сообщения
объединенного
института
ядерных
исследований
дубна

E2 - 12100

P.Exner

BOUNDED ENERGY APPROXIMATION TO AN UNSTABLE QUANTUM SYSTEM

1979

E2 - 12100

P.Exner

BOUNDED ENERGY APPROXIMATION TO AN UNSTABLE QUANTUM SYSTEM

Экснер П. **E2 - 12100**

Энергетическое приближение при описании нестабильных квантовых систем

В рамках общего кинематического подхода к описанию нестабильных квантовых систем исследуется приближение при помощи так называемых состояний с ограниченной энергией. Получены приближения явного физического значения для пространства состояний, редуцированного эволюционного оператора, законов распада и других характеристик нестабильной системы. Результаты показывают, что известные трудности с полугрупповым условием Вейскопфа-Вигнера являются несущественными.

Работа выпрлнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1979

Exner P. **E2 - 12100**

Bounded Energy Approximation to an Unstable Quantum System

An approximation using the so-called bounded energy states is studied within the framework of general kinematical concept of unstable quantum systems. Approximations of a clear physical meaning are obtained for the state space, reduced evolution operator, decay laws and other characteristics of the unstable system. The results show that the well-known troubles with the Weisskopf-Wigner semigraup condition are not essential.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1979

1. INTRODUCTION

The quantum kinematical concept of unstable systems represents itself a matter of interest for a long time. Most attention has been paid to the time evolution. The usual Weisskopf-Wigner description gives a simple universal framework in which the dynamics of decays can be studied. It imposes, however, the semigroup condition on the reduced evolution operator; it is well-known that such an assumption and some analogous conditions lead to total Hamiltonian H containing the whole real axis in its spectrum - see, e.g., [1-6]. The semigroup condition (or WW-condition) may be thus regarded only as an approximation.

The deviations are probably negligible from the practical point of view. Let us mention some related results. It was recognized twenty years ago that the decay laws corresponding to semibounded H have a power-like asymptotics for $t \to \infty$. In realistic physical models, however, this effect takes place at $\gtrsim 10$ lifetimes and thus it cannot be tested experimentally - cf., e.g., [7-9]. On the other hand, the initial decay rate was shown to be zero for finite energy states [10,11]. This quantity is not directly measurable, but a behaviour of the decay law for small t is interesting, especially if one assumes an unstable system suffering from frequently repeated measurements [12-16]. Deviations from the exponential decay law can be "amplified" in such specific experimental conditions, but the model calculation together with discussion of a typical experiment given in ref.[17] show that they are still too small.

It is, of course, possible to treat this problem from the more general point of view. One can be, for example, interested in the state space or in the reduced evolution operator. The latter is interesting especially in connection with the inverse decay problem[2]. One can take as a starting information for it either the continuous semigroup of contractions (WW-condition) or some continuous positive definite operator-valued function, which is in a some sense near to this semigroup, but corresponds to semibounded total Hamiltonian. A relation between the minimal unitary dilations referring to these two cases in not apriori clear. Let us mention that Williams[1] has formulated in essential the same problem discussing unitary dilation of the Zwanziger representation.

After preliminaries we shall introduce (in Section 3) the notion of bounded energy state. Using it we shall be able to deduce (in Section 4) statements concerning approximation of the reduced evolution operator and decay laws, mostly for the case when the state Hilbert space $\mathcal{H}_u$ of the unstable system is finite-dimensional. In the case $\dim \mathcal{H}_u = \infty$ we shall obtain an interesting mathematical problem (validity of the assumption (S) - see below), which up to our knowledge is not solved. Further we shall show the form of minimal unitary dilation corresponding to the approximative reduced evolution operator. In the last section we discuss the physical interpretation of the approximation, especially from the point of view of WW-condition.

2. PRELIMINARIES

Discussion of the kinematical concept of unstable quantum system usually starts with the following assumptions[1-4]:

(i) the state Hilbert space $\mathcal{H}_u$ of an unstable system is a proper subspace of some Hilbert space $\mathcal{H}$,

(ii) a strongly continuous unitary representation $U(.)$ of one-parameter group of time translations is realized on $\mathcal{H}$, $U(t) = \exp(-iHt)$, H being the total Hamiltonian, and $\mathcal{H}_u$ is not an invariant subspace of $U(t)$ for any $t > 0$.

