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ON THE CALCULATION OF CRITICAL EXPONENTS BY THE METHODS OF QUANTUM FIELD THEORY

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ON THE CALCULATION OF CRITICAL EXPONENTS BY THE METHODS OF QUANTUM FIELD THEORY

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Казаков Д.И., и др. E2 - 12249

О вычислении критических индексов методами квантовой теории поля

Функция Гелл-Манна-Лоу и аномальные размерности квантовополевой модели ϕ^4 вычислены в четырехпетлевом приближении в формализме размерной ренормировки. На их основе найдены коэффициенты ϵ-разложения для критических индексов до степени ϵ^4 включительно. Для обработки рядов ϵ-разложения использован метод суммирования, включающий модифицированное преобразование Бореля и конформное отображение. Полученные значения критических индексов находятся в хорошем согласии с экспериментом и с результатами других теоретических подходов.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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On the Calculation of Critical Exponents by the Methods of Quantum Field Theory

The Gell-Mann-Low function and the anomalous dimensions of the quantum field model $\mathcal{L}_{int} = -\frac{(4\pi)^2}{4!} g(\vec{\phi}^2)^2$ are calculated in four-loop approximation using dimensional renormalization scheme. Proceeding from them the coefficients of ϵ-expansions for critical exponents are found up to ϵ^4. These expansions are treated by a summation method based on a modified Borel transform and a conformal mapping. The obtained values of critical exponents are in good agreement with the experimental data and the results of other theoretical approaches.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1979

1. Introduction

Deep analogies between statistical physics and quantum field theory /1/ can be efficiently used to obtain quantitative predictions on the behaviour of statistical systems in the neighbourhood of phase transition /2/. The main part in this approach is assigned to the renormalization group methods /3/ and ε - expansion /4/. Calculating the ordinary Feynman diagrams of quantum φ^4 model in $4-2\varepsilon$ dimensions and solving the renormalization group equations, we can express the critical exponents of phase transitions in the form of the series in ε. The physical (three-dimensional) case corresponds to the value of $\varepsilon = 1/2$.

Much progress in this direction has been achieved by the authors of papers /5,6/. They succeeded in calculating the contributions of all three-loop and some four-loop diagrams. However, in view of the asymptotic character of the obtained series in ε, further progress requires not only taking into account the diagrams of higher orders but also using the methods of "improvement" and summation of these series. The realization of this program is the aim of the present paper.

In recent years several authors /7,8/ have developed simple and efficient methods of computing the contributions of Feynman diagrams to the renormalization group functions. Application of this technique enabled us to compute analytically the contributions of all relevant diagrams and to complete four-loop calculations in the φ^4 model.

This gives us an opportunity to construct the series in ε up to ε^4 for all critical exponents. For these series we apply the method of summation we have developed earlier[9] including the modified Borel transform and conformal mapping of the integrand.

The values of critical exponents obtained by this method are in good agreement with the experimental data as well as with the results of other theoretical approaches[10].

2. Renormalization Group Functions of the φ^4 Model in Four-Loop Approximation

The present section is a pure field-theoretical part of our paper. It contains the calculation of renormalization group functions (anomalous dimensions and Gell-Mann-Low function) of the φ^4 model in the framework of dimensional renormalization.

We consider $O(n)$ -symmetrical model of n -component scalar field with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi^a\partial_\mu\varphi^a - \frac{m_B^2}{2}\varphi^a\varphi^a - \frac{16\pi^2}{4!}g_B\left(\varphi^a\varphi^a\right)^2, \quad a = 1,2,\dots,n. \tag{1}$$

To calculate the Feynman diagrams we use dimensional regularization and the procedure of minimal subtractions or, in other words, the dimensional renormalization scheme[11]. Namely, from each divergent integral we subtract only singular terms of its Laurent expansion in ε , where $\varepsilon=(4-d)/2$ and d is the dimension of space-time. In terms of the renormalization constants this means that they are expanded into the series in the inverse powers of ε ,

$$Z\left(\frac{1}{\varepsilon},g\right) = 1 + \sum_{\nu=1}^{\infty}\frac{c_\nu(g)}{\varepsilon^\nu}. \tag{2}$$

Renormalized Green functions Γ_R are obtained from the regularized ones through the following limiting procedure

$$\Gamma_R(\{p\},\mu^2,m^2,g) = \lim_{\varepsilon\to 0} Z_\Gamma\left(\frac{1}{\varepsilon},g\right)\Gamma(\{p\},m_B^2,g_B,\varepsilon), \tag{3}$$

where

$$m_B^2 = Z_m\left(\frac{1}{\varepsilon},g\right)m^2, \quad g_B = (\mu^2)^\varepsilon g Z_4\left(\frac{1}{\varepsilon},g\right)Z_2^{-2}\left(\frac{1}{\varepsilon},g\right). \tag{4}$$

Here $\{p\}$ are the momentum arguments of the Green function Γ , μ is the renormalization parameter and Z_m , Z_4 and Z_2 are the renormalization constants of the mass, four-vertex Γ_4 and inverse propagator $D^{-1}\equiv\Gamma_2$, respectively. Z_m and g_B allow the expansions analogous to (2):

$$Z_m\left(\frac{1}{\varepsilon},g\right) = 1 + \sum_{\nu=1}^{\infty}\frac{b_\nu(g)}{\varepsilon^\nu}, \tag{5}$$

$$g_B = (\mu^2)^\varepsilon\left(g + \sum_{\nu=1}^{\infty}\frac{a_\nu(g)}{\varepsilon^\nu}\right). \tag{6}$$

