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0(4) EXPANSION ON THE LIGHT-CONE

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0(4) разложение на световом конусе

В работе рассмотрен матричный элемент коммутатора электромагнитных или слабых токов. Целью работы является выяснение структуры разложения на световом конусе в рамках общих принципов квантовой теории поля. Применяется представление Иоста-Лемана и разложение спектральной функции в ряд по многочленам Гегенбауера. Доказывается сходимость разложения по группе (0(4), при этом нет ограничения на окружность светового конуса. Обсуждается связь между коэффициентами разложения и моментами структурных функций глубоконеупругого рассеяния.

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0(4) Expansion on the Light-Cone

We investigated the structure of the light-cone expansion of matrix elements of current commutators for electromagnetic or weak currents on the basis of general Quantum Field Theory. For this aim, the DJL representation and the expansion of the spectral function in terms of Gegenbauer polynomials are used. The convergence of the 0(4) expansion is proved. The connection between the expansion coefficients and the moments of the structure functions of deep-inelastic scattering is discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

The dynamical treatment of deep inelastic lepton hadron scattering on the basis of quantum chromodynamics relies in an essential way on light-cone expansions of current products. For scalar currents it is assumed that the following expansion of this operator product is valid near the light-cone /1/

$$j(x)\,j(0) \underset{x^2\approx 0}{=} \sum_{n=0}^{\infty} C_n^1(x^2)\; O_{\mu_1\cdots\mu_n}(0)\, x^{\mu_1}\cdots x^{\mu_n}. \qquad (1.1)$$

The light-cone coefficients $C_n^1(x^2)$ are generalized functions which have singularities at $x^2=0$ in general. The local operators $O_{\mu_1\cdots\mu_n}$ are characterized by Lorentz spin n and scale dimension d_n. Without further assumptions such an expansion cannot be proven in the framework of general QFT. Also in perturbation theory only recently serious results have been obtained /2/.

Here we investigate the existence of the corresponding expansion for the matrix elements itself on the basis of general Quantum Field Theory

$$\langle p|[j(x),j(0)]|p\rangle = \varepsilon(x^0)\sum_n C_{2n}(x^2)\, C_{2n}^1\!\left(\frac{x^0}{\sqrt{x^2}}\right)(-x^2)^n \qquad (1.2)$$

as well as connections between the light-cone coefficients and the experimentally observable moments. The main result of this work is the proof that the light-cone expansion (1.2) converges for all values of x. The main lines of these investigations are presented in the second section.

One important point of this proof are the expansions of DJL spectral functions and test functions in terms of Gegenbauer polynomials, details of which are given in an Appendix. In the following section the relations between light-cone coefficients and moments are studied. Here a chain of formulas is derived which is very similar to a set of equations given in a previous paper /3/.
With the help of these results the following conclusion can be drawn: Whereas in the non asymptotical region a correspondence between moments and light-cone coefficients exists one-to-one relations cannot be proven for these asymptotic expressions without using positivity of spectral functions. In sec.4 an extension of these results to non forward scattering is discussed. It is interesting, that the representation of the moments with the help of the absorptive part is more complicated as expected.

2. Light-Cone Expansion for Matrix Elements

In this section expansions of the matrix elements of current commutators $C(x,p) = \langle p|[j(x),j(0)]|p\rangle$, j - scalar current, are studied. It is more convienent to use the symmetrized commutator defined by /5/

$$\left(C(x,p), \varphi(x_0,\vec{x}) \right) = \left(\bar{C}(x^2,\vec{x}^2), \phi(x^2,\vec{x}) \right) \tag{2.1}$$

$$\phi(x^2,\vec{x}) = \left(2\sqrt{x^2+\vec{x}^2}\right)^{-1}\left(\varphi(\sqrt{x^2+\vec{x}^2},\vec{x}) - \varphi(-\sqrt{x^2+\vec{x}^2},\vec{x})\right)$$

$\varphi(x_0,\vec{x}) \in S(R^4)$, $\phi(x^2,\vec{x}) \in S_+ \otimes S(R^3)$. Simple expansions of this commutator

$$\bar{C}(x^2,\vec{x}^2) = \sum (\vec{x})^{2n} f_{2n}(x^2) \quad , \quad f_{2n}(x^2) \in S_+' \tag{2.3}$$

have been investigated in a previous paper /3/. It has been shown that after an integration of eq.(2.2) with test functions $\varphi(x^2) \in S_+$ this relation is nothing else but a Taylor expansion of the entire function $\bar{C}^{\varphi}(\vec{x}^2) = (\bar{C}(x^2,\vec{x}^2), \varphi(x^2))$.
For physical application it is more appropriate to have an expansion in terms of spherical harmonics which have a direct correspondence to the operator product expansion (1.1). Furthermore for such expansions exist simpler connections between light- cone coefficients and experimentally observable moments.
The direct way to get the desired expansion of the symmetrized commutator $\bar{C}(x^2,\vec{x}^2)$ starts with an expansion of the entire analytical function $\bar{C}^{\varphi}(\vec{x}^2)$. Here the variable $\vec{x}$ has to be considered as a part of an euclidean four vector x_E with $x_E^2 = \vec{x}^2 + x_4^2$. The parametrization of the space R^4 with spherical coordinates reads

$$\begin{aligned} x_1 &= |x_E| \sin\theta \sin\nu \sin\mu \,, & x_3 &= |x_E| \sin\theta \cos\nu \,, \\ x_2 &= |x_E| \sin\theta \sin\nu \cos\mu \,, & x_4 &= |x_E| \cos\theta \,. \end{aligned} \tag{2.4}$$

The spherical harmonics are /6/

$$Y_{K\ell m}(\theta,\nu,\mu) = \gamma_{K\ell m}\, C^{1+\ell}_{K-\ell}(\cos\theta) \sin^{\ell}\theta \; P^m_{\ell}(\cos\nu)\, e^{im\mu} \tag{2.5}$$

$$\gamma_{K\ell m} = \frac{2^{2(m+\ell)}}{\sqrt{2}\,\pi} \sqrt{\frac{(K+1)(1+2\ell)(K-\ell)!\,(\ell!)^2(\ell-m)!\,\Gamma^2(\frac{1}{2}+m)}{\pi\,(K+\ell+1)!\,(\ell+m)!}}$$

and the corresponding harmonical polynomials are

$H_{\kappa\ell m}(x) = x^{\kappa} Y_{\kappa\ell m}(\theta,\gamma,\mu)$ (2.6) . In the special case of forward scattering $\bar{C}$ depends on x^2 and $\vec{x}^2$ only, so that the spherical harmonics with k even, l=m=0 are sufficient to expand $\bar{C}^{\varphi}(\vec{x}^2)$. Because of the orthogonality relation of the remaining Gegenbauer polynomials

$$\int_0^{\pi} d\theta \sin^2\theta \, C_n^1(\cos\theta) \, C_r^1(\cos\theta) = \tfrac{1}{2}\pi \, \delta_{nr} \tag{2.7}$$

it is easy to show the following expansion:

$$\bar{C}^{\varphi}(\vec{x}^2) = \sum_n C_{2n}^{\varphi}(x_E^2)\,(x_E^2)^n \, C_{2n}^1\left(\frac{x_4}{|x_E|}\right) , \tag{2.8a}$$

$$C_{2n}^{\varphi}(x_E^2) = (x_E^2)^{-n} \frac{2}{\pi} \int_{-1}^{+1} d\cos\theta \sqrt{1-\cos^2\theta} \; \bar{C}^{\varphi}(x_E^2 \sin^2\theta) \tag{2.8b}$$