Let us further mention briefly some important notions (for details see again[1-4]). Time evolution on the space $\mathcal{H}_u$ itself is governed by the operator-valued function V: $V(t) = E_u U(t) E_u$, where E_u is a projection, $E_u \mathcal{H} = \mathcal{H}_u$. Assuming the system to be prepared at $t = 0$ in a state described by a density matrix ρ, $\mathrm{Ran}\,\rho \subset \mathcal{H}_u$, we define the decay law P_ρ as

$$P_\rho(t) = \mathrm{Tr}\{V^*(t)V(t)\rho\} = \mathrm{Tr}\{U^*(t)E_u U(t)\rho\}. \qquad (1a)$$

The density matrix ρ is a positive trace class operator so that there exists an orthonormal basis $\{\phi_k\} \subset \mathcal{H}_u$, $\rho\phi_k = w_k \phi_k$, $\sum_k w_k = 1$. The decay law can be then expressed in the form

$$P_\rho(t) = \sum_k w_k P_k(t), \quad P_k(t) = \|V(t)\phi_k\|^2 . \qquad (1b)$$

Let $E_H(.)$ denote the spectral measure of the Hamiltonian and $E_\lambda = E_H((-\infty, \lambda])$. A density matrix ρ is said to describe a <u>finite energy state</u> if the integral

$$\langle H \rangle_\rho = \int_{\mathbf{R}} \lambda \, d\mu_\rho(\lambda), \quad \mu_\rho(\lambda) = \mathrm{Tr}(\rho E_\lambda), \qquad (2)$$

converges. The set of all finite energy states (for given H) will be denoted as $M(H)$. The following assertion holds[11]:

Proposition 1 If $\rho \in M(H)$, then $\dot{P}_\rho(0) = 0$.

If the reduced evolution operator $V(.)$ is known, one can try to find a tripple $\{\mathcal{H}, U(.), E_u\}$ such that $V(t) = E_u U(t) \restriction \mathcal{H}_u$; this is the so-called inverse decay problem [2]. It can be solved by means of the theory of unitary dilations [18]; a solution exists if and only if $V(.)$ is a weakly continuous positive definite operator-valued function, $V(0) = E_u$. Moreover, under the minimality condition

$$[\bigcup_{t \in \boldsymbol{R}} U(t) \mathcal{H}_u]_{lin} = \mathcal{H} \qquad (3)$$

this solution (called minimal unitary dilation) is unique up to an isometric isomorphism.

It is further known that any reduced evolution operator $V(.)$ can be expressed as

$$(\psi, V(t)\phi) = \int_{\boldsymbol{R}} e^{-i\lambda t} \, d(\psi, F_\lambda \phi)$$

for any $\psi, \phi \in \mathcal{H}_u$, where $\{F_\lambda\}$ is one-parameter non-decreasing family of Hermitean operators, which is bounded and weakly continuous on the right, $F_\lambda = E_u E_\lambda \restriction \mathcal{H}_u$ (see [4], Theorem 3.b). The energy support of $V(.)$ is the set $\sigma[V] = \{\lambda \in \boldsymbol{R} : F_{\lambda+\epsilon} - F_{\lambda-\epsilon} \neq 0$ for any $\epsilon > 0\}$. This set coincides with spectrum of Hamiltonian [4].

Proposition 2: It holds $\sigma(H) = \sigma[V]$.

3. BOUNDED ENERGY STATES

A density matrix ρ is said to describe a bounded energy state if the Lebesgue-Stieltjes measure generated by the function $\mu_\rho(.)$ (see (2)) has a bounded support. In other words, there exists a positive b such that

$$\mu_\rho(\lambda) = \begin{cases} 0 & \dots \ \lambda \leq -b \\ 1 & \dots \ \lambda \geq b \end{cases}$$

The set of all bounded energy states will be denoted as $B(H)$.

Proposition 3: To any state ρ there exists a one-parameter family $\{\rho_b\} \subset B(H)$ such that $\lim_{b\to\infty} Tr|\rho - \rho_b| = 0$.