The functions $a_\nu(g)$, $b_\nu(g)$ and $c_\nu(g)$ (their mass-independence was proven in[12]) can be unambiguously calculated in perturbation theory by requiring that the limit $\varepsilon\to 0$ in (3) exists. The quantities $a_1(g)$, $b_1(g)$ and $c_1(g)$ are related to the functions entering into the differential renormalization group equation

$$\left[\mu^2\frac{\partial}{\partial\mu^2} + \beta(g)\frac{\partial}{\partial g} + \gamma_m(g)m^2\frac{\partial}{\partial m^2} - \gamma_\Gamma(g)\right]\Gamma_R(\{p\},\mu^2,m^2,g) \tag{7}$$

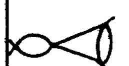
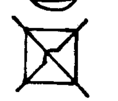
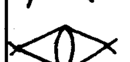
by the following relations:

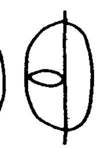
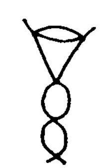
$$\beta(g) \equiv \left.\frac{\partial g}{\partial \ln\mu^2}\right|_{g_B} + \varepsilon g = \left(g\frac{\partial}{\partial g} - 1\right)a_1(g), \tag{8}$$

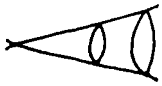
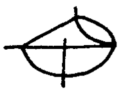
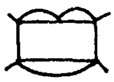
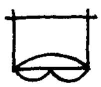
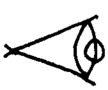
$$\gamma_m(g) \equiv \left.\frac{\partial \ln m^2}{\partial \ln\mu^2}\right|_{m_B^2,g_B} = g\frac{\partial}{\partial g}b_1(g), \tag{9}$$

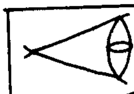
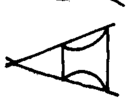
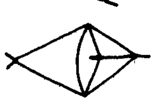
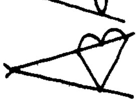
$$\gamma_\Gamma(g) \equiv \left.\frac{\partial \ln Z_\Gamma}{\partial \ln\mu^2}\right|_{m_B^2,g_B} = -g\frac{\partial}{\partial g}c_1(g). \tag{10}$$

Table 1. The contribution of a diagram to an appropriate renormalization constant is given by a product of expressions from the second and third columns in the case of Z_2 and Z_4 and from the second and fourth columns in the case of Z_{φ^2}.

Diagram	Singular contribution	Combinatorical factor for Z_2 and Z_4	Combinatorical factor for Z_{φ^2}
	$\frac{1}{\varepsilon}$	$\frac{3}{2}\,\frac{n+8}{9}\,g$	$\frac{1}{2}\,\frac{n+2}{3}\,g$
	$-\frac{1}{4\varepsilon}$	$\frac{1}{6}\,\frac{n+2}{3}\,g^2$	
	$-\frac{1}{\varepsilon^2}$	$-\frac{3}{4}\,\frac{n^2+6n+20}{27}\,g^2$	$-\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^2$
	$-\frac{1-\varepsilon}{2\varepsilon^2}$	$-3\,\frac{5n+22}{27}\,g^2$	$-\frac{1}{2}\,\frac{n+2}{3}\,g^2$
	$\frac{1-\varepsilon/2}{6\varepsilon^2}$	$-\frac{1}{4}\,\frac{n^2+10n+16}{27}\,g^3$	
	$\frac{1}{\varepsilon^3}$	$\frac{3}{8}\,\frac{n^3+8n^2+24n+48}{81}\,g^3$	$\frac{1}{8}\,\frac{(n+2)^3}{27}\,g^3$
	$\frac{1-\varepsilon}{2\varepsilon^3}$	$\frac{3}{2}\,\frac{3n^2+22n+56}{81}\,g^3$	$\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^3$
	$\frac{1-9/4\,\varepsilon}{6\varepsilon^2}$	$\frac{1}{2}\,\frac{n^2+10n+16}{27}\,g^3$	$\frac{1}{6}\,\frac{(n+2)^2}{9}\,g^3$
	$\frac{2\zeta(3)}{\varepsilon}$	$\frac{5n+22}{27}\,g^3$	
	$\frac{1-\varepsilon-\varepsilon^2}{3\varepsilon^3}$	$\frac{3}{2}\,\frac{3n^2+22n+56}{81}\,g^3$	$\frac{1}{4}\,\frac{(n+2)(n+8)}{27}\,g^3$
	$\frac{1-\varepsilon-\varepsilon^2}{3\varepsilon^3}$	$\frac{3}{2}\,\frac{n^2+20n+60}{81}\,g^3$	
	$\frac{1-3\varepsilon+4\varepsilon^2}{6\varepsilon^3}$	$6\,\frac{n^2+20n+60}{81}\,g^3$	$\frac{(n+2)(n+8)}{27}\,g^3$
	$\frac{1-2\varepsilon+\varepsilon^2}{3\varepsilon^3}$	$\frac{3}{4}\,\frac{3n^2+22n+56}{81}\,g^3$	$\frac{1}{4}\,\frac{(n+2)^2}{9}\,g^3$

