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THE TRANSFORMATION COEFFICIENTS
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**A NEW METHOD OF OBTAINING
THE TRANSFORMATION COEFFICIENTS
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Новый способ вывола коэффициента преобразования
трехчастичной базисной функции

Исследуется новый способ вывола коэффициентов преобразования
трехчастичных функций в шестимерном пространстве. Этот способ являет-
ся обобщением предложенных прежде методов. Применяется теоретико-
групповое представление. Полученный результат включает в себя как
частный случай выражения для коэффициентов преобразования, известные
раньше.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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A New Method of Obtaining the Transformation
Coefficients for the Three-Body Basis Functions

A new method is proposed for obtaining the coefficient
 $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi$ corresponding to transformations of the
three-body basis function in the six-dimensional space.

The investigation has been performed at the Laboratory of
Theoretical Physics, JINR.

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In the quantum mechanical three-body problem
there are several ways for constructing basis
functions. One of the easiest is to choose the
set of the so-called tree-functions $\Phi_{KJM}^{j_1 j_2}(\vec{\eta}, \vec{\xi})/1/$.
The transformations of the Jacobi coordinates
generate transformations of the basis functions:

$$\Phi_{KJM}^{j'_1 j'_2}(\vec{\eta}', \vec{\xi}') = \sum_{j_1 j_2} \langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi \Phi_{KJM}^{j_1 j_2}(\vec{\eta}, \vec{\xi}). \quad (1)$$

The coefficients $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi$, i.e., matrix
elements of these transformations, were first
obtained by Raynal and Revai^{/2/}, and Efros and
Smorodinsky^{/3/}. The results, however, looked
different to such an extent, that it made neces-
sary to prove the identity of the two forms of
 $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi$. This was done in paper^{/4/}. Look-
ing for a common form of the two versions, we
found a much simpler method of obtaining the
transformation coefficient, which, by the way,
allows quite surprising group theoretical gene-
ralizations. In the following we present this
new way of constructing $\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi$.

By definition, we can write

$$\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^\phi = \int (\Phi_{KJM}^{j'_1 j'_2}(\vec{\eta}', \vec{\xi}'))^* \Phi_{KJM}^{j_1 j_2}(\vec{\eta}, \vec{\xi}) d\vec{\eta} d\vec{\xi}. \quad (2)$$

Making use of the orthonormality of the basis functions, we can formally consider the matrix element of unity instead of the overlap integral:

$$\langle j_1 j_2 | j_1 j_2 \rangle_{KJM}^{\phi} = \langle j_1 j_2 | 1 | j_1 j_2 \rangle_{KJM}^{\phi} \quad (3)$$

That means, that one can avoid a rather lengthy procedure of straightforward calculation of (2). Instead of that, one has to find such an expansion of unity into a series, in which $\langle j_1 j_2 | j_1 j_2 \rangle_{KJM}^{\phi}$ occurs as a coefficient. For that purpose it is necessary to write down 1 in a form which can be expanded in terms of the basis functions corresponding to J and M.

The explicit form of the basis function

$\Phi_{KJM}^{j_1 j_2}(\vec{\eta}, \vec{\xi})$ is the following:

$$\Phi_{KJM}^{j_1 j_2}(\vec{\eta}, \vec{\xi}) = Y_{JM}^{j_1 j_2}(\vec{m}, \vec{n}) \Psi_{Kj_1 j_2}(\Phi) \quad (4)$$

The part of this function, corresponding to J, M is

$$Y_{JM}^{j_1 j_2}(\vec{m}, \vec{n}) = \sum_{m_1 m_2} C_{j_1 m_1 j_2 m_2}^{JM} Y_{j_1 m_1}(\vec{m}) Y_{j_2 m_2}(\vec{n}) \quad (5)$$

where

$$\vec{m} = \frac{\vec{\eta}}{\eta}, \quad \vec{n} = \frac{\vec{\xi}}{\xi},$$

and j_1, j_2 are the angular momenta, corresponding to $\vec{\eta}$ and $\vec{\xi}$.

