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Еникова М.М., Карлуковски В.И. **E2 - 12257**

О существовании сингулярных решений в киральных теориях

Рассматривается SU(2)×SU(2) –симметричная киральная модель для пионного поля. Расширена совокупность решений полевых уравнений, найденная в предшествующей работе, путем включения решения с особенностями, сосредоточенными на мировых линиях. Можно рассматривать эти особенности как источники пионного поля. Это позволяет получить решение неоднородных полевых уравнений с некоторым типом источников в пространстве Минковского.

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Enikova M.M., Karloukovski V.I. **E2 - 12257**

On the Existence of Singular Solutions in Chiral Theories

The nonlinear SU(2)xSU(2) symmetric chiral model for the pion field is discussed. We enlarge the family of solutions to the field equations previously obtained by including solutions with world line singularities. The last can be regarded as sources of the pion field. This allows one to write down solutions to the inhomogeneous field equations with certain external sources in the Minkowski space-time.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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In a recent work [1,2] we have written explicitly a large family of finite-energy solutions to the field equations (in 3+1 dimensional space-time) in the Schwinger-Weinberg realization [3,4] of the chiral SU(2)xSU(2) symmetry. Let us recall that the Lagrangian and the field equations have in this case the form

$$\mathcal{L} = \frac{1}{2} g_{ab}(\pi)\, \partial_\mu \pi^a \partial^\mu \pi^b \tag{1}$$

and

$$g_{an} \Box \pi^n + \frac{1}{2}\left(\frac{\partial g_{an}}{\partial \pi^m} + \frac{\partial g_{ma}}{\partial \pi^n} - \frac{\partial g_{mn}}{\partial \pi^a}\right) \partial_\mu \pi^m \partial^\mu \pi^n = 0, \tag{2}$$

where

$$\begin{aligned} g_{ab}(\pi) &= d_1(\pi^2)\, \delta_{ab} + d_2(\pi^2)\pi_a \pi_b, \\ d_2(\pi^2) &= F_\pi^2 [\eta\pi^2 + f^2(\pi^2)]^{-2}, \\ d_2(\pi^2) &= -F_\pi^2 (\eta\pi^2 + f^2)^{-2} (\eta + 4ff' - 4\pi^2 f'^2). \end{aligned} \tag{3}$$

The sign $\eta = \pm 1$ is introduced to distinguish the cases of the compact $(\eta = +1)$ and noncompact $(\eta = -1)$ chirality [5]. Equation (2) can be written in the form

$$\Box \pi^a + \Gamma^a_{mn}\, \partial_\mu \pi^m \partial^\mu \pi^n = 0, \tag{4}$$

where Γ^a_{mn} is the Christoffel symbol.

The solutions we have found in [1,2] can be written in a simpler form by choosing the arbitrary function $f(\pi^2)$ in (3) (which specifies the parametrization of the pion field) to be

$$f(\pi^2) = \sqrt{F_\pi^2 - \eta\pi^2} \; . \qquad (5)$$

Then to every solution $\tau(x)$ of the wave equation

$$\Box\tau(x) = 0 \qquad (6)$$

there corresponds a five-parameter family of solutions

$$\pi(x) = F_\pi [A\cos\tau(x) + B\sin\tau(x)] \qquad (7)$$

to the field equations (2) or (4). Here A and B are two arbitrary constant isovectors obeying the constraint

$$\eta(A^2 + B^2) - (A\times B)^2 = 1. \qquad (8)$$

In the case $\eta = -1$ both B and τ should be pure imaginary in (7) and (8).

The solution (7) may naturally be cast in a four-dimensional form by adding to the isovectors $\pi(x)$, A, and B a zeroth component. We define in this way the four-dimensional vectors $n(x)$, a, and b, respectively.

$$n(x) = a\cos\tau(x) + b\sin\tau(x) \qquad (9)$$

normalized in the flat four-dimensional-isospin-space metric $\eta_{\alpha\beta}$ ($\eta_{00} = 1$, $\eta_{ab} = \eta\delta_{ab}$, $\eta_{0a} = 0$, $a,b = 1,2,3$)

$$n^2 = 1, \quad a^2 = 1, \quad b^2 = 1. \qquad (10)$$

Then (8) becomes

$$ab = 0 \qquad (11)$$

and appears to be a consistency condition for (10). The isovectors $\pi(x)$, A and B are three-dimensional projections of $n(x)$, a, and b,

$$\pi_j(x) = F_\pi n_j(x), \quad A_j = a_j, \quad B_j = b_j, \quad j = 1,2,3. \qquad (12)$$

Any two parametrizations of the pion field defined by the choice of the two functions $f(\pi^2)$ and $\tilde f(\tilde\pi^2)$ are related by (cf. [4])

$$\tilde\pi_a = \pi_a\phi(\pi^2), \quad \tilde f(\tilde\pi^2) = f(\pi^2)\phi(\pi^2) \qquad (13)$$

which implies

$$\tilde f(\tilde\pi^2)(\tilde\pi^2)^{-1/2} = f(\pi^2)(\pi^2)^{-1/2} \qquad (14)$$

and allows one to perform reparametrizations of the pion field.