	$-\frac{1-\varepsilon/2-5/4\,\varepsilon^2}{8\varepsilon^3}$	$\frac{1}{8}\,\frac{n^3+8n^2+32n+40}{81}\,g^4$	
	$-\frac{1-5\varepsilon}{32\varepsilon^2}$	$\frac{1}{12}\,\frac{3n^2+12n+12}{27}\,g^4$	
	$-\frac{1-7/2\,\varepsilon+\frac{13}{4}\varepsilon^2}{24\varepsilon^3}$	$\frac{1}{4}\,\frac{5n^2+32n+44}{81}\,g^4$	
	$-\frac{1-2\varepsilon+4\varepsilon^2}{12\varepsilon^3}$	$\frac{1}{4}\,\frac{5n^2+32n+44}{81}\,g^4$	
	$-\frac{1}{\varepsilon^4}$	$-\frac{3}{16}\,\frac{n^4+10n^3+40n^2+80n+112}{243}\,g^4$	$-\frac{1}{16}\,\frac{(n+2)^4}{81}\,g^4$
	$-\frac{1-\varepsilon}{2\varepsilon^4}$	$-\frac{3}{4}\,\frac{3n^3+24n^2+80n+136}{243}\,g^4$	$-\frac{1}{8}\,\frac{(n+2)^3}{27}\,g^4$
	$-\frac{1-2\varepsilon+\varepsilon^2}{4\varepsilon^4}$	$-\frac{3}{4}\,\frac{11n^2+76n+156}{243}\,g^4$	
	$-\frac{1-\varepsilon-\varepsilon^2}{3\varepsilon^4}$	$-\frac{3}{4}\,\frac{n^3+18n^2+80n+144}{243}\,g^4$	$-\frac{1}{8}\,\frac{(n+2)^2(n+8)}{81}\,g^4$
	$-\frac{1-9/4\,\varepsilon}{6\varepsilon^3}$	$-\frac{1}{2}\,\frac{n^3+8n^2+32n+40}{81}\,g^4$	$-\frac{1}{6}\,\frac{(n+2)^3}{27}\,g^4$
	$-\frac{1-3\varepsilon+4\varepsilon^2}{6\varepsilon^4}$	$-3\,\frac{n^3+14n^2+76n+152}{243}\,g^4$	$-\frac{1}{2}\,\frac{(n+2)^2(n+8)}{81}\,g^4$

	$-\frac{1-2\varepsilon+\varepsilon^2}{3\varepsilon^4}$	$-\frac{3}{4}\frac{3n^3+24n^2+80n+136}{243}g^4$	$-\frac{1}{4}\frac{(n+2)^3}{27}g^4$
	$-\frac{1-5\varepsilon+13\varepsilon^2-11\varepsilon^3+6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\frac{n^3+14n^2+76n+152}{243}g^4$	$-\frac{1}{2}\frac{(n+2)^2(n+8)}{81}g^4$
	$-\frac{5-10\varepsilon-\varepsilon^2+6\varepsilon^3}{24\varepsilon^4}$	$-\frac{3}{2}\frac{11n^2+76n+156}{243}g^4$	$-\frac{1}{4}\frac{(n+2)^2}{9}g^4$
	$\frac{5\zeta(5)}{\varepsilon}$	$-3\frac{2n^2+55n+186}{243}g^4$	
	$-\frac{\zeta(3)-3(\zeta(3)-\frac{1}{2}\zeta(4))\varepsilon}{2\varepsilon^2}$	$-6\frac{5n^2+62n+176}{243}g^4$	
	$-\frac{1-\frac{8}{3}\varepsilon+\frac{5}{3}\varepsilon^2+\frac{8}{3}\varepsilon^3}{8\varepsilon^4}$	$-6\frac{7n^2+72n+164}{243}g^4$	
	$-\frac{1-\varepsilon-\varepsilon^2-\varepsilon^3+2\zeta(3)\varepsilon^3}{4\varepsilon^4}$	$-\frac{3}{2}\frac{n^3+14n^2+76n+152}{243}g^4$	
	$-\frac{1-2\varepsilon-2\varepsilon^2+6\varepsilon^3-6\zeta(3)\varepsilon^3}{6\varepsilon^4}$	$-\frac{3}{8}\frac{n^3+18n^2+80n+144}{243}g^4$	$-\frac{1}{8}\frac{(n+2)^2(n+8)}{81}g^4$
	$-\frac{1-3\varepsilon+\frac{7}{2}\varepsilon^2}{12\varepsilon^3}$	$-\frac{3}{4}\frac{n^3+18n^2+96n+128}{243}g^4$	$-\frac{1}{4}\frac{(n+2)^2(n+8)}{81}g^4$
	$-\frac{1-\frac{11}{2}\varepsilon+\frac{121}{12}\varepsilon^2}{16\varepsilon^3}$	$-\frac{5n^2+32n+44}{81}g^4$	$-\frac{1}{6}\frac{(n+2)^2}{9}g^4$