Proof: Let us assume the projections $E^{(b)} = E_H(\Delta_b)$, $\Delta_b = (-b, b)$, for all $b > 0$. To any state ρ we define

$$\rho_b = n_b E^{(b)} \rho E^{(b)}, \quad n_b^{-1} = Tr(\rho E^{(b)}), \qquad (4)$$

for those b for which $n_b^{-1} > 0$; clearly $\rho_b \in B(H)$. According to the definition it holds $s\text{-}\lim_{b\to\infty} E^{(b)} = I$ so that $\lim_{b\to\infty} n_b = 1$. Using further properties of the trace norm:

$$Tr|BC| \leq \|B\| \, Tr|C|, \quad Tr|C^+| = Tr|C| \qquad (5)$$

for all bounded B and any C of the trace class, we can make the following estimate

$$Tr|\rho - \rho_b| \leq 2 Tr|(I - E^{(b)})\rho| + n_b - 1.$$

According to the polar decomposition theorem there exists a partial isometry Z such that $|(I - E^{(b)})\rho| = Z^+(I - E^{(b)})\rho$. Clearly $s\text{-}\lim_{b\to\infty} Z^+(I - E^{(b)}) = 0$, which gives the desired relation. QED

Every bounded energy state belongs to $M(H)$. For finite energy states Proposition 1 holds; the most substantial point is the existence of the derivative $\dot{P}_\rho(0)$ (notice that the continuous function $P_\rho(.)$ is maximal at $t = 0$). One can prove a much stronger assertion for bounded energy states:

Proposition 4: If $\rho \in B(H)$, then the decay law P_ρ is a restriction to positive real axis of a function analytic in the whole plane.

Proof: If $\rho \in B(H)$, there exists $b > 0$ such that $\rho = E^{(b)}\rho E^{(b)}$. The decay law can be then expressed as

$$P_\rho(t) = \mathrm{Tr}\{U_b^+(t) E_u U_b(t)\rho\}, \qquad (6a)$$

where

$$U_b(t) = E^{(b)}U(t) = \exp(-iH_b t), \; H_b = E^{(b)}H. \qquad (6b)$$

The operator H_b is bounded, $||H_b|| \le b$, and therefore all derivatives of the function $U_b(.)$ are also bounded. By a simple induction one can find the derivatives

$$P_\rho^{(n)}(t) = \sum_{k=0}^{n} \binom{n}{k} i^{2k-n} \mathrm{Tr}\{H_b^k E_u H_b^{n-k} U_b(t)\rho U_b^+(t)\}; \qquad (7)$$

the first one of the relations (5) implies

$$|P_\rho^{(n)}(t)| \le \sum_{k=0}^{n} \binom{n}{k} ||H_b^k||\,||H_b^{n-k}|| \le \sum_{k=0}^{n} \binom{n}{k} b^n = (2b)^n .$$

Let us further define the function

$$f: f(z) = \sum_{n=0}^{\infty} P^{(n)}(0)\frac{z^n}{n!}, \quad z \in \mathbf{C}. \qquad (8)$$

The last inequality gives $|f(z)| \le \exp(2b|z|)$, thus the series has an infinite radius convergence and the function f is analytic in **C** due to theorems of Abel and Weierstrass /19/.

It remains to prove that (8) is the Taylor series of P_ρ for real t. A short calculation shows that

$$P_\rho(t) - \sum_{n=0}^{N} P_\rho^{(n)}(0)\frac{t^n}{n!} = \mathrm{Tr}\{E_u(U_b(t) - U_{b,N}(t))\rho U_b^+(t)\} +$$

$$+ \mathrm{Tr}\{E_u \sum_{r=0}^{N} \frac{1}{r!}(-iH_b t)^r \rho(U_b^+(t) - U_{b,N-r}^+(t))\},$$

where

$$U_{b,N}(t) = \sum_{r=0}^{N} \frac{1}{r!}(-iH_b t)^r .$$

This can be further estimated as

$$|P_\rho(t) - \sum_{n=0}^{N} P^{(n)}(0)\frac{t^n}{n!}| \le ||U_b(t) - U_{b,N}(t)|| +$$

$$+ ||\sum_{r=0}^{N}(U_b^+(t) - U_{b,N-r}^+(t))E_u \frac{1}{r!}(-iH_b t)^r|| \le$$

$$\le ||U_b(t) - U_{b,N}(t)|| + \sum_{r=0}^{N} ||U_b(t) - U_{b,N-r}(t)|| \frac{(bt)^r}{r!} .$$