$$\Psi_{Kj_1 j_2}(\Phi) = N_{Kj_1 j_2} (\sin \Phi)^{j_1} (\cos \Phi)^{j_2} P_{\frac{K-j_1-j_2}{2}}^{(j_1+\frac{1}{2}, j_2+\frac{1}{2})}(\cos 2\Phi) \quad (6)$$

where we used $\xi^2 + \eta^2 = 1$ and therefore could introduce

$$\cos \Phi = \xi, \quad \sin \Phi = \eta.$$

The normalization factor $N_{Kj_1 j_2}$ is well-known from the theory of Jacobi polynomials:

$$N_{Kj_1 j_2} = \left[\frac{2(K+2) \Gamma(\frac{K-j_1-j_2}{2} + 1) \Gamma(\frac{K+j_1+j_2}{2} + 2)}{\Gamma(\frac{K-j_1+j_2}{2} + \frac{3}{2}) \Gamma(\frac{K+j_1-j_2}{2} + \frac{3}{2})} \right]^{\frac{1}{2}} \quad (7)$$

Let us introduce now two three-dimensional vectors \vec{P}_i and \vec{Q}_i having the properties

$$\vec{P}_i = \vec{Q}_i = 1$$

and

(8)

$$\vec{P}_i \vec{Q}_i = 0.$$

We can write then

$$1 = e^{2(P_i^2 - Q_i^2)} \quad (9)$$

where the factor 2 is introduced for the sake of convenience. For reasons, which will become clear later, we consider in the following $i\vec{Q}_i$ instead of \vec{Q}_i . Further, we substitute one of the P_i -s and Q_i -s by P_k and Q_k , which are connected with the original vectors by the orthogonal transformation

$$i\vec{Q}_k = \cos \phi i\vec{Q}_i + \sin \phi \vec{P}_i \quad (10)$$

$$\vec{P}_k = \sin \phi i\vec{Q}_i - \cos \phi \vec{P}_i$$

(Because of the orthogonality of this transformation the properties (8) are invariant). Thus, instead of (9), we have

$$e^{-2P_i P_k \cos \phi - 2Q_i Q_k \cos \phi + 2iP_i Q_k \sin \phi + 2iQ_i P_k \sin \phi} \quad (11)$$

Using the plane-wave expansion

$$e^{i\vec{p}\cdot\vec{x}} = \sum_{\lambda\mu} i^\lambda j_\lambda(p x) Y_{\lambda\mu}^*(p) Y_{\lambda\mu}(x)$$

we can re-write (11) in the form

$$\sum_{pqrs} (-1)^{\frac{p+r+q+s}{2}} j_p(2iP_i P_k \cos\phi) j_r(2Q_i P_k \sin\phi) j_q(2P_i Q_k \sin\phi) j_s(2iQ_i Q_k \cos\phi) \quad (12)$$

$$\cdot Y_{p\pi}^*(\hat{P}_i) Y_{q\chi}^*(\hat{P}_i) Y_{r\rho}^*(\hat{Q}_i) Y_{s\sigma}^*(\hat{Q}_i) Y_{i p \pi}(\hat{P}_i) Y_{r \rho}(\hat{P}_i) Y_{r \rho}(\hat{P}_i) Y_{q \chi}(\hat{Q}_i) Y_{k s \sigma}(\hat{Q}_i),$$

where $\hat{P}_i, \hat{Q}_i, \hat{P}_k$ and \hat{Q}_k denote the three-dimensional polar angles.

Remember now the so-called Bateman expansion, which is usually given in the form (^{5/} p. 974, f. 8.442 (2)).

$$J_\nu(az) J_\mu(bz) = \frac{2}{\Gamma(\mu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k (az)^{2k} \Gamma(-k, -\nu-k; \mu+1; \frac{b^2}{a^2})}{k! \Gamma(\nu+k+1)}$$

and note, that the hypergeometrical function can be substituted by the Jacobi polynomial (^{5/} p. 1050, f. 8.962 (4)):

$$P_k^{(\nu, \mu)}(x) = \frac{\Gamma(k+1+\mu)}{k! \Gamma(1+\mu)} \left(\frac{x-1}{2}\right)^k F(-k, -k-\nu; \mu+1; \frac{x+1}{x-1})$$