	$-\frac{1-4\varepsilon+5\varepsilon^2+6\varepsilon^3-12\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\frac{11n^2+76n+156}{243}g^4$	$-\frac{1}{4}\frac{(n+2)(5n+22)}{81}g^4$
	$-\frac{1-\varepsilon-\varepsilon^2-\varepsilon^3+2\zeta(3)\varepsilon^3}{4\varepsilon^4}$	$-\frac{3}{4}\frac{3n^3+24n^2+80n+136}{243}g^4$	$-\frac{1}{8}\frac{(n+2)(n^2+6n+20)}{81}g^4$
	$-\frac{5-\frac{10}{3}\varepsilon-\frac{37}{4}\varepsilon^2}{48\varepsilon^3}$	$-\frac{5n^2+32n+44}{81}g^4$	$-\frac{1}{6}\frac{(n+2)^2}{9}g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3-6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\frac{n^3+10n^2+72n+160}{243}g^4$	$-\frac{1}{4}\frac{(n+2)(n^2+6n+20)}{81}g^4$
	$-\frac{3\zeta(3)-3(\zeta(3)+\frac{1}{2}\zeta(4))\varepsilon}{2\varepsilon^2}$	$-3\frac{5n^2+62n+176}{243}g^4$	$-\frac{1}{2}\frac{(n+2)(5n+22)}{81}g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3}{12\varepsilon^4}$	$-\frac{3}{2}\frac{7n^2+72n+164}{243}g^4$	$-\frac{1}{4}\frac{(n+2)(5n+22)}{81}g^4$
	$-\frac{1-\frac{8}{3}\varepsilon+\frac{5}{3}\varepsilon^2+\frac{8}{3}\varepsilon^3}{8\varepsilon^4}$	$-3\frac{11n^2+76n+156}{243}g^4$	$-\frac{1}{2}\frac{(n+2)(5n+22)}{81}g^4$
	$-\frac{1-3\varepsilon+\varepsilon^2+5\varepsilon^3-6\zeta(3)\varepsilon^3}{12\varepsilon^4}$	$-3\frac{n^3+14n^2+76n+152}{243}g^4$	$-\frac{1}{2}\frac{(n+2)(n^2+6n+20)}{81}g^4$
	$-\frac{1-6\varepsilon+19\varepsilon^2-30\varepsilon^3+24\zeta(3)\varepsilon^3}{24\varepsilon^4}$	$-6\frac{7n^2+72n+164}{243}g^4$	$-\frac{(n+2)(5n+22)}{81}g^4$
	$-\frac{1-6\varepsilon+19\varepsilon^2-30\varepsilon^3}{24\varepsilon^4}$	$-6\frac{7n^2+72n+164}{243}g^4$	$-\frac{(n+2)(5n+22)}{81}g^4$

Hence, to find the Gell-Mann-Low function $\beta(g)$ and anomalous dimensions $\gamma(g)$ we have to know the coefficients at $1/\varepsilon$ in the renormalization constant expansions in the inverse powers of ε.

To perform the corresponding calculations in four-loop approximation in the theory (1) we use the methods developed in papers/7,8/. The functions $\beta(g)$, $\gamma_2(g)$ and $\gamma_4(g)$ were computed according to eqs.(8),(10). As for $\gamma_m(g)$ it turns out to be more convenient to use the equality $\gamma_m(g)=\gamma_2(g)-\gamma_{\varphi^2}(g)$ following from $Z_m=Z_{\varphi^2}Z_2^{-1}$ where $\gamma_{\varphi^2}(g)=\partial \ln Z_{\varphi^2}/\partial \ln\mu\,|_{m_B,g_B}$ and Z_{φ^2} is the renormalization constant of the two-point Green function with φ^2 insertion, i.e., $\langle \varphi(x)\varphi(0)\int dy\, \vec{\varphi}^2(y)\rangle$. The contributions of various diagrams to Z_2 (for propagator), Z_4 and Z_{φ^2} (for four-vertex) are given in Table 1. For the renormalization group functions we obtain the following expansions:

$$\gamma_2(g)=\frac{g^2}{36}(n+2)-\frac{g^3(n+2)(n+8)}{16\cdot 27}+\frac{5g^4(n+2)}{64\cdot 81}\cdot(-n^2+18n+100)+O(g^5), \tag{11}$$

$$\gamma_m(g)=\frac{g}{6}(n+2)-\frac{5g^2}{36}(n+2)+\frac{g^3(n+2)(5n+37)}{72}-\frac{g^4(n+2)}{64\cdot 243}\Big[-n^2+7578n+31060+48\zeta(3)(3n^2+10n+68)+288\zeta(4)(5n+22)\Big]+O(g^5) \tag{12}$$

$$\gamma_4(g)=-\frac{g}{6}(n+8)+\frac{g^2}{9}(5n+22)-\frac{g^3}{16\cdot 27}\Big[35n^2+942n+2992+96\zeta(3)(5n+22)\Big]+\frac{g^4}{8\cdot 243}\Big[-5n^3+6640n^2+20624n+49912+24\zeta(3)(63n^2+764n+2332)-72\zeta(4)\cdot(5n^2+62n+176)+480\zeta(5)(2n^2+55n+186)\Big]+O(g^5), \tag{13}$$

$$\beta(g)=-g\big(\gamma_4(g)-2\gamma_2(g)\big)=$$
$$=\frac{g^2}{6}(n+8)-\frac{g^3}{6}(3n+14)+\frac{g^4}{16\cdot 27}\Big[33n^2+922n+2960+96\zeta(3)(5n+22)\Big]-\frac{g^5}{32\cdot 243}\Big[-5n^3+6320n^2+80456n+196648+96\zeta(3)(63n^2+764n+2332)-288\zeta(4)(5n^2+62n+176)+1920\zeta(5)(2n^2+55n+186)\Big]+O(g^6). \tag{14}$$

In particular, for $n=1$ we have

$$\gamma_2(g)=\frac{g^2}{12}-\frac{g^3}{16}+\frac{65}{192}g^4, \tag{15}$$

$$\gamma_m(g)=\frac{g}{2}-\frac{5g^2}{12}+\frac{7}{4}g^3-\frac{3}{4}g^4\Big(\frac{159}{16}+\zeta(3)+2\zeta(4)\Big), \tag{16}$$

$$\gamma_4(g)=-\frac{3}{2}g+3g^2-g^3\Big(\frac{147}{16}+6\zeta(3)\Big)+g^4\Big(\frac{297}{8}+39\zeta(3)-9\zeta(4)+60\zeta(5)\Big), \tag{17}$$

$$\beta(g)=\frac{3}{2}g^2-\frac{17}{6}g^3+g^4\Big(\frac{145}{16}+6\zeta(3)\Big)-g^5\Big(\frac{3499}{96}+39\zeta(3)-9\zeta(4)+60\zeta(5)\Big). \tag{18}$$

The obtained expansions of the renormalization group functions can be used in quantum field theory (in four-dimensional space-time) for the investigation of ultraviolet behaviour of the Green functions. They can also be utilized in different attempts to go beyond the perturbation theory by continuating its results into the region $g \gtrsim 1$ of coupling constant, as it was made for example, in papers/9,14/. In the present paper we use the obtained results to determine the critical exponents of phase transitions in the framework of field-theoretic approach to critical phenomena based on the ε-expansion.