After dropping the test function the following series is obtained

$$\bar{C}(x^2,\vec{x}^2) = \sum_n C_{2n}(x^2,x_E^2)\,(x_E^2)^n \, C_{2n}^1\left(\frac{x_4}{|x_E|}\right), \tag{2.9a}$$

$$C_{2n}(x^2,x_E^2) = (x_E^2)^{-n} \frac{2}{\pi} \int_{-1}^{+1} d\cos\theta \sin\theta \; \bar{C}(x^2, x_E^2 \sin^2\theta). \tag{2.9b}$$

This would be the desired expansion if we could identify $x_E^2 = -x^2,\ x_0^2 = -x_4^2$. Each term in eq. (2.9a) is an analytical function with respect to the variables x_E^2 and x_4^2 , so that an analytical continuation to values $x_E^2 < 0$ can be done and the identification $x_E^2 = -x^2,\ x_0^2 = -x_4^2$ is possible. In this way it is however impossible to prove the convergence of this expansion in the Minkowski space, i.e., in that region where it should be applied.

For this reason we justify this expansion in another way. The starting point is the DJL representation for the symmetrized commutator

$$\bar{C}(x^2,\vec{x}^2) = \frac{(-i)}{2\pi}\int_0^{\infty} d\lambda^2 \; \bar{D}(x^2,\lambda^2)\,\tilde{\Psi}(x,\lambda^2) \tag{2.10}$$

$$\bar{D}(x^2,\lambda^2) = \frac{1}{2\pi}\frac{\partial}{\partial x^2}\left(\theta(x^2)\,J_0(\lambda\sqrt{x^2})\right) \tag{2.11}$$

$$\tilde{\Psi}(x,\lambda^2) = \int d^4u \, e^{-iux} \, \Psi(u,\lambda^2) . \tag{2.12}$$

In the case of forward scattering there exists always a Jost-Lehmann spectral function $\Psi(u,\lambda^2) = \delta(u_0)\,\Psi_J(\vec{u},\lambda^2)$ with the support restriction $|\vec{u}| \le 1,\ \lambda^2 \ge (1-\sqrt{1-\vec{u}^2})^2$. The function $\bar{C}^{\varphi}$ reads now

$$\bar{C}^{\varphi}(\vec{x}^2) = -\frac{i}{2\pi}\int_0^{\infty} d\lambda^2 \int_0^{\infty} dy^2 \, \varphi(y^2)\,\bar{D}(y^2,\lambda^2)\,\tilde{\Psi}_J(\vec{x},\lambda^2) \tag{2.13}$$

$$= \frac{1}{4i\pi^2}\int d\vec{u}\, e^{i\vec{u}\vec{x}} \left(\Psi_J(\vec{u},\lambda^2),\ \varphi_B(\lambda^2)\right) = \int d\vec{u}\, e^{i\vec{u}\vec{x}} \, \chi(\vec{u}),$$

where

$$\varphi_B(\lambda^2) = \left(\frac{\partial}{\partial x^2}\left(\theta(x^2)\,J_0(\lambda\sqrt{x^2})\right),\ \varphi(x^2)\right) \tag{2.14}$$

is again a test function of the space S_+. With regard to the O(4) expansion we interpret the variable $\vec{u}$ as the first three components of a variable u_E acting in an euclidean space. Then the euclidean function $\chi_E(u_E) = \delta(u_4)\,\chi(\vec{u})$ leads to the same result for $\bar{C}^{\varphi}(\vec{x}^2)$ if we take as Fourier transform

$$\bar{C}^{\varphi}(\vec{x}^2) = \int du_4 \, d\vec{u} \; e^{i(\vec{u}\vec{x}+u_4x_4)} \, \chi_E(u_E) , \tag{2.15}$$

The next step consists in a O(4) expansion of the function $\chi_E(u_E)$ which is a generalized function over the space S. For this purpose the O(4) expansion of test functions from the space $S(R^4)$ must be considered first (Appendix)

$$\varphi(u_E) = \sum \varphi_{\kappa\ell m}(u_E^2)\, H_{\kappa\ell m}(u_E) \tag{2.16a}$$

$$\varphi_{\kappa\ell m}(u_E^2) = |u_E|^{-\kappa}\int_{S_3} d\Omega\, Y_{\kappa\ell m}(u/|u_E|)\,\varphi(u_E), \qquad (2.16b)$$

This series converges in the topology of the space $S(R^4)$ (see Appendix). Now it is easy to set up

$$\chi_E(u_E) = \sum_{\kappa,\ell,m} \chi_{\kappa\ell m}(u_E^2)\, H_{\kappa\ell m}(u_E)\,\frac{1}{u_E^{2\kappa}}, \qquad (2.17a)$$

where the expansion coefficients $\chi_{\kappa\ell m}(u_E^2)$ are defined by

$$(\chi_{\kappa\ell m}(u_E^2), \varphi(u_E^2)) = (\chi_E(u_E), H_{\kappa\ell m}(u_E)\varphi(u_E^2)). \qquad (2.17b)$$

The series (2.17a) is convergent in S' and has to be understood in the sense

$$(\chi_E(u_E), \varphi(u_E)) = \sum_{\kappa\ell m}(\chi_E(u_E), H_{\kappa\ell m}(u_E)\varphi_{\kappa\ell m}(u_E^2))$$

$$= \sum_{\kappa\ell m}(\chi_{\kappa\ell m}(u_E^2)\,\varphi_{\kappa\ell m}(u_E^2)) \qquad (2.18)$$

$$= \sum_{\kappa\ell m}(\chi_{\kappa\ell m}(u_E^2)(u_E^2)^{-\kappa}H_{\kappa\ell m}(u_E), \varphi(u_E))$$