It holds

$$||U_b(t) - U_{b,N-r}(t)|| \le 1 + \sum_{j=0}^{N-r}\frac{1}{j!}||H_b t||^j \le 1 + \exp(bt),$$

and therefore the second term in the last estimate can be majorized by the convergent series $\sum_{r=0}^{\infty}(1+\exp(bt))\frac{(bt)^r}{r!}$ and converges uniformly with respect to N. Boundedness of H_b implies $U_b(t) = u - \lim_{N\to\infty} U_{b,N}(t)$ for all t, so that finally we get

$$\lim_{N\to\infty} |P_\rho(t) - \sum_{n=P}^{N} P_\rho^{(n)}(0)\frac{t^n}{n!}| \le \lim_{N\to\infty} ||U_b(t) - U_{b,N}(t)|| +$$

$$+ \sum_{r=0}^{\infty}\frac{(bt)^r}{r!} \lim_{N\to\infty} ||U_b(t) - U_{b,N-r}(t)|| = 0,$$

which means $P_\rho(t) = f(t)$ for all real t. QED

4. APPROXIMATIVE DESCRIPTION OF AN UNSTABLE SYSTEM

According to Proposition 3 any state can be approximated by bounded energy states. Now we shall study the information this approximation could bring for description of unstable systems.

Let ρ be a state of such a system, $\mathrm{Ran}\,\rho \subset \mathcal{H}_u$. Notice firstly that the states ρ_b given by (4) generally do not fulfill $\mathrm{Ran}\,\rho_b \subset \mathcal{H}_u$. It means that we cannot hold $\mathcal{H}_u$ as a state Hilbert space, otherwise we should be faced to serious interpretative difficulties ($P_{\rho b}(0) \neq 1$ etc.). Thus we choose (for given b) the *approximative state Hilbert space* $\mathcal{H}_u^b = \overline{E^{(b)}\mathcal{H}_u} = \overline{\mathrm{Ran}\,E^{(b)}E_u}$; the corresponding projection will be denoted as $E_u^{(b)}$. This choice of $\mathcal{H}_u^b$ has a clear physical motivation; we postpone a discussion of this point to the last section. It holds obviously $\mathrm{Ran}\,\rho_b \subset \mathrm{Ran}\,E^{(b)}E_u \subset \mathcal{H}_u^b$. We introduce further the *approximative reduced evolution operator* by

$$V_b:\ V_b(t) = E_u^{(b)}U(t)E_u^{(b)}, \quad t \in \mathbf{R}. \tag{9}$$

The subspaces $\mathcal{H}_u, \mathcal{H}_u^b$ in $\mathcal{H}$ are generally different, since the projections $E^{(b)}, E_u$ need not commute. The main problem is in what sense the one-parameter family $\{\mathcal{H}_u^b\}$ approximates $\mathcal{H}_u$. Let us accept for a while the following assumptions

$$\operatorname*{s-lim}_{b\to\infty} E_u^{(b)} = E_u, \tag{S}$$

$$\operatorname*{u-lim}_{b\to\infty} E_u^{(b)} = E_u; \tag{U}$$

validity of them we shall discuss a little later. The corresponding approximations to the reduced evolution operator and the decay laws are given by the following assertion:

<u>Theorem 1</u>: (i) If the assumption (S) holds, then $\operatorname*{s-lim}_{b\to\infty} V_b(t) = V(t)$ and $\lim_{b\to\infty} P_{\rho_b}(t) = P_\rho(t)$ for all $t \in \mathbf{R}$;

(ii) If the assumption (U) holds, then $\operatorname*{u-lim}_{b\to\infty} V_b(t) = V(t)$ and $\lim_{b\to\infty} P_{\rho_b}(t) = P_\rho(t)$ uniformly in $\mathbf{R}$.

<u>Proof</u>: (i) The relation $\operatorname*{s-lim}_{b\to\infty} V_b(t) = V(t)$ follows directly from (S). It further implies $\operatorname*{s-lim}_{b\to\infty} P_b(t) = P(t)$, where we have denoted $P_b(t) = V_b^+(t)V_b(t)$, $P(t) = V^+(t)V(t)$. This relation together with the simple estimate

$$|P_{\rho_b}(t) - P_\rho(t)| \le \mathrm{Tr}|\rho_b - \rho| + |\mathrm{Tr}((P_b(t) - P(t))\rho)|$$

and Proposition 3 prove the remaining part of (i).
(ii) The statement follows from the estimates

$$\|V_b(t) - V(t)\| \le 2\|E_u^{(b)} - E_u\|,$$

$$|P_{\rho_b}(t) - P_\rho(t)| \le \mathrm{Tr}|\rho_b - \rho| + 4\|E_u^{(b)} - E_u\|. \qquad \text{QED}$$