3. Critical Exponents and ε-Expansion

The renormalization group method being transferred from quantum field theory to statistical mechanics has been very successfull for the description of critical behaviour of various systems in the neighbourhood of second order phase transition. Scaling invariance taking place in critical phenomena finds its natural interpretation in terms of renormalization group: Scaling behaviour near the critical point caused by the appearance of long-range order can be described in terms of Euclidean quantum field theory possessing an infra-red stable fixed point. The fixed point g_0 of renormalization group equation

$$p^2 \frac{\partial}{\partial p^2} \bar{g}\left(\frac{p^2}{\mu^2}, g\right) = \beta\left(\bar{g}\left(\frac{p^2}{\mu^2}, g\right)\right) \tag{19}$$

is determined by the equation $\beta(g_0)=0$ and is called infra-red stable one, if $\beta'(g_0)>0$. In this case effective charge $\bar{g}(p^2/\mu^2, g)$ tends to g_0 when the momentum argument p^2 tends to zero, i.e., on large distances. In the presence of infra-red stable fixed point the dimensionless Green functions obey the scaling laws for small p^2 /3/

$$\Gamma_R \underset{p^2\to 0}{\sim} (p^2)^{-\gamma_r(g_0)} \tag{20}$$

with the powers equal to the value of anomalous dimensions at $g=g_0$.

A systematic approach to the description of critical phenomena in terms of quantum field theory of φ^4 model has been developed in papers /15,16/. There have been found direct relations between the anomalous dimension (20) and critical exponents, characterizing scaling behaviour of statistical quantities in the neighbourhood of critical temperature T_c. Thus, for example, at $T-T_c$ asymptotics of correlation function $\Gamma(\vec{x})$ for $|\vec{x}|\to\infty$ is defined by the exponent η,

$$\Gamma(\vec{x}) \underset{|\vec{x}|\to\infty}{\sim} \frac{1}{|\vec{x}|^{d-2+\eta}}. \tag{21}$$

Correlation length ξ for $t=T-T_c\to 0$ obeys the following scaling law,

$$\xi \underset{t\to 0}{\sim} t^{-\nu}(1+const\cdot t^{\omega\nu}+\dots), \tag{22}$$

where ω characterizes the deviation from scaling. All other critical exponents can be expressed in terms of η and ν /16/ For example, critical exponent γ determining the power behaviour of susceptibility χ,

$$\chi \underset{t\to 0}{\sim} t^{-\gamma}(1+const\cdot t^{\omega\nu}+\dots) \tag{23}$$

is related with η and ν by the equation

$$\gamma = (2-\eta)\nu. \tag{24}$$

Based on the analogy between grand partition function and quantum field generating functional, we can establish the following identities between statistical and field-theoretical quantities: /16/ The quantum field $\vec{\varphi}$ is identified with the order parameter, the temperature difference $T-T_c$ with the square of the bare mass m_B^2, the correlation function $\Gamma(\vec{x})$ with the propagator $\langle\varphi(x)\varphi(0)\rangle$, the susceptibility χ with the quantity $D(0)$, where $D(p^2)=\int dx\, e^{ipx}\langle\varphi(x)\varphi(0)\rangle$ is the Fourier transform of the propagator, and the inverse correlation length ξ^{-1} corresponds to the physical mass m_{ph} which defines the position of the pole of $D(p^2)$: $D^{-1}(p^2=-m_{ph}^2)=0$. To find the critical exponents on the basis of φ^4 theory we have to investigate quantitatively the power behaviour of m_{ph} and $D(0)$ for $m_B^2\to 0$ and also the behaviour of $D(p^2)$ in the limit $p^2\to 0$ at $m_B^2=0$. The renormalization group equations reduce this analysis to finding of anomalous dimensions $\gamma_m(g)$ and $\gamma_2(g)$ at the infra-red stable fixed point $g=g_0$.

It can easily be seen from eq.(14), that β-function vanishes at g =0. Thus for d =4 an infra-red stable fixed point g =0 and all anomalous dimensions $\gamma(g_0)$ also vanish. This means that in the limit $p^2\to 0$ interaction disappears, i.e., the φ^4 theory in four dimensions behaves like a free theory in the infra-red region.

The idea of ε-expansion [4] is the following: We start from the $d=4$ case being a zero approximation and then construct a perturbation theory in powers of $2\varepsilon = 4-d$. To pass then to the physically interesting case $d=3$ we must put $2\varepsilon=1$. Indeed, performing renormalizations in the φ^4 model at $\varepsilon \neq 0$ one should use the β-function

$$\beta_\varepsilon(g) = -\varepsilon g + \beta(g), \tag{25}$$

where $\beta(g)$ is the four-dimensional Gell-Mann-Low function. The function $\beta_\varepsilon(g)$ vanishes at some point $g_0(\varepsilon)$ and has a positive derivative at this point. For ε small infra-red stable fixed point $g_0(\varepsilon)$ is of an order of ε, that enables us, reexpanding $\gamma(g_0(\varepsilon))$ in power series in ε, to express all critical exponents in terms of these new expansions, the so-called ε-expansions.