In our special case the function $\chi_E(u_E)$ depends on $u_4, \vec{u}^2$ only so that the special harmonical polynomials $H_{\kappa\ell m}$ with $k=2n$, $l=m=0$ appear in the expansion

$$\chi_E(u_E) = \sum_{n=0}^{\infty} g_{2n}(u_E^2)\, C_{2n}^1\left(\frac{u_4}{|u_E|}\right)(u_E^2)^n, \quad g_{2n} = \frac{\chi_{2n00}}{u^{4n}} \qquad (2.19)$$

Contributions localized at n=0 have to be considered separately. Having the expansion of the function χ_E it is possible to prove the existence of the Gegenbauer expansion of its Fourier transform $\bar{C}^{\varphi}(\vec{x}^2)$ in the complex x-plane. We start with

$$\tau_N(u_E) = \chi_E(u_E) - \sum_{n=0}^{N} g_{2n}(u_E^2)\, C_{2n}^1\left(\frac{u_4}{|u_E|}\right)(u_E^2)^n \qquad (2.20)$$

which converges to zero $\lim_{N\to\infty}\tau_N(u_E) = 0$ in $S'(R^4)$. Because of the finite range of the variable u also $\tau_N(u_E)e^{-y_E u_E}$ converges to zero in $S'(R^4)$ for all finite values of y. For this reason the Fourier transform of $\tau_N(u_E)e^{-y_E u_E}$ exists and has the same property

$$\int d^4u_E\, e^{iu_E(x_E+iy_E)}\left(\chi_E(u_E) - \sum_{n=0}^{N} g_{2n}(u_E^2)\, C_{2n}^1\left(\frac{u_4}{|u_E|}\right)u_E^{2n}\right) \underset{N\to\infty}{\to} 0 \qquad (2.21)$$

This means that an expansion of $\bar{C}^{\varphi}$ exists and converges for all finite values of x_E, $x_E \in C_4$. The following calculations show that this series is the desired Gegenbauer expansion

(2.8a)

$$\bar{C}^{\varphi}(\vec{x}^2) = \int d^4u_E\, e^{iu_E x_E}\sum_{n=0}^{\infty} g_{2n}(u_E^2)\, C_{2n}^1\left(\frac{u_4}{|u_E|}\right)u_E^{2n}$$

$$= \sum_{n=0}^{\infty}\int d^4u_E\, e^{iu_E x_E}\, g_{2n}(u_E^2)\, C_{2n}^1\left(\frac{u_4}{|u_E|}\right)u_E^{2n}$$

$$= \sum_{n=0}^{\infty} H_{2\kappa 00}\left(\frac{\partial}{i\partial x}\right)\int d^4u_E\, e^{iu_E x_E}\, g_{2n}(u_E^2) \qquad (2.22)$$

$$= \sum_{n=0}^{\infty} x_E^{2\kappa}\, C_{2n}^1\left(\frac{x_4}{|x_E|}\right)\left(\frac{2}{i}\frac{\partial}{\partial x_E^2}\right)^{2n}\tilde{g}_{2n}(x_E^2)$$

Hereby formula

$$H_{2\kappa nm}\left(\frac{\partial}{i\partial q}\right)f(q^2) = H_{2\kappa nm}\left(\frac{2}{i}\frac{\partial}{\partial q^2}\right)^{2\kappa}f(q^2) \qquad (2.23)$$

has been applied. The continuation to the Minkowski space contains no further difficulties. Eq. (2.22) has to be compared with eq. (2.8a) which however converges now for all finite complex values of x. If the test functions $\varphi(x^2)$ are omitted whereby the coefficients $C_{2n}(x^2, x_E^2)$ appear instead of $C_{2n}^{\varphi}(x_E^2)$ then the expansion (1.2) is obtained by setting $C_{2n}(x^2, x_E^2 = -x^2) = C_{2n}(x^2)$.

Finally one remark concerning the character of the light-cone coefficients is added. We will show that $C_{2n}(x^2)$ are not generalized functions of the space S'_+. At first we discuss the expansion coefficients $\bar{C}^{\varphi}_{2n}$ given by eq.(2.8b)

$$C^{\varphi}_{2n}(x^2_E) = (x^2_E)^{-n} \frac{2}{\pi} \int_{-1}^{+1} d\alpha \sqrt{1-\alpha^2}\, \bar{C}^{\varphi}(x^2_E(1-\alpha^2)) \qquad (2.8b)$$

From the Jost Lehmann representation for $\bar{C}(x_0^2, \vec{x}^2)$ it follows that $\bar{C}^{\varphi}$ is an entire analytical function with order and type smaller than one, so that the estimate

$$|\bar{C}^{\varphi}(\vec{x}^2)| \leq C_{\delta,\varphi} (1 + \mathrm{Re}\sqrt{\vec{x}^2})^{\rho}\, e^{(1+\delta)|\mathrm{Im}\sqrt{\vec{x}^2}|}$$

$C_{\delta,\varphi}$ const., $\delta > 0$, ρ integer

is fulfilled. Because of the finite range of the integration in eq.(2.8b) the coefficients have the same analytical properties and allow a similar estimate

$$|C^{\varphi}_{2n}(x^2_E)| \leq C'_{\delta,\varphi} (1 + \mathrm{Re}\sqrt{x^2_E})^{\rho}\, e^{(1+\delta)|\mathrm{Im}\sqrt{\vec{x}^2}|} .$$

Continuing C^{φ}_{2n} to Minkowski space $x^2 = -x^2_E < 0$ it is obvious that C^{φ}_{2n} may exponentially grow for $x^2 \to \infty$. Consequently after omitting the test functions and identifying the variables $C^{\varphi}_{2n}(-x^2) \to C_{2n}(x^2, -x^2) = C_{2n}(x^2)$ the light-cone coefficients C_{2n} are not generalized functions over the space S'_+. A satisfactory definition of these functionals needs test functions which are exponenetially falling at infinity, e.g.

$$|\varphi(x^2)| = \max_{q \leq \rho} \sup_{x^2 \in R_+} \left| (1+x^2)^{\rho}\, e^{(1+\delta)\sqrt{x^2}} \left(\frac{\partial}{\partial x^2}\right)^{q} \varphi(x^2) \right| < \infty .$$

$\delta > 1$, ρ, q integer.