Let us turn now to the question of validity of the above assumptions. We shall start with the auxiliary statement:

<u>Lemma</u>: The inequality $\dim \mathcal{H}_u^b \le \dim \mathcal{H}_u$ holds for all $b > 0$. Moreover, if $\dim \mathcal{H}_u < \infty$, then for any b large enough $\dim \mathcal{H}_u^b = \dim \mathcal{H}_u$.

<u>Proof</u>: Linear dependence of vectors $\chi_1, \dots, \chi_n \in \mathcal{H}_u$ implies linear dependence of $E^{(b)}\chi_1, \dots, E^{(b)}\chi_n$; by negation we obtain $\dim \mathcal{H}_u^b = \dim \overline{E^{(b)}\mathcal{H}_u} \le \dim \mathcal{H}_u$. Let further $\{\phi_1, \dots, \phi_n\}$ denote an orthonormal

basis in $\mathcal{H}_u$, and let us assume that for all $b>0$ the Gram determinant $\Gamma(E^{(b)}\phi_1,\dots,E^{(b)}\phi_n)=0$. According to the definition of $E^{(b)}$ this determinant is a continuous function of $E^{(b)}$; thus we obtain $\Gamma(\phi_1,\dots,\phi_n)=\lim_{b\to\infty}\Gamma(E^{(b)}\phi_1,\dots,E^{(b)}\phi_n)=0$, which contradicts the assumed linear independence of $\phi_1,\dots,\phi_n$. QED

We shall now solve the problem for $\mathcal{H}_u<\infty$. In this case the following statement holds:

Theorem 2: If $\dim\mathcal{H}_u<\infty$, then both the assumptions (U) and (S) are valid.

Proof: Let $\{\phi_1,\dots,\phi_N\}$ be an orthonormal basis in $\mathcal{H}_u$. According to the proved lemma for all b large enough the vectors $E^{(b)}\phi_1,\dots,E^{(b)}\phi_N$ are linearly independent and span therefore the N-dimensional subspace $\mathrm{Ran}\,E^{(b)}E_u=\mathcal{H}_u^b$ in $\mathcal{H}$. We denote by $\{\psi_1^b,\dots,\psi_N^b\}$ the basis obtained by Gram-Schmidt orthogonalization from $\{E^{(b)}\phi_1,\dots,E^{(b)}\phi_N\}$:

$$\psi_n^b=\frac{\tilde\psi_n}{\|\tilde\psi\|},\quad \tilde\psi_n=E^{(b)}\phi_n-\sum_{k=1}^{n-1}(\psi_k^b,E^{(b)}\phi_n)\psi_k^b,$$

$n=1,2,\dots,N$. With the help of relations $E^{(b)}\psi_k^b=\psi_k^b$ and orthonormality of $\phi_1,\dots,\phi_N$ we obtain

$$\tilde\psi_n=E^{(b)}\phi_n-\sum_{k=1}^{n-1}(\psi_k^b-\phi_k,\phi_n)\psi_k^b. \qquad (*)$$

Then the following estimate is possible

$$\|E^{(b)}\phi_n\|-\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|\le\|\tilde\psi_n\|\le\|E^{(b)}\phi_n\|+\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|. \qquad (**)$$

To any δ, $0<\delta<2^{2-3N}$, there exists b_0 such that for all $b>b_0$ the inequalities

$$\|\phi_n-E^{(b)}\phi_n\|<\delta,\quad n=1,2,\dots,N, \qquad (10a)$$

hold (since $s-\lim_{b\to\infty}E^{(b)}=I$).). We shall prove that this implies

$$\|\psi_n^b-\phi_n\|<2^{3n-1}\delta,\quad n=1,2,\dots,N. \qquad (10b)$$

The inequalities (10a) give $|1-\|E^{(b)}\phi_n\||\le\|\phi_n-E^{(b)}\phi_n\|<\delta$; using further (*) and (**) we obtain

$$\|\psi_n^b-\phi_n\|\le\frac{1}{\|\tilde\psi_n\|}\{|1-\|\tilde\psi_n\||+\|\phi_n-E^{(b)}\phi_n\|+\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|\}<$$

$$<\frac{|1-\|E^{(b)}\phi_n\||+\delta+2\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|}{\|E^{(b)}\phi_n\|-\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|},$$

i.e.,

$$\|\psi_n^b-\phi_n\|<2\,\frac{\delta+\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|}{1-\delta-\sum_{k=1}^{n-1}\|\psi_k^b-\phi_k\|}. \qquad (10c)$$