We present now the explicit relations between the critical exponents and anomalous dimensions $\gamma(g_0)$. The corresponding formulas have been obtained in [16]. However, we think it to be expedient to reproduce here the derivation of these relations in the framework of dimensional regularization scheme used throughout this paper.

Taking the Fourier transform of (21) and comparing it with the asymptotic expression of the propagator

$$D(p^2) \underset{p^2\to 0}{\sim} (p^2)^{-1+\gamma_2(g_0)} \tag{26}$$

we get

$$\eta = 2\gamma_2(g_0). \tag{27}$$

The other exponent ν is connected with the physical mass m_{ph}. The latter is the renormalization invariant and obeys the renormalization group equation without anomalous dimensions,

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + \beta_\varepsilon(g)\frac{\partial}{\partial g} + \gamma_m(g)m^2\frac{\partial}{\partial m^2}\right) m^2_{ph}(m^2,\mu^2,g) = 0. \tag{28}$$

According to the theorem of homogeneous functions,

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + m^2\frac{\partial}{\partial m^2}\right) m^2_{Ph} = m^2_{Ph}. \tag{29}$$

Then, excluding $\mu^2\frac{\partial}{\partial\mu^2}$ from (28) with the help of (29), we have

$$\left[m^2\frac{\partial}{\partial m^2} - \frac{\beta_\varepsilon(g)}{1-\gamma_m(g)}\frac{\partial}{\partial g} - \frac{1}{1-\gamma_m(g)}\right] m^2_{Ph} = 0. \tag{30}$$

Using the standard methods of analysis of such equations [3], we come to the conclusion that

$$m^2_{Ph} \underset{m^2\to 0}{\sim} (m^2)^{\frac{1}{1-\gamma_m(g_0)}}. \tag{31}$$

So far as μ and g are fixed we have

$$m^2 \sim m_B^2 \equiv t. \tag{32}$$

Hence

$$\xi^{-1} \equiv m_{Ph} \underset{t\to 0}{\sim} t^{\frac{1}{2(1-\gamma_m(g_0))}}. \tag{33}$$

and, comparing (33) with (22), we find

$$\nu = \frac{1}{2(1-\gamma_m(g_0))}. \tag{34}$$

Taking into account the second term in the expansion of the effective charge near the fixed point,

$$\bar g\left(\frac{p^2}{\mu^2}, g\right) = g_0 + const\cdot\left(\frac{p^2}{\mu^2}\right)^{\beta'(g_0)} + \dots, \tag{35}$$

we can find the correction to the solution (31) which determines the exponent ω:

$$\omega = 2\beta'_\varepsilon(g_0). \tag{36}$$

Let us also derive equation (24). For $D(0)$ the following renormalization group equation is true:

$$\left[\mu^2 \frac{\partial}{\partial \mu^2} + \beta_\varepsilon(g) \frac{\partial}{\partial g} + \gamma_m(g) m^2 \frac{\partial}{\partial m^2} + \gamma_2(g) \right] D(0) = 0. \tag{37}$$

Proceeding in the same way as for eq.(28), we come to

$$D(0) \underset{m^2 \to 0}{\sim} (m^2)^{-\frac{1-\gamma_2(g_0)}{1-\gamma_m(g_0)}} \tag{38}$$

and therefore

$$\gamma = \frac{1-\gamma_2(g_0)}{1-\gamma_m(g_0)} = (2-\eta)\nu. \tag{39}$$

Now we are in a position to write down the ε -expansions up to ε^4 for $g_0(\varepsilon)$ and critical exponents η , ν and ω on the basis of eqs.(27),(34) and (36) and the functions $\gamma_2(g)$, $\gamma_m(g)$ and $\beta(g)$ calculated in four-loop approximation. It should be noticed that the coefficients of ε -expansions are independent of the renormalization procedure. Our results which are given below are in agreement with those of paper /6/ where the corresponding calculations have been performed up to ε^3 for $g_0(\varepsilon), \nu$ and ω and up to ε^4 for η . In order to obtain the standard form of ε -expansion we choose the expansion parameter to be 2ε .

$$g_0(\varepsilon) = \frac{3}{n+8}(2\varepsilon) + \frac{9(3n+14)}{(n+8)^3}(2\varepsilon)^2 + \frac{3(2\varepsilon)^3}{8(n+8)^5}\Big[-33n^3 + 110n^2 + 1760n + 4544 - 96\zeta(3)(n+8)(5n+22)\Big] + \frac{(2\varepsilon)^4}{16(n+8)^7}\Big[-5n^5 - 2670n^4 - 5584n^3 + 52784n^2 + 309312n + 529792 + 1920\zeta(5)(n+8)^2 \cdot (2n^2+55n+186) - 288\zeta(4)(n+8)^3(5n+22) - 96\zeta(3)(-63n^4 - 422n^3 + 4452n^2 + 39432n + 72512)\Big] + O(\varepsilon^5), \tag{40}$$

$$\eta = \frac{(n+2)}{2(n+8)^2}(2\varepsilon)^2\Big[1 + \frac{2\varepsilon}{4(n+8)^2}(-n^2+56n+272) + \frac{(2\varepsilon)^2}{16(n+8)^4} \cdot \big(-5n^4 - 230n^3 + 1124n^2 + 17920n + 46144 - 384\zeta(3)(n+8)(5n+22)\big)\Big] + O(\varepsilon^5), \tag{41}$$