3. Connections between Light-cone Coefficients and Moments

The relations between light-cone coefficients and moments consitute a connection between quantum field theoretical quantities and observable structure functions. In the case of O(4) expansions these relations have been given in /4/ at first. Here they are obtained without restriction to OPE. With similar arguments as in the foregoing section the light-cone expansion of matrix elements of T-products of currents can be established:

$$T(x,p) = \langle p| T j(x) j(0) |p\rangle = \sum_n (x^2_E)^n C^1_{2n}\left(\frac{x_4}{|x_E|}\right) T_{2n}(x^2, x^2_E) . \qquad (3.1)$$

The differences are: in formula (2.13) $\bar{D}(y^2, \lambda^2)$ must be replaced by $D_c(y^2, \lambda^2)$. The analogously defined function $\varphi_c(\lambda^2) = 4i\pi^2 \int dy^2\, D(y^2, \lambda^2)$ is not a test function. Therefore for rising spectral functions subtractions have to be taken into account and one has to be careful in omitting of test functions. In the following we assume that subtractions are not needed and consider the T-product at euclidean values

$$T(x_E, p) = \sum (x^2_E)^n C^1_{2n}\left(\frac{x_4}{|x_E|}\right) T_{2n}(x^2_E) , \quad T_{2n}(x^2_E) = T_{2n}(-x^2_E, x^2_E) \qquad (3.2)$$

The moments have to be primarily defined as expansion coefficients of the T-product at euclidean momenta $q^2_E = -q^2 = Q^2$

$$T(q^2_E) = \frac{1}{\pi} \sum \frac{(-1)^n}{(q^2_E)^n} C^1_{2n}\left(\frac{q_4}{|q_E|}\right) \mu_{2n}(q^2_E) \qquad (3.3)$$

$$\mu_{2n}(q_E^2) = \frac{(-1)^n}{2\pi}(q_E^2)^n \int d\omega_q \, C_{2n}^1\left(\frac{q_4}{|q_E|}\right) T(q_E)$$

Substituting the Fourier transform

$$\tilde{T}(q_E) = \int dx_E \, e^{-iq_E x_E} \, T(x_E) \tag{3.4}$$

into eq.(3.3) and applying eq. (2.23) we get

$$\mu_{2n}(q_E^2) = \pi \, 2^{2n} (q_E^2)^{2n} \left(\frac{\partial}{\partial q_E^2}\right)^{2n} \tilde{T}_{2n}(q_E^2),$$

$$\tilde{T}_{2n}(q_E^2) = \int dx_E \, e^{-iq_E x_E} \, T_{2n}(x_E^2). \tag{3.5}$$

It is remarkable, that all contributions to the n-th moment come from the n-th light-cone coefficient only. For each finite q_E^2 exists a correspondence between the moments and the light-cone coefficients. It is however not clear what happens in the limit $q_E^2 \to \infty$. In a previous paper /3/ we have shown that in this limit the light-cone singularities determine the asymptotical behaviour of the moments whereas the asymptotical behaviour of the moments must not determine the light-cone singularities if only spectrality and causality are taken into account. Despite of the simpler structure of the O(4) expansion the same conclusions as in /3/ must be drawn here.

To see this ,we apply representations with euclidean spectral functions, so that

$$\bar{C}(x^2, \vec{x}^2) = -\frac{i}{2\pi}\int_0^\infty d\lambda^2 \, \bar{D}(x^2, \lambda^2) \, \tilde{\Psi}(\vec{x}, \lambda^2) \tag{3.6}$$

$$\tilde{\Psi}(\vec{x}, \lambda^2) = \int d^4u_E \, e^{iu_E x_E} \, \Psi(u_E, \lambda^2)$$

$$T(q_E) = \frac{1}{\pi}\int d^4u_E \, d\lambda^2 \, \frac{1}{(q_E - u_E)^2 + \lambda^2} \, \Psi(u_E, \lambda^2) \tag{3.7}$$

Simple calculation give:

$$\bar{C}(x^2, \vec{x}^2) = \sum (-x^2)^n \, C_{2n}^1\left(\frac{x_4}{\sqrt{x^2}}\right) C_{2n}(x^2, -x^2)$$

$$C_{2n}(x^2, -x^2) = -\frac{i}{4\pi^2}\int_0^\infty d\lambda^2 \left[\frac{\partial}{\partial x^2}\left(\theta(x^2) \, J_0(\lambda\sqrt{x^2})\right)\right] h_{2n}(\lambda^2, -x^2)$$

$$h_{2n}(\lambda^2, -x^2) = 2(-1)^n \pi^2 \int_0^1 du_E^2 \, (u_E^2)^{2n+1} \, \frac{J_{2n+1}(u_E\sqrt{-x^2}) \, \Psi_{2n}(u_E^2, \lambda^2)}{(u_E\sqrt{-x^2})^{2n+1}}$$

$$\hat{h}_{2n}(\lambda^2, Q^2) = \frac{\pi^4 2^{2n+2}}{2n+1}(-1)^n \int_0^1 du_E^2 \, \frac{(u_E^2)^{2n+1} \, \Psi_{2n}(u_E^2, \lambda^2 - u_E^2)}{\left(1 + \sqrt{1 + \frac{4Q^2 u_E^2}{(Q^2+\lambda^2)^2}}\right)^{2n+1}}$$

$$\mu_{2n}(Q^2) = (Q^2)^{2n} \int_0^\infty d\lambda^2 \, \frac{\hat{h}_{2n}(Q^2, \lambda^2)}{(Q^2 + \lambda^2)^{2n+1}} \tag{3.8}$$

$$\Psi(\vec{u}, \lambda^2) = \sum C_{2n}^1\left(\frac{u_4}{|u_E|}\right) (u_E^2)^n \, \Psi_{2n}(u_E^2, \lambda^2)$$

Here the continuation to Minkowski values $x^2 = -x_E^2$ and $q_E^2 = Q^2$ has already been done. As it is expected ,the connections are valid for each n separately. The resulting chain (3.8) is very similar to eqs. (3.2),(2.6) in ref. /3/. In distinction to /3/ the functions $C_{2n}(x^2, -x^2)$, $h_{2n}(\lambda^2, -x^2)$, $\hat{h}_{2n}(\lambda^2, Q^2)$ depend on a second variable in a very special manner however. For the investigation of asymptotical connections these limits in the second variables can be performed independently, so that we have to discuss the relations with $C_{2n}(x^2, 0)$

$h_{2n}(x^2, 0)$, $\hat{h}_{2n}(x^2, \infty)$, the additional variables fixed to 0 or ∞, respectively.