The relation (10b) follows now from (10a) by induction. For an arbitrary vector $\psi\in\mathcal{H}$ the inequalities

$$\|E_u^{(b)}\psi-E_u\psi\|=\|\sum_{n=1}^{N}(\psi_n^b,\psi)\psi_n^b-\sum_{n=1}^{N}(\phi_n,\psi)\phi_n\|\le$$

$$\le\sum_{n=1}^{N}|(\psi_n^b,\psi)|\,\|\psi_n^b-\phi_n\|+\sum_{n=1}^{N}|(\psi_n^b-\phi_n,\psi)|\le$$

$$\le(N+1)\|\psi\|\sum_{n=1}^{N}\|\psi_n^b-\phi_n\|$$

hold, which give together with (10b)

$$\|E_u^{(b)}-E_u\|<2^{3N}(N+1)\delta.$$

Thus the assumption (U) (and consequently also (S)) is valid. QED

The case $\dim\mathcal{H}_u=\infty$ is more complicated. The assumption (U) needs not hold here:

Example: Let us take $\mathcal{H} = L^2(\mathbf{R})$, $\mathcal{H}_u = L^2((-\infty, 0))$ and $U: (U(t)\phi)(x) = \phi(x-t)$. The generator $H = -i\frac{d}{dx}$ can be expressed as $H = F^{-1}QF$, where $(Q\phi)(x) = x\phi(x)$ and F is the Fourier-Plancherel operator. Let us assume for example the unit vector

$$\psi_a \in \mathcal{H}_u : \psi_a(x) = \begin{cases} a^{-1} & \dots\ -a^2 \le x \le 0 \\ 0 & \dots\ x < -a^2 \ \text{ or } \ x > 0 \end{cases}$$

One easily verifies that $\|E^{(b)}\psi_a\|^2 = \frac{1}{\pi}\int_{-y_0}^{y_0} y^{-2}\sin^2 y\, dy$, where $y_0 = \frac{1}{2}ba^2$. A rough estimate then gives $\|E^{(b)}\psi_a\|^2 \le \frac{1}{\pi}ba^2$. For any $\psi \in \mathcal{H}_u$ we have $E^{(b)}\psi = E^{(b)}E_u\psi \in \operatorname{Ran} E^{(b)}E_u \subset \mathcal{H}^b_u$, i.e., $E^{(b)}_u E^{(b)}\psi = E^{(b)}\psi$, further the relation (*) from the proof of the following Theorem 3 gives $E^{(b)}_u\psi = E^{(b)}\psi$. Then $\|E^{(b)}_u\psi_a - E_u\psi_a\|^2 = \|E^{(b)}\psi_a - \psi_a\|^2 \ge 1 - \frac{1}{\pi}ba^2$.

Any non-zero number may be chosen for a, thus for arbitrary b we have $\|E^{(b)}_u - E_u\| \ge 1$.

One can ask whether the weaker assumption (S) is generally valid for an infinitedimensional (separable) Hilbert space $\mathcal{H}_u$. Up to our knowledge, solution of this problem is not known. Let us notice that this is the interesting mathematical problem independently of specific circumstances of its formulation.

We shall return now to the question mentioned in the introduction, namely what the above described approximation means for the inverse decay problem. The following statement holds:

Theorem 3: Let $\{\mathcal{H}, U(.), E_u\}$ be the minimal unitary dilation of V(.), then the minimal unitary dilation of the approximative reduced evolution operator $V_b(.)$ equals $\{\mathcal{H}^b, U_b(.), E^{(b)}_u\}$, where $\mathcal{H}^b = E^{(b)}\mathcal{H}$ and $U_b(.)$ is given by (6b).