$$\frac{1}{\nu} = 2 - \frac{n+2}{n+8}(2\varepsilon) - \frac{n+2}{2(n+8)^3}(13n+44)(2\varepsilon)^2 - \frac{(n+2)(2\varepsilon)^3}{8(n+8)^5}\Big[-3n^3 + 452n^2 + 2672n + 5312 - 96\zeta(3)(n+8)(5n+22)\Big] - \frac{(n+2)(2\varepsilon)^4}{32(n+8)^7}\Big[-3n^5 - 398n^4 + 12900n^3 + 81552n^2 + 219968n + 357120 + 1280\zeta(5)(n+8)^2 \cdot (2n^2+55n+186) - 288\zeta(4)(n+8)^3(5n+22) - 16\zeta(3)(n+8) \cdot (3n^4 - 194n^3 + 148n^2 + 9472n + 19488)\Big] + O(\varepsilon^5), \tag{42}$$

$$\omega = (2\varepsilon) - \frac{3(3n+14)}{(n+8)^2}(2\varepsilon)^2 + \frac{(2\varepsilon)^3}{4(n+8)^4}\Big[33n^3 + 538n^2 + 4288n + 9568 + 96\zeta(3)(n+8)(5n+22)\Big] - \frac{(2\varepsilon)^4}{16(n+8)^6}\Big[-5n^5 + 1488n^4 + 46616n^3 + 419528n^2 + 1750080n + 2599552 + 32\zeta(3)(n+8)(189n^3 + 1644n^2 + 5748n + 11616) + 1920\zeta(5)(n+8)^2(2n^2+55n+186) - 288\zeta(4)(n+8)^3(5n+22)\Big] + O(\varepsilon^5). \tag{43}$$

4. Summation of the ε -Expansion Series

It is well known that perturbation theory series in the coupling constant g are asymptotical. In recent years the technique of estimating the high order coefficients has been developed /17/. The series of ε -expansion, arising when solving the equation $\beta_\varepsilon(g_0(\varepsilon)) = 0$, are also asymptotical. As it was shown in paper /18/ asymptotic estimates of the coefficients of perturbation series in g lead to the following estimates of the coefficients of high orders of ε -expansion $f(2\varepsilon) = \sum_k (-2\varepsilon)^k f_k$:

$$f_k \underset{k \to \infty}{\sim} k!\, a^k k^b c\,, \tag{44}$$

where f stands for g_0 , η , $1/\nu$, or ω and parameters a and b are respectively

$$a=\frac{3}{n+8}\quad,\quad b=\begin{cases}4+\frac{n}{2} & \text{for } g_o,\\ 3+\frac{n}{2} & \text{for } \eta,\\ 4+\frac{n}{2} & \text{for } 1/\nu,\\ 5+\frac{n}{2} & \text{for } \omega.\end{cases}\tag{45}$$

From (45) it follows that ε -expansion series have zero radius of convergence. Therefore, the direct substitution of $\varepsilon = 1/2$ into (41)-(43) cannot lead to any reliable conclusion about the values of critical exponents at physical point $d = 3$. To come to the value $\varepsilon = 1/2$ we use the method for summation of the asymptotical series we have developed earlier/9/. It takes into account the exact coefficients of lower orders (40)-(43) and, on the other hand, the information obtained from asymptotic estimates (44),(45). It is based on a modified Borel transform

$$f(2\varepsilon)=\int_0^\infty \frac{dx}{2\varepsilon a}\, e^{-\frac{x}{2\varepsilon a}}\left(\frac{x}{2\varepsilon a}\right)^{b+3/2} B(x)\,.\tag{46}$$

Then

$$B(x)=\sum_k (-x)^k B_k\;,\qquad B_k=\frac{f_k}{a^k\Gamma(k+b+\frac{5}{2})}\;\widetilde{\scriptstyle k\to\infty}\;\frac{c}{k^{3/2}}\,.\tag{47}$$

The series (47) for $B(x)$ possesses the unit circle of convergence. The function $B(x)$, as it follows from (44), is free of singularities in the integration region $[0,\infty)$ and has a square root branch point/19/ at $x=-1$. In order to perform the analytical continuation of $B(x)$ beyond the unit circle, we use conformal mapping $x\to w$ with

$$w(x)=\frac{\sqrt{1+x}-1}{\sqrt{1+x}+1}\,.\tag{48}$$

It maps the integration region $[0,\infty)$ into the interior of the unit circle while the cut $(-\infty,-1]$ is mapped into its boundary. Inside this circle the series in w , obtained by reexpansion of the function $B(x(w))$, is convergent. The coefficient at w^N is determined by the coefficients f_k of the initial ε -expansion with $k\leqslant N$. Therefore, if we truncate the series in w at N's term

$$B(x)\approx\left(\frac{x}{w}\right)^{\lambda}\sum_k^N B_k^{(\lambda)} w^k,\tag{49}$$

we obtain for $f(2\varepsilon)$ an approximate expression $f_N(2\varepsilon)$ corresponding to N -th order of pertubtation theory.