4. Generalizations to Non Forward Scattering

Here we consider some obvious extensions of the results of the foregoing sections to nonforward scattering. The convergence of light-cone expansions can be proved similarly as long as a Jost Lehmann representation is valid .

$$\bar{C}(x^2, \vec{x}) = \sum_{\kappa, l} C_{\kappa l}(x_E^2) Y_{\kappa l 0}(\eta) (x_E^2)^{\frac{\kappa}{2}} , \qquad (4.1)$$

$$C_{\kappa l}(x^2, x_E^2) = (x_E^2)^{-\frac{\kappa}{2}} \int d\omega_{(\eta)} \bar{C}(x^2, \vec{x}) \bar{Y}_{\kappa l 0}(\eta) . \qquad (4.2)$$

The amplitude in momentum space has the expansion

$$T(q_E) = \sum_{\kappa, l} (q_E^2)^{-\frac{\kappa}{2}} Y_{\kappa l 0}\left(\frac{q}{|q_E|}\right) \mu_{\kappa l}(q_E^2) \sqrt{2}\, i^{\kappa} , \qquad (4.3)$$

$$\mu_{\kappa l}(q_E) = \int d\omega_{(q)} T(q_E) \bar{Y}_{\kappa l 0}\left(\frac{q}{|q_E|}\right) (q_E^2)^{\frac{\kappa}{2}} i^{-\kappa} \frac{1}{\sqrt{2}} . \qquad (4.4)$$

If a light-cone expansion for the amplitude in x space is written down

$$T(x_E^2, \vec{x}) = \sum T_{\kappa l}(x_E^2)(x_E^2)^{\frac{\kappa}{2}} Y_{\kappa l 0}(\eta) \sqrt{2}\, \pi , \qquad (4.5)$$

then quite analogously to eq.(3.5) the Christ Hasslacher formula follows

$$\mu_{\kappa l}(q_E^2) = \pi\, 2^{\kappa} (q_E^2)^{\kappa} \left(\frac{\partial}{\partial q_E^2}\right)^{\kappa} \tilde{T}_{\kappa l}(q_E^2) , \ k \text{ even}! \qquad (4.6)$$

Of course chains connecting light-cone singularities and moments exist too.

One interesting point is the expression of the moments in terms of structure functions. The simplest way to derive such equations exploits dispersion relations. If we consider the symmetrized amplitude

$$T^S(q, P, k) = \frac{1}{2}\left(T(q_1, P_1, q_2, P_2) + T(-q_2, P_1, -q_1, P_2)\right)$$

$$q = \frac{1}{2}(q_1 + q_2), \ P = \frac{1}{2}(P_1 + P_2), \ k = P_1 - P_2$$

then the subtracted dispersion relation

$$T^S(q, P, k) = T_0^S + P_{\frac{k}{2}} + \frac{1}{\pi} \int_{q_E^2 + \frac{t}{4}}^{\infty} dq_0' \, W^S(q_0', \vec{q}, P, k) \left(\frac{q_0^2 + q_E^2}{q_0'^2 + q_E^2}\right)^{\frac{k}{2}} \left(\frac{1}{q_0' - q_0} + \frac{1}{q_0' + q_0}\right)$$

k even

leads to the expression for the moments

$$\mu_{\kappa l} = c_{\kappa l} \int_{\frac{1}{2q_E}(q_E^2 + \frac{t}{4})}^{\infty} dx \left((ix)^2 - 1\right)^{1/4} Q_{k+\frac{1}{2}}^{l+\frac{1}{2}}(ix) \int_{-1}^{+1} dz \, P_{l}(z) W(q_0', \vec{q}, P, k) + d_{\kappa l}$$

$$k, l \text{ even}, \qquad x = \frac{q_0'}{|q_E|} \quad z = \frac{q_3}{|\vec{q}|} \ ; \ c_{\kappa l}, d_{\kappa l} \text{ constants} ,$$

which has to be compared with

$$\mu_{\kappa} = \int_{\xi > 0} d\xi \, \xi^{\kappa - 1} \left(1 + \frac{\xi^2}{Q^2}\right) W(q) \cdot$$

Only in the case $l=0$ there appears the Nachtmann variable $\xi = \frac{Q^2}{q_0 + \sqrt{Q^2 + q_0^2}}$ because of $q_E^{\kappa+1}\left(\left(i\frac{q_0}{q_E} - 1\right)^{1/4} Q_{\kappa+\frac{1}{2}}^{l+\frac{1}{2}}\left(i\frac{q_0}{q_E}\right)\right) \sim \left((q_0^2 + q_E^2)^{\frac{1}{2}} - q_0\right)^{\kappa+1}$.

5. Conclusions

The O(4) expansion of matrix elements corresponds to the operator product expansion on the light-cone. Therefore we are more interested in this expansion then in the group

Finally one remark concerning the character of the light-cone coefficients is added. We will show that $C_{2n}(x^2)$ are not generalized functions of the space S'_+. At first we discuss the expansion coefficients $\bar{C}^{\varphi}_{2n}$ given by eq.(2.8b)

$$C^{\varphi}_{2n}(x^2_E) = (x^2_E)^{-n} \frac{2}{\pi} \int_{-1}^{+1} d\alpha \sqrt{1-\alpha^2}\, \bar{C}^{\varphi}(x^2_E(1-\alpha^2)) \qquad (2.8b)$$

From the Jost Lehmann representation for $\bar{C}(x_0^2, \vec{x}^2)$ it follows that $\bar{C}^{\varphi}$ is an entire analytical function with order and type smaller than one, so that the estimate

$$|\bar{C}^{\varphi}(\vec{x}^2)| \leq C_{\delta,\varphi} (1 + \mathrm{Re}\sqrt{\vec{x}^2})^{\rho}\, e^{(1+\delta)|\mathrm{Im}\sqrt{\vec{x}^2}|}$$

$C_{\delta,\varphi}$ const., $\delta > 0$, ρ integer

is fulfilled. Because of the finite range of the integration in eq.(2.8b) the coefficients have the same analytical properties and allow a similar estimate

$$|C^{\varphi}_{2n}(x^2_E)| \leq C'_{\delta,\varphi} (1 + \mathrm{Re}\sqrt{x^2_E})^{\rho}\, e^{(1+\delta)|\mathrm{Im}\sqrt{\vec{x}^2}|}.$$

Continuing C^{φ}_{2n} to Minkowski space $x^2 = -x^2_E < 0$ it is obvious that C^{φ}_{2n} may exponentially grow for $x^2 \to \infty$. Consequently after omitting the test functions and identifying the variables $C^{\varphi}_{2n}(-x^2) \to C_{2n}(x^2, -x^2) = C_{2n}(x^2)$ the light-cone coefficients C_{2n} are not generalized functions over the space S'_+. A satisfactory definition of these functionals needs test functions which are exponenetially falling at infinity, e.g.

$$|\varphi(x^2)| = \max_{q \leq p}\ \sup_{x^2 \in R_+} \left| (1+x^2)^{\rho}\, e^{(1+\delta)\sqrt{x^2}} \left(\frac{\partial}{\partial x^2}\right)^q \varphi(x^2) \right| < \infty.$$

$\delta > 1$, ρ, q integer.