Proof: The projection $E^{(b)}$ reduces U(t) and therefore $\{U_b(t)\}$ is the unitary group on $\mathcal{H}^b$. It holds

$E^{(b)}\phi = \phi$ for any $\phi \in \operatorname{Ran} E^{(b)}E_u$, further to arbitrary $\psi \in \mathcal{H}^b_u$ there exists a sequence $\{\phi_k\} \subset \operatorname{Ran} E^{(b)}E_u$, $\phi_k = E^{(b)}\phi_k \to \psi$; it means $E^{(b)}\psi = \psi$. On the other hand, if $\chi \in (\mathcal{H}^b_u)^{\perp}$, then $E^{(b)}_u\chi = 0 = E^{(b)}E^{(b)}_u\chi$, so that together

$$E^{(b)}E^{(b)}_u = E^{(b)}_u = E^{(b)}_u E^{(b)}. \qquad (*)$$

Using this relation we obtain

$$V_b(t) = E^{(b)}_u U(t) E^{(b)}_u = E^{(b)}_u U_b(t) E^{(b)}_u.$$

Thus $\{\mathcal{H}^b, U_b(.), E^{(b)}_u\}$ is a unitary dilation of $V_b(.)$; it remains to prove its minimality. Let us assume existence of non-zero $\chi \in \mathcal{H}^b$ such that for all $t \in \mathbf{R}$, $\psi \in \mathcal{H}^b_u$ the equality $(\chi, U_b(t)\psi) = 0$ holds. Especially this is true for all $\psi \in \operatorname{Ran} E^{(b)}E_u$, i.e., $0 = (\chi, E^{(b)}U(t)E^{(b)}\phi) = (E^{(b)}\chi, U(t)\phi)$ for any $\phi \in \mathcal{H}_u$. It holds $E^{(b)}\chi = \chi$ since $\chi \in \mathcal{H}^b$ so that finally we get $(\chi, U(t)\phi) = 0$ for all $t \in \mathbf{R}$, $\psi \in \mathcal{H}_u$, which contradicts the assumed minimality of the tripple $\{\mathcal{H}, U(.), E_u\}$. QED

5. DISCUSSION

Let us turn now to the physical interpretation of obtained results. The approximation (4) and the corresponding choice of the approximative state space can be simply understood in the following way: a system prepared in a state ρ, $\operatorname{Ran}\rho \subset \mathcal{H}_u$, passes through the energy filter which is open for values from Δ_b*.

* The choice of Δ_b in the proof of Proposition 3 is only a matter of convenience. Obviously any other family $\{\Delta_b\}$ of Borel sets may be used such that $s-\lim_{b\to\infty} E_H(\Delta_b) = I$.

The output of such an operation is the state ρ_b, $\mathrm{Ran}\,\rho_b \subset \mathcal{H}^b_u$. The filter may be placed anywhere between the preparation of state ρ and the next measurement, since $E^{(b)}$ commutes with the evolution operator. Without loss of generality we can assume that this operation follows immediately after the preparation of ρ, and that they together form the preparation of the state ρ_b (at a given instant).

Now the proved assertions show the relations between description of the unstable system prepared in this way and that corresponding to the case when the energy filter is absent. In a lot of applications we have $\mathcal{H}_u$ (and therefore also $\mathcal{H}^b_u$) finite-dimensional, then all the statements may be used. For $\dim \mathcal{H}_u = \infty$ we have Propositions 3, 4 and Theorem 3; validity of the other conclusions is conditioned by the assumption (S).

We know that the "unphysical" semigroup condition for $V(\cdot)$ is in a very good agreement with experimental experience. Because of this fact the previous considerations are not only academical. We can take (at least for $\dim \mathcal{H}_u < \infty$) the approximative state space in such a way that the corresponding reduced evolution operator has not the unpleasant property (since $\sigma(H_b) = \sigma[V_b] \subset \Lambda_b$ due to Proposition 2 and Theorem 3) and at the same time it approximates the continuous contractive semigroup (in the sense of operator norm and uniformly with respect to t for $\dim \mathcal{H}_u < \infty$). Conversely, one can interpret the WW-condition as a (very good) approximation of the true physical description of the time evolution for unstable systems*.

* We left alone the question whether the true physical states of an unstable system can be expressed as ρ_b (with sharp cut-offs) or in some more sophisticated way.

