their timelike character follows from condition (3) (b).

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