or by Wigner's d-function. Substituting

$$j_\lambda(x) = \sqrt{\frac{\pi}{2x}} J_{\lambda+\frac{1}{2}}(x),$$

we can apply these formulae to the spherical Bessel functions which occur in (12). It is easy to see, that the step from the hypergeometrical function to the Jacobi polynomial is possible

only, if we consider the imaginary vector iQ_i . Replacing the lengths of $\vec{P}_i, \vec{Q}_i, \vec{P}_k, \vec{Q}_k$ by unity, we can write then

$$j_p(2i\cos\phi) j_q(2\sin\phi) = \frac{\pi}{4} \sum_{K_1} (-1)^{K_1} \Gamma^{\frac{q}{2} + \frac{1}{2}} (\sin\phi \cos\phi)^{-\frac{1}{2}} d^{\frac{K_1+1}{2}} \frac{(\cos 2\phi)}{\frac{p+q+1}{2}, \frac{q-p}{2}} \cdot \left[\left(\frac{K_1-p+q}{2} + \frac{1}{2} \right)! \left(\frac{K_1+p-q}{2} + \frac{1}{2} \right)! \left(\frac{K_1+p+q}{2} + 1 \right)! \left(\frac{K_1-p-q}{2} \right)! \right]^{-\frac{1}{2}},$$

and

$$j_s(2i\cos\phi) j_r(2\sin\phi) = \frac{\pi}{4} \sum_{K_2} (-1)^{K_2} \Gamma^{\frac{s}{2} + \frac{1}{2}} (\sin\phi \cos\phi)^{-\frac{1}{2}} d^{\frac{K_2+1}{2}} \frac{(\cos 2\phi)}{\frac{s+r+1}{2}, \frac{r-s}{2}} \cdot \left[\left(\frac{K_2-r+s}{2} + \frac{1}{2} \right)! \left(\frac{K_2+r-s}{2} + \frac{1}{2} \right)! \left(\frac{K_2+r+s}{2} + 1 \right)! \left(\frac{K_2-r-s}{2} \right)! \right]^{-\frac{1}{2}}.$$

$$\cdot \left[\left(\frac{K_2-r+s}{2} + \frac{1}{2} \right)! \left(\frac{K_2+r-s}{2} + \frac{1}{2} \right)! \left(\frac{K_2+r+s}{2} + 1 \right)! \left(\frac{K_2-r-s}{2} \right)! \right]^{-\frac{1}{2}} \quad (13)$$

Returning now to the matrix element between two states (5), we obtain a long but simple expression for (3):

$$\frac{\pi}{8} \sum_{m_1 m_2} C_{j_1 m_1}^{JM} j_1^{m_1} j_2^{m_2} C_{j_1' m_1'}^{JM} j_1'^{m_1'} j_2'^{m_2'} \int \sum_{pqrs} \sum_{K_1 K_2} (-1)^{K_1 + \frac{q}{2} - \frac{r}{2} + 1} \pi \rho \chi \sigma$$

$$\times (\sin 2\phi)^{-1} d^{\frac{K_1+1}{2}} \frac{(\cos 2\phi)}{\frac{p+q+1}{2}, \frac{q-p}{2}} d^{\frac{K_2+1}{2}} \frac{(\cos 2\phi)}{\frac{s+r+1}{2}, \frac{r-s}{2}} (\cos 2\phi),$$

$$\times \left[\left(\frac{K_1 - p + q}{2} + \frac{1}{2} \right)! \left(\frac{K_1 + p - q}{2} + \frac{1}{2} \right)! \left(\frac{K_1 + p + q}{2} + 1 \right)! \left(\frac{K_1 - p - q}{2} \right)! \right]^{-\frac{1}{2}}$$

$$\times \left[\left(\frac{K_2 - r + s}{2} + \frac{1}{2} \right)! \left(\frac{K_2 + r - s}{2} + \frac{1}{2} \right)! \left(\frac{K_2 + r + s}{2} + 1 \right)! \left(\frac{K_2 - r - s}{2} \right)! \right]^{-\frac{1}{2}} \times$$