As shown in [1,2], the energy

$$H \equiv P^\circ = \frac{1}{2}\int g_{ab}\left[\frac{\partial\pi^a}{\partial x_0}\frac{\partial\pi^b}{\partial x_0} + \nabla\pi^a\,\nabla\pi^b\right]d^3x \qquad (15)$$

which for the solution (7) is

$$H = \frac{1}{2}\eta F_\pi^2 \int\left[\left(\frac{\partial\tau}{\partial x_0}\right)^2 + (\nabla\tau)^2\right]d^3x \qquad (16)$$

can readily be rendered finite by the choice of $\tau(x)$.

Simple formulas also exist for the isospin and axial-vector charges

$$Q_a^V = J_a \int \frac{\partial \tau}{\partial x_0} d^3x, \quad Q_a^A = K_a \int \frac{\partial \tau}{\partial x_0} d^3x, \tag{17}$$

where

$$K_c = (a \wedge b)_{0c}, \quad J_c = \frac{1}{2} \epsilon_{cmn} (a \wedge b)_{mn}, \tag{18}$$

The aim of the present note is to enlarge this family of solutions (regular throughout the Minkowski space) including certain class of singular solutions and, in particular, solutions with a singular (world) line γ

$$\pi = \pi(x; \gamma) \tag{19}$$

which would play in some sense the role of classical Green's functions. We mean by this such functions (19) which upon inserting in the left-hand side of the field equation, would give δ-functions, $\delta^3(x - \xi)$ in the right-hand side.

In the nonlinear case, however, one should be careful in attempting to carry over this type of singular solutions involving distributions. The distributions are inherent in the linear analysis and are known to be a concept inconsistent, in general, with nonlinearity and multiplication. So it is not quite clear what should one understand by "left-hand side" of a nonlinear field equation. Should it be (2) or (4), or (2) multiplied by $f\delta^a_c + (f - 2\pi^2 f')^{-1} (\eta + 2ff')\pi^a \pi_c$, for instance, in which case it becomes

$$(\eta\pi^2 + f^2)^{-1} (f\delta_{cn} - 2f'\pi_n \pi_c)[\Box\pi^n + \Gamma^n_{ab} \partial_\mu \pi^a \partial^\mu \pi^b] = 0, \tag{20}$$

Here we propose to use Eq. (20) in order to give a precise meaning to the point-source solution (19).

On the manifold of the functions of the form (7) or (9), with $\tau(x)$ arbitrary, the left-hand side of Eq. (20) takes the form $K_c \Box \tau(x)$. If we now choose $\tau(x)$ in (7) or (9) to be, for instance, the static solution

$$\tau(x; \xi) = \frac{1}{4\pi} [(x - \xi)^2]^{-1/2} \tag{21}$$

of the equation

$$\Box\tau(x) = -\Delta\tau(x) = \delta^3(x - \xi) \tag{22}$$

then

$$n(x; \xi) = a \cos\tau(x; \xi) + b \sin\tau(x; \xi) \tag{23}$$

will be a five-parameter family of singular solutions to equation (20) with $f(\pi^2)$ given by (5). The singularity of such a solution is on a line parallel to the time axis. These solutions can be cast in a Lorentz covariant form recovering the unit time-like vector; let us denote it by $\ell (\ell^2 = 1)$, in (21)

$$\tau(x; \xi, \ell) = \frac{1}{4\pi} \{[\ell, (x - \xi)]^2 - (x - \xi)^2\}^{-1/2}. \tag{24}$$

Now the world line of the singularity is γ: $(x^\circ, x) = (t, \xi + t\ell/\ell_0)$.

There exists, further, a whole family of singular solutions, oscillating with time about the static singular solution, which can be obtained merely by applying an additional oscillating exponential factor to (24)

$$\tau(x-\xi; \ell, k) = \frac{1}{4\pi} \{[\ell(x-\xi)]^2 - (x-\xi)^2\}^{-1/2} \exp ik[\ell(x-\xi) - \sqrt{(\ell(x-\xi))^2 - (x-\xi)^2}]. \tag{25}$$

The function (25) satisfies the equation

$$\Box\tau(x) = \delta^3(x - \xi + \frac{1}{\ell_0}(x^\circ - \xi^\circ)\ell). \tag{26}$$

The field energy of all these solutions with the (world) line singularity is infinite exactly in the same fashion as the energy of the Coulomb field of a point charge in classical electrodynamics. They can play, however, an important analogous role in the classical chiral mesodynamics serving to introduce external sources. If one smears, for example, (25) by a suitable function

$$\tau(x;s,\ell,k) = \int_{\Sigma} s(\xi)\tau(x-\xi;\ell,k)\,d^3\xi \qquad (27)$$

on the space-like surface Σ (let it be $\Sigma:\xi_0=0$) one obtaines a family of solutions

$$\pi(x;s,\ell,k) = F_\pi[A\cos\tau(x;s,\ ,k) + B\sin\tau(x;s,\ell,k)] \qquad (28)$$

to the inhomogeneous field equations

$$g_{an}[\Box\pi^n + \Gamma^n_{ab}\partial_\mu\pi^a\partial^\mu\pi^b] = p_a\, s(0, x + \frac{x_0}{\ell_0}\ell\ ), \qquad (29)$$

where

$$p_a = f^{-1}[\delta^c_a - (\eta\pi^2 + f^2)^{-1}\ (\eta + 2ff')\pi^c\pi_a]K_c\,. \qquad (30)$$

To complete the analogy with classical electrodynamics one should establish the analogue of the Lorentz force. It would determine the action of the field on the singularities and govern their motion (world lines). Then one would have a coupled set of equations for the field and its sources.

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