3. Connections between Light-cone Coefficients and Moments

The relations between light-cone coefficients and moments consitute a connection between quantum field theoretical quantities and observable structure functions. In the case of O(4) expansions these relations have been given in /4/ at first. Here they are obtained without restriction to OPE. With similar arguments as in the foregoing section the light-cone expansion of matrix elements of T-products of currents can be established:

$$T(x,p) = \langle p | T j(x) j(0) | p \rangle = \sum_n (x^2_E)^n C^1_{2n}\left(\frac{x_4}{|x_E|}\right) T_{2n}(x^2, x^2_E). \qquad (3.1)$$

The differences are: in formula (2.13) $\bar{D}(y^2, \lambda^2)$ must be replaced by $D_c(y^2, \lambda^2)$. The analogously defined function $\varphi_c(\lambda^2) = 4i\pi^2 \int dy^2\, D(y^2, \lambda^2)$ is not a test function. Therefore for rising spectral functions subtractions have to be taken into account and one has to be careful in omitting of test functions. In the following we assume that subtractions are not needed and consider the T-product at euclidean values

$$T(x_E, p) = \sum (x^2_E)^n C^1_{2n}\left(\frac{x_4}{|x_E|}\right) T_{2n}(x^2_E), \quad T_{2n}(x^2_E) = T_{2n}(-x^2_E, x^2_E) \qquad (3.2)$$

The moments have to be primarily defined as expansion coefficients of the T-product at euclidean momenta $q^2_E = -q^2 = Q^2$

$$T(q^2_E) = \frac{1}{\pi} \sum \frac{(-1)^n}{(q^2_E)^n} C^1_{2n}\left(\frac{q_4}{|q_E|}\right) \mu_{2n}(q^2_E) \qquad (3.3)$$

$$\mu_{2n}(q_E^2) = \frac{(-1)^n}{2\pi}(q_E^2)^n \int d\omega_q \, C_{2n}^1\left(\frac{q_4}{|q_E|}\right) T(q_E)$$

Substituting the Fourier transform

$$\tilde{T}(q_E) = \int dx_E \, e^{-iq_E x_E} \, T(x_E) \tag{3.4}$$

into eq.(3.3) and applying eq. (2.23) we get

$$\begin{aligned} \mu_{2n}(q_E^2) &= \pi \, 2^{2n} (q_E^2)^{2n} \left(\frac{\partial}{\partial q_E^2}\right)^{2n} \tilde{T}_{2n}(q_E^2) \, , \\ \tilde{T}_{2n}(q_E^2) &= \int dx_E \, e^{-iq_E x_E} \, T_{2n}(x_E^2) \, . \end{aligned} \tag{3.5}$$

It is remarkable, that all contributions to the n-th moment come from the n-th light-cone coefficient only. For each finite q_E^2 exists a correspondence between the moments and the light-cone coefficients. It is however not clear what happens in the limit $q_E^2 \to \infty$. In a previous paper /3/ we have shown that in this limit the light-cone singularities determine the asymptotical behaviour of the moments whereas the asymptotical behaviour of the moments must not determine the light-cone singularities if only spectrality and causality are taken into account. Despite of the simpler structure of the O(4) expansion the same conclusions as in /3/ must be drawn here.

To see this ,we apply representations with euclidean spectral functions, so that

$$\bar{C}(x^2, \vec{x}^2) = -\frac{i}{2\pi} \int_0^\infty d\lambda^2 \, \bar{D}(x^2, \lambda^2) \, \tilde{\Psi}(\vec{x}, \lambda^2) \tag{3.6}$$

$$\tilde{\Psi}(\vec{x}, \lambda^2) = \int d^4u_E \, e^{iu_E x_E} \, \Psi(u_E, \lambda^2)$$

$$T(q_E) = \frac{1}{\pi} \int d^4u_E \, d\lambda^2 \, \frac{1}{(q_E - u_E)^2 + \lambda^2} \, \Psi(u_E, \lambda^2) \tag{3.7}$$

Simple calculation give:

$$\bar{C}(x^2, \vec{x}^2) = \sum (-x^2)^n \, C_{2n}^1\left(\frac{x_4}{\sqrt{x^2}}\right) C_{2n}(x^2, -x^2)$$

$$C_{2n}(x^2, -x^2) = -\frac{i}{4\pi^2} \int_0^\infty d\lambda^2 \left[\frac{\partial}{\partial x^2}\left(\theta(x^2) \, J_0(\lambda\sqrt{x^2})\right)\right] h_{2n}(\lambda^2, -x^2)$$

$$h_{2n}(\lambda^2, -x^2) = 2(-1)^n \pi^2 \int_0^1 du_E^2 \, (u_E^2)^{2n+1} \, \frac{J_{2n+1}(u_E\sqrt{-x^2})}{(u_E\sqrt{-x^2})^{2n+1}} \, \Psi_{2n}(u_E^2, \lambda^2)$$

$$\hat{h}_{2n}(\lambda^2, Q^2) = \frac{\pi^4 \, 2^{2n+2}}{2n+1} (-1)^n \int_0^1 du_E^2 \, \frac{(u_E^2)^{2n+1} \, \Psi_{2n}(u_E^2, \lambda^2 - u_E^2)}{\left(1 + \sqrt{1 + \frac{4Q^2 u_E^2}{(Q^2 + \lambda^2)^2}}\right)^{2n+1}}$$

$$\mu_{2n}(Q^2) = (Q^2)^{2n} \int_0^\infty d\lambda^2 \, \frac{\hat{h}_{2n}(Q^2, \lambda^2)}{(Q^2 + \lambda^2)^{2n+1}} \tag{3.8}$$

$$\Psi(\vec{u}, \lambda^2) = \sum C_{2n}^1\left(\frac{u_4}{|u_E|}\right) (u_E^2)^n \, \Psi_{2n}(u_E^2, \lambda^2)$$

Here the continuation to Minkowski values $x^2 = -x_E^2$ and $q_E^2 = Q^2$ has already been done. As it is expected ,the connections are valid for each n separately. The resulting chain (3.8) is very similar to eqs. (3.2),(2.6) in ref. /3/. In distinction to /3/ the functions $C_{2n}(x^2, -x^2)$, $h_{2n}(\lambda^2, -x^2)$, $\hat{h}_{2n}(\lambda^2, Q^2)$ depend on a second variable in a very special manner however. For the investigation of asymptotical connections these limits in the second variables can be performed independently, so that we have to discuss the relations with $C_{2n}(x^2, 0)$

$h_{2n}(x^2, 0)$, $\hat{h}_{2n}(x^2, \infty)$, the additional variables fixed to 0 or ∞, respectively.