A little more general point of view could be also used. Some time ago Haag has noticed [20] that in any real experiment we do not determine the state as a point in the state space, but as some $*$-weak neighbourhood*. Thus an experiment does not tell us that the system is in a state $\rho^{(0)}$, but that it is in a state ρ such that for Hermitean operators $A_1, \dots, A_n$ and positive numbers $\epsilon_1, \dots, \epsilon_n$ we have

$$|\mathrm{Tr}((\rho - \rho^{(0)})A_k)| < \epsilon_k, \quad k = 1,2,\dots, n.$$

The family $\{\rho_b\}$ approximates ρ in the sense of trace norm topology, which is, of course, stronger than the $*$-weak topology on the state space: for any Hermitean A we have

$$\mathrm{Tr}((\rho_b - \rho)A)| \leq ||A||\,\mathrm{Tr}|\rho_b - \rho|, \qquad \text{i.e.,}$$

$$\lim_{b\to\infty} \mathrm{Tr}((\rho_b - \rho)A) = 0.$$

It means that for b large enough one cannot distinguish experimentally ρ_b from ρ. Especially we are not able to decide experimentally whether a given state is a bounded energy state or not (cf. the analogous problem for finite energy states in [10,11]).

From the practical point of view it is difficult to distinguish ρ_b from ρ at all. Let us mention the model[17] calculation of the charged

*This statement is used in [20] to a purpose which is different from ours. The state space at this place means the Banach space in which states are elements of the positive cone with the unit norm - cf. e.g. [6].

kaon decay in the hydrogen bubble chamber; the differences between ρ and ρ_h there are too small to be measured even if we choose h as small as it is experimentally possible.* One cannot, of course, exclude a possibility that a decay will be discovered in which WW-condition would not be fulfilled even approximately. In our opinion, however, such a fact could not cause any serious theoretical troubles. The main importance of our conclusions is that they give a deeper understanding for physical use of the "unphysical" Weisskopf-Wigner condition.

ACKNOWLEDGEMENT

The author is truly indebted to Dr. J.Stará and Dr. M.Havlíček for discussions. Thanks are also due to Dr. J.Dolejší for critical reading of the manuscript and to Prof. G.Lassner, Dr. M.I.Shirokov and Dr. L.Dadashev for some comments.

REFERENCES

1. Williams D.N., Commun.Math.Phys. 1971, 21, p. 314.
2. Horwitz L.P., Lavita J., Marchand J.-P. J.Math.Phys. 1971, 12, p. 2537.
3. Sinha K. Helv.Phys.Acta, 1972, 45, p. 619.
4. Exner P. Commun.Math.Phys. 1976, 50, p. 1.
5. Misra B., Sinha K. Helv.Phys.Acta, 1977, 50, p. 99.

*Results of this model are interesting in connection with the effect recently called "Zeno's paradox" (cf. [21] and also [12]): they show that in real observations of decays we are very far from the paradoxical situation.

6. Davies E.B. Quantum Theory of Open Systems, Academic Press, London 1976.
7. Khalfin L.A. JETP, 1957, 33, p. 1371.
8. Matthews P.T., Salam A. Phys.Rev., 1958, 115, p. 1079.
9. Höhler G. Zs.f.Physik, 1958, 152, p. 542.
10. Havlíček M., Exner P. Czech.J.Phys., 1973. B23, p. 594.
11. Exner P. Czech.J.Phys. 1976, B26, p. 976.
12. Beskow A., Nilsson J. Arkiv for Fysik, 1967, 34, p. 561.
13. Ekstein H., Siegert H.J.F. Ann. of Phys., 1971, 68, p. 509.
14. Fonda L. et al. N.Cim., 1973, 15A, p. 689.
15. Degasperis A., Fonda L., Ghirardi G.C. N.Cim., 1974, 21A, p. 471.
16. Exner P. Czech.J.Phys. 1977, B27, p. 117, 233, 361.
17. Dolejší J., Exner P. Czech. J.Phys., 1977, B27, p. 855.
18. Sz.-Nagy B., Foias C. Harmonic Analysis of Operators on Hilbert Space, North-Holland Publ. Co, Amsterdam, 1970.
19. Sveshnikov A.G., Tikhonov A.N. The theory of functions of a Complex Variable, "Mir", M., 1971.
20. Haag R : in Mathematics of Contemporary Phy-Physics (ed. R.F.Streater), Academic Press, New York, 1972.
21. Misra B., Sudarshan E.C.G. J.Math.Phys., 1977, 18, p. 756.

Received by Publishing Department
on December 18 1978.