(14)

$$\times Y_{p\pi}^* (\hat{P}_i) Y_{q\chi}^* (\hat{P}_i) Y_{j_1 m_1} (\hat{P}_i) Y_{r\rho}^* (\hat{Q}_i) Y_{s\sigma}^* (\hat{Q}_i) Y_{j_2 m_2} (\hat{Q}_i) \times$$

$$\times Y_{p\pi} (\hat{P}_k) Y_{r\rho} (\hat{P}_k) Y_{j_1 m_1}^* (\hat{P}_k) \times$$

$$\times Y_{q\chi} (\hat{Q}_k) Y_{s\sigma} (\hat{Q}_k) Y_{j_2 m_2}^* (\hat{Q}_k) d\hat{P}_i d\hat{Q}_i d\hat{P}_k d\hat{Q}_k.$$

Using the well-known properties of the spherical functions

$$Y_{\ell m}^* (\theta, \phi) = (-1)^m Y_{\ell -m} (\theta, \phi)$$

and

$$\int_0^{2\pi} d\phi \int_0^\pi d\theta \sin\theta Y_{\ell_1 m_1} (\theta, \phi) Y_{\ell_2 m_2} (\theta, \phi) Y_{\ell_3 m_3}^* (\theta, \phi) =$$

$$= \left[\frac{(2\ell_1 + 1)(2\ell_2 + 1)}{4\pi(2\ell_3 + 1)} \right]^{\frac{1}{2}} C_{\ell_1 0 \ell_2 0}^{\ell_3 0} C_{\ell_1 m_1 \ell_2 m_2}^{\ell_3 m_3}$$

(see /6/ p. 124, f. (1) and p. 131, f. (5)),
(14) can be rewritten in the form

$$\sum_{\substack{p\pi q\sigma \\ \pi\rho\kappa\sigma}} \sum_{\substack{m_1' m_2' \\ m_1 m_2}} \sum_{K_1 K_2} C_{j_1 m_1 j_2 m_2}^{JM} C_{j_1' m_1' j_2' m_2'}^{JM} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} \times$$

$$\times C_{p\pi r\rho}^{j_1 m_1} C_{q\chi s\sigma}^{j_2 m_2} C_{p_0 q_0}^{j_1 0} C_{r_0 s_0}^{j_2 0} C_{p_0 r_0}^{j_1 0} C_{q_0 s_0}^{j_2 0} \times$$

$$\times \left[\left(\frac{K - p + q + 1}{2} \right)! \left(\frac{K + p - q + 1}{2} \right)! \left(\frac{K - p - q}{2} \right)! \left(\frac{K + p + q}{2} + 1 \right)! \right]^{-\frac{1}{2}}$$

$$\times \left[\left(\frac{K_2 - s + r + 1}{2} \right)! \left(\frac{K_2 + s - r + 1}{2} \right)! \left(\frac{K_2 - s - r}{2} \right)! \left(\frac{K_2 + s + r}{2} + 1 \right)! \right]^{-\frac{1}{2}} \times$$

$$\times (\sin 2\phi)^{-1} d_{\frac{p+q+1}{2}, \frac{p-q}{2}}^{\frac{K_1+1}{2}} (\cos 2\phi) d_{\frac{r+s+1}{2}, \frac{s-r}{2}}^{\frac{K_2+1}{2}} (\cos 2\phi). \quad (15)$$

The parameters K_1 and K_2 are fixed by the condition $K = K_1 + K_2$. Instead of the sum over K_1, K_2 it is enough to consider a term with a definite K . The summation of the Clebsch-Gordan coefficients leads to the 9j coefficient

$$\sum_{m_1 m_{1k}} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} C_{j_3 m_3 j_4 m_4}^{j_3 m_3 j_4 m_4} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} C_{j_3 m_3 j_4 m_4}^{j_3 m_3 j_4 m_4} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} C_{j_3 m_3 j_4 m_4}^{j_3 m_3 j_4 m_4} =$$

$$\times C_{j_1 m_1 j_3 m_3}^{j_1 m_1 j_3 m_3} C_{j_2 m_2 j_4 m_4}^{j_2 m_2 j_4 m_4} C_{j_1 m_1 j_2 m_2}^{j_1 m_1 j_2 m_2} C_{j_3 m_3 j_4 m_4}^{j_3 m_3 j_4 m_4} =$$

$$= \delta_{jj'} \delta_{mm'} \left[(2j_{12} + 1)(2j_{13} + 1)(2j_{24} + 1)(2j_{34} + 1) \right]^{\frac{1}{2}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & j \end{Bmatrix}$$