4. Generalizations to Non Forward Scattering

Here we consider some obvious extensions of the results of the foregoing sections to nonforward scattering. The convergence of light-cone expansions can be proved similarly as long as a Jost Lehmann representation is valid .

$$\bar{C}(x^2, \vec{x}) = \sum_{\kappa, l} C_{\kappa l}(x_E^2) Y_{\kappa l 0}(\eta) (x_E^2)^{\frac{\kappa}{2}} , \qquad (4.1)$$

$$C_{\kappa l}(x^2, x_E^2) = (x_E^2)^{-\frac{\kappa}{2}} \int d\omega_{(\eta)} \bar{C}(x^2, \vec{x}) \bar{Y}_{\kappa l 0}(\eta) . \qquad (4.2)$$

The amplitude in momentum space has the expansion

$$T(q_E) = \sum_{\kappa, l} (q_E^2)^{-\frac{\kappa}{2}} Y_{\kappa l 0}\left(\frac{q}{|q_E|}\right) \mu_{\kappa l}(q_E^2) \sqrt{2}\, i^{\kappa} , \qquad (4.3)$$

$$\mu_{\kappa l}(q_E) = \int d\omega_{(\eta)} T(q_E) \bar{Y}_{\kappa l 0}\left(\frac{q}{|q_E|}\right) (q_E^2)^{\frac{\kappa}{2}} i^{-\kappa} \frac{1}{\sqrt{2}} . \qquad (4.4)$$

If a light-cone expansion for the amplitude in x space is written down

$$T(x_E^2, \vec{x}) = \sum T_{\kappa l}(x_E^2) (x_E^2)^{\frac{\kappa}{2}} Y_{\kappa l 0}(\eta) \sqrt{2}\, \pi , \qquad (4.5)$$

then quite analogously to eq.(3.5) the Christ Hasslacher formula follows

$$\mu_{\kappa l}(q_E^2) = \pi\, 2^{\kappa} (q_E^2)^{\kappa} \left(\frac{\partial}{\partial q_E^2}\right)^{\kappa} \tilde{T}_{\kappa l}(q_E^2) , \ \kappa \text{ even!} \qquad (4.6)$$

Of course chains connecting light-cone singularities and moments exist too.

One interesting point is the expression of the moments in terms of structure functions. The simplest way to derive such equations exploits dispersion relations. If we consider the symmetrized amplitude

$$T^S(q, p, k) = \frac{1}{2} \left(T(q_1, p_1, q_2, p_2) + T(-q_2, p_1, -q_1, p_2) \right)$$

$$q = \frac{1}{2}(q_1 + q_2), \quad p = \frac{1}{2}(p_1 + p_2), \quad k = p_1 - p_2$$

then the subtracted dispersion relation

$$T^S(q, p, k) = T_0^S + P_{\frac{\kappa}{2}} + \frac{1}{\pi} \int_{q_E^2 + \frac{t}{4}}^{\infty} dq_0' \, W^S(q_0', \vec{q}, p, k) \left(\frac{q_0^2 + q_E^2}{q_0'^2 + q_E^2} \right)^{\frac{\kappa}{2}} \left(\frac{1}{q_0' - q_0} + \frac{1}{q_0' + q_0} \right)$$

κ even

leads to the expression for the moments

$$\mu_{\kappa l} = c_{\kappa l} \int_{\frac{1}{2q_E}(q_E^2 + \frac{t}{4})}^{\infty} dx \, \left((ix)^2 - 1 \right)^{\frac{1}{4}} Q_{\kappa + \frac{1}{2}}^{l + \frac{1}{2}}(ix) \int_{-1}^{+1} dz \, P_l(z) W(q_0', \vec{q}, p, k) + d_{\kappa l}$$

κ, l even, $\quad x = \frac{q_0'}{|q_E|} \quad z = \frac{q_3}{|\vec{q}|}$; $c_{\kappa l}, d_{\kappa l}$ *constants* ,

which has to be compared with

$$\mu_\kappa = \int_{\xi > 0} d\xi \, \xi^{\kappa - 1} \left(1 + \frac{\xi^2}{Q^2} \right) W(q) .$$

Only in the case $l=0$ there appears the Nachtmann variable $\xi = \frac{Q^2}{q_0 + \sqrt{Q^2 + q_0^2}}$ because of $q_E^{\kappa+1} \left(\left(i\frac{q_0}{q_E} - 1 \right)^{\frac{1}{4}} Q_{\kappa + \frac{1}{2}}^{l + \frac{1}{2}}\left(i\frac{q_0}{q_E} \right) \right) \sim \left((q_0^2 + q_E^2)^{\frac{1}{2}} - q_0 \right)^{\kappa+1}$.

5. Conclusions

The O(4) expansion of matrix elements corresponds to the operator product expansion on the light-cone. Therefore we are more interested in this expansion then in the group theoretically more favoured O(3.1) expansion /7/ which has no such counterpart.

It has been shown that this O(4) expansion converges in the full complex x plane. The light-cone coefficients are generalized functions which in general do not belong to the space S'_+. The moments are primarily defined as expansion coefficients of the T-product. Because of the group theoretical origin of the O(4) expansion each moment corresponds to one light-cone coeffcient only. On the other hand the old result remains true: The q-limit of the light-cone coefficients determines the asymptotical behaviour of the moments whereas the asymptotical behaviour of the moments must not determine the q-limit of the light-cone coefficients if only spectrality and causality are taken into account. Generalizations to nonforward scattering can be obtained if a three dimensional Jost-Lehmann representation is valid.

We would like to thank V.A.Matveev, A.N.Tavkhelidze and B.I.Zavialov for fruitful discussions.

Appendix

Expansion of Functions of the Space $S(R_4)$ in Terms of Harmonical Polynomials

Here the expansion of test functions φ in terms of harmonical polynomials

$$\varphi(x) = \sum_{k,l,m} \varphi_{klm}(x^2)\, H_{klm}(x) \qquad (A.1)$$

$$\varphi_{klm} = (x^2)^{-\frac{k}{2}} \int d\omega_{(\eta)}\, \bar{Y}_{klm}(\eta)\, \varphi(x)\,, \quad \eta = \frac{x}{|x|} \qquad (A.2)$$

is investigated. The important point is to show the convergence of $\sum_{klm} \varphi_{klm}(x^2) H_{klm}(x)$ in $S(R_4)$. Then it follows easily that this sum represents the function $\varphi(x)$ in each point.
The proof consists in an estimate of each term of this series leading to its majorization /8,9/. Let us remember some facts about spherical harmonics in four dimensions. They form a complete and closed set of orthogonal functions on a sphere

$$\int_{S^3} \bar{Y}_{k_1 l_1 m_1}(\eta)\, Y_{k_2 l_2 m_2}(\eta)\, d\omega_{(\eta)} = \delta_{k_1 k_2}\, \delta_{l_1 l_2}\, \delta_{m_1 m_2}$$

$$d\omega_{(\eta)} = \sin^2\theta\, d\theta\, \sin\nu\, d\nu\, d\mu \qquad (A.3)$$

For each k exist $N(k) = (k+1)^2$ linear independent harmonical polynomials which are eigenvectors of the spherical part Δ_η of the four dimensional Laplace operator $\Delta = r^{-3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} + r^{-2}\Delta_\eta$

$$\Delta_\eta H_{klm}(x) = -k(k+2)\, H_{klm}(x) \qquad (A.4)$$

Furthermore they fulfill the integral representation /6/

theoretically more favoured O(3.1) expansion /7/ which has no such counterpart.