Of great importance here is the specific value of the parameter λ introduced in (49). It determines the power of asymptotic behaviour of $f_N(2\varepsilon)$ for large ε ,

$$f_N(2\varepsilon)\;\widetilde{\scriptstyle \varepsilon\to\infty}\;(2\varepsilon)^{\lambda}.\tag{50}$$

For the solvable models with the known asymptotic behaviour in the coupling constant g , we find that the best convergence of the sequence of approximants $f_N(g)$ to the true function $f(g)$ is achieved only for the choice of λ , which is consistent with asymptotics of $f(g)$ for $g\to\infty$ /9/. As far as the behaviour of the critical exponents for large ε is unknown, we fix λ just from the requirement of steepest convergence of our approximation procedure.

Consider the set of quantities

$$\Delta_N = 1-\frac{f_N(1)}{f_{N-1}(1)}\tag{51}$$

characterizing the relative variation of f_N when taking into account the next term of perturbation theory. If we guess the true asymptotic behaviour of $f(2\varepsilon)$, the relative errors Δ_N should decrease very fast. Numerical analysis shows the existence of a sharp minimum of $|\Delta_N|$ at the definite values of λ , which appear to be different for each critical exponent. We have checked this method on solvable models and obtained very hopeful results/9/. In our case an application of this method leads to the following values of λ :

$$\lambda = \begin{cases} 1{,}2 \div 1{,}4 & \text{for } g_0 , \\ 2 \div 3 & \text{for } \eta , \\ 1{,}0 \div 1{,}3 & \text{for } 1/\nu , \\ 0{,}7 \div 0{,}9 & \text{for } \omega . \end{cases} \qquad (52)$$

To define the value of λ we used also the other method proposed in paper[20]. It is based on the fact that the asymptotics of $f_N(2\varepsilon)$ for large ε is defined by the asymptotics of the coefficients $B_k^{(\lambda)}$ for $k \to \infty$, which was estimated numerically. The values of λ obtained by this method lie within the bounds (52).

Now using the ε-expansions (41)-(43) and the values of λ (52) we can calculate the critical exponents at the point $2\varepsilon = 1$ according to equations (46) and (49) for $N = 4$. The results of calculations are summarized in Table 2. The value of errors are taken to be equal to $\pm \Delta_4 \cdot f_4$ (1). We present also the value of $g_0(2\varepsilon)$ for $\varepsilon = 1/2$:

	$n = 1$	$n = 2$	$n = 3$
$g_0(1) =$	$0{,}488 \pm 0{,}006$	$0{,}435 \pm 0{,}006$	$0{,}392 \pm 0{,}006$

It is interesting to observe the variations of the values of critical exponents with the number of terms of perturbation theory taken into account. Below we write down (in the case $n = 1$) the values of the exponents obtained both by direct substitution of $2\varepsilon = 1$ into (41),(42) and by using summation methods described in this section (respectively $\eta_N^{PT}(1)$, $\nu_N^{PT}(1)$ and $\eta_N(1), \nu_N(1)$).

N	1	2	3	4
$\eta_N^{PT}(1)$	-	0,0185	0,0372	0,0289
$\eta_N(1)$	-	0,0320	0,0332	0,0333
$\nu_N^{PT}(1)$	0,6	0,645	0,595	0,731
$\nu_N(1)$	0,620	0,626	0,625	0,628

As one can see from this table, an application of special

Table 2. Comparison of our results (the first column) with the calculations in φ^4 model in 3 dimensions (the second column), with high temperature expansion in 3-dimensional Ising ($n = 1$) and Heisenberg ($n = 3$) models (the third column) and with the experiment (the fourth column). The numbers given in the second, third and fourth columns are taken from paper [10].

		$n = 1$		
η	$0{,}0333 \pm 0{,}0001$	$0{,}0315 \pm 0{,}0025$		$0{,}016 \pm 0{,}014$
ν	$0{,}628 \pm 0{,}002$	$0{,}6300 \pm 0{,}0008$	$0{,}638^{+0{,}002}_{-0{,}008}$	$0{,}625 \pm 0{,}005$
ω	$0{,}781 \pm 0{,}015$	$0{,}782 \pm 0{,}010$		
		$n = 2$		
η	$0{,}0352 \pm 0{,}0001$	$0{,}0335 \pm 0{,}0025$		
ν	$0{,}666 \pm 0{,}004$	$0{,}6693 \pm 0{,}0010$		$0{,}675 \pm 0{,}001$
ω	$0{,}777 \pm 0{,}015$	$0{,}778 \pm 0{,}008$		
		$n = 3$		
η	$0{,}0354 \pm 0{,}0001$	$0{,}0340 \pm 0{,}0025$	$0{,}043 \pm 0{,}014$	
ν	$0{,}700 \pm 0{,}007$	$0{,}7054 \pm 0{,}0011$	$0{,}715^{+0{,}025}_{-0{,}015}$	
ω	$0{,}779 \pm 0{,}007$	$0{,}779 \pm 0{,}006$		

methods of summation drastically improves the approximating properties of perturbation theory.

The summation technique developed in this paper has been also used to find the values of the critical exponents at $2\varepsilon = 2$ which corresponds[21] (for $n = 1$) to the two-dimensional Ising model allowing an exact solution. For the exponents η and ν we obtain the results: $\eta_4(2) = 0{,}18$ and $\nu_4(2) = 0{,}92$ to be compared with the exact values $1/4$ and 1, respectively.

Concluding this section we would like to note that the corrections $\sim \varepsilon^4$ by themselves, i.e.,without an application of special summation methods, do not improve the agreement with the values of critical exponents known from the literature, while the method proposed above improves this agreement drastically. It is an additional argument in favour of its efficiency and gives a direct confirmation of applicability of quantum field theory approach based on ε -expansion to the evaluation of critical exponents.

A c k n o w l e d g e m e n t s

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