Substituting this expression into (15), we obtain the following form for the transformation coefficient:

$$\sum_{pqrs} \left[\left(\frac{K_1 - p + q + 1}{2} \right)! \left(\frac{K_1 + p - q + 1}{2} \right)! \left(\frac{K_1 + p + q}{2} + 1 \right)! \left(\frac{K_1 - p - q}{2} \right)! \right]^{-\frac{1}{2}} \times$$

$$\times \left[\left(\frac{K_2 - r + s + 1}{2} \right)! \left(\frac{K_2 + r - s + 1}{2} \right)! \left(\frac{K_2 + r + s}{2} + 1 \right)! \left(\frac{K_2 - r - s}{2} \right)! \right]^{-\frac{1}{2}} \times$$

$$\times \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\} (\sin 2\phi)^{-1} d^{\frac{K_1 + 1}{2}} (\cos 2\phi) d^{\frac{K_2 + 1}{2}} \frac{q + p + 1}{2} \frac{q - p}{2} \frac{r + s + 1}{2} \frac{s - r}{2} \quad (16)$$

where we introduced the notation

$$\left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\} = [(2j_1 + 1)(2j_2 + 1)(2j'_1 + 1)(2j'_2 + 1)]^{\frac{1}{2}} \times$$

$$\times (2p + 1)(2q + 1)(2s + 1)(2r + 1) \times$$

$$\times \begin{pmatrix} p & q & j_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} s & r & j_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} p & r & j'_1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} q & s & j'_2 \\ 0 & 0 & 0 \end{pmatrix} \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\} \quad (17)$$

Considering the whole basis function (4) instead of (5), we can write down the final form of the orthonormal transformation coefficient:

$$\langle j'_1 j'_2 | j_1 j_2 \rangle_{KJM}^{\phi} = \frac{\pi}{2} \left(\frac{K_1 - j_1}{2} \right)! \left(\frac{K_1 + j_1 + 1}{2} \right)! \left(\frac{K_2 - j_2}{2} \right)! \left(\frac{K_2 + j_2 + 1}{2} \right)! \times$$

$$\times \frac{a_{Kj'_1 j'_2}}{a_{Kj_1 j_2}} \sum_{pqrs} \left[\left(\frac{K_1 - p + q + 1}{2} \right)! \left(\frac{K_1 + p - q + 1}{2} \right)! \left(\frac{K_1 - p - q}{2} \right)! \left(\frac{K_1 + p + q}{2} + 1 \right)! \right]^{-\frac{1}{2}} \times$$

$$\times \left[\left(\frac{K_2 - s + r + 1}{2} \right)! \left(\frac{K_2 + s - r + 1}{2} \right)! \left(\frac{K_2 + r - s}{2} \right)! \left(\frac{K_2 + r + s}{2} + 1 \right)! \right]^{-\frac{1}{2}} \times$$

$$\times \left\{ \begin{matrix} p & r & j'_1 \\ q & s & j'_2 \\ j_1 & j_2 & J \end{matrix} \right\} (\sin 2\phi)^{-1} d^{\frac{K_1 + 1}{2}} (\cos 2\phi) d^{\frac{K_2 + 1}{2}} \frac{q + p + 1}{2} \frac{p - q}{2} \frac{r + s + 1}{2} \frac{s - r}{2} \quad (18)$$

with

$$a_{Kj_1 j_2} = \left[\left(\frac{K - j_1 - j_2}{2} \right)! \left(\frac{K + j_1 + j_2 + 1}{2} \right)! \left(\frac{K - j_1 + j_2 + 1}{2} \right)! \left(\frac{K + j_1 - j_2 + 1}{2} \right)! \right]^{\frac{1}{2}}$$

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