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$$d\omega_{(\eta)} = \sin^2\theta\, d\theta\, \sin\nu\, d\nu\, d\mu \tag{A.3}$$

For each k exist $N(k) = (k+1)^2$ linear independent harmonical polynomials which are eigenvectors of the spherical part Δ_η of the four dimensional Laplace operator $\Delta = r^{-3}\frac{\partial}{\partial r} r^3 \frac{\partial}{\partial r} + r^{-2}\Delta_\eta$

$$\Delta_\eta H_{klm}(x) = -k(k+2)\, H_{klm}(x) \tag{A.4}$$

Furthermore they fulfill the integral representation /6/

$$H_{\kappa\ell m}(x) = \frac{2^{\kappa}(\kappa+1)}{\pi^3}\int_{S^3} d\omega(\eta)\,(\eta x)^{\kappa}\, Y_{\kappa\ell m}(\eta) \qquad (A.5)$$

and for the spherical harmonics the addition theorem

$$\sum_{\ell,m} \bar{Y}_{\kappa\ell m}(\eta)\, Y_{\kappa\ell m}(\xi) = \frac{(\kappa+1)^2}{2\pi^2}\, P_{\kappa}(\xi\eta) \qquad (A.6)$$

is valid. For $\xi=\eta$ this yields $|Y_{\kappa\ell m}| \le \kappa+1\,\frac{1}{\pi\sqrt{2}}$ (A.7) because of the normalization property of the Legendre polynomials $P_{\kappa}(1)=1$. Besides the usual space S with the topology given by

$$\|\varphi(x)\|_{p,q} = \sup_{R^4}\,(1+x^2)^{p}\,|D^{q}\varphi(x)|$$

$$q=\sum_{i=1}^{4} q_i\,, \quad D^{q} = \left(\frac{\partial}{\partial x_1}\right)^{q_1}+\cdots+\left(\frac{\partial}{\partial x_4}\right)^{q_4}\,, \quad |x|=r \qquad (A.8)$$

the space $S(R_+)$ is used, described by the semi norm system

$$\|\varphi(t)\|^{(\kappa)}_{p,q,s} = \sup_{R_+}\, t^{\frac{(k-s)_+}{2}}\,(1+t)^{p}\,|\varphi^{(q)}(t)|\,, \quad (k-s)_+ = \max(k-s,0). \qquad (A.9)$$

At first it will be shown that the expansion coefficients $\varphi_{\kappa\ell m}(x^2)$ of the function $\varphi\in S(R_4)$ are elements of the space $S(R_+)$. By construction it is clear that $\tilde{\varphi}_{\kappa\ell m}(r) = \int_{S^3} d\omega(\eta)\, Y_{\kappa\ell m}\,\varphi(x)$ belongs to the space $S(R)$. To show that the function $\varphi_{\kappa\ell m}$ is also an element of the space $S(R_+)$ the properties of $\tilde{\varphi}_{\kappa\ell m}$ at $r=0$ must be investigated. For computing the derivatives $\left(\frac{\partial}{\partial r}\right)^{s}\tilde{\varphi}_{\kappa\ell m}$ in eq. (A.10) we apply the chain rule

$$\left(\frac{\partial}{\partial r}\right)^{s} = \left(\frac{x_i}{r}\frac{\partial}{\partial x_i}\right)^{s} = \sum \eta_{i_1}\cdots\eta_{i_s}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_s}}\,, \quad \eta_i = \frac{x_i}{r} \qquad (A.11)$$

so that

$$\left(\frac{\partial}{\partial r}\right)^{s}\tilde{\varphi}_{\kappa\ell m}(0) = \int_{S^3} d\omega(\eta)\,\bar{Y}_{\kappa\ell m}(\eta)\sum \eta_{i_1}\cdots\eta_{i_s}\frac{\partial}{\partial x_{i_1}}\cdots\frac{\partial}{\partial x_{i_s}}\,\varphi(r,\eta)\Big|_{\eta=0} \qquad (A.12)$$

If additionally the inequality $s<\kappa,\ \kappa>0$ is taken into account then the orthogonality relations (A.3) give the desired result $\left(\frac{\partial}{\partial r}\right)^{s}\tilde{\varphi}_{\kappa\ell m}(0) = 0$. From this it is obvious that the even expansion coefficients $\varphi_{\kappa\ell m}(x^2) = r^{-\kappa}\tilde{\varphi}_{\kappa\ell m}(r)$ belong to the space $S(R_+)$.

As next step the semi norms $\|\varphi_{\kappa\ell m}\|^{(\kappa)}_{pqs}$ have to be estimated. At first they can be related to the $S(R^1)$ semi norms (A.8)

$$\|\varphi_{\kappa\ell m}(r^2)\|^{(\kappa)}_{pqs} = \sup\left|r^{2(\kappa-\ell)_+}(1+r^2)^{p}\left(\frac{1}{2r}\frac{\partial}{\partial r}\right)^{q}\varphi_{\kappa\ell m}(r^2)\right|$$

$$\le C\,(\kappa+1)^{q}\max_{s_1\le q}\|r^{\kappa-2q-s}\,\varphi_{\kappa\ell m}(r)\|_{p+q,s_1}$$

$$\text{for } k\ge 2q+s \qquad (A.13)$$

and

$$\|\varphi_{\kappa\ell m}(r^2)\|^{(\kappa)}_{pqs} \le C\,(\kappa+1)^{q}\max_{s_1\le\frac{(\kappa-\ell)_+}{2}}\left\|\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{q-\frac{(k-\ell)_+}{2}}\varphi_{\kappa\ell m}(r)\right\|_{p+q,s_1}$$

$$\text{for } k<2q+s \qquad (A.14)$$

A further estimate with respect to the $S(R_4)$ semi norms is obtained if the bounds for the spherical functions (A.8) and the definition (A.2) of the expansion coefficients are used.

$$\|\varphi_{\kappa\ell m}(r^2)\|^{(\kappa)}_{pqs} \le C\,(\kappa+1)^{q}\max_{|s_1|\le 3q+s}\|\varphi(x)\|_{p+q\ s_1} \qquad (A.15)$$

As further auxiliary step we estimate the derivatives of the harmonical polynomials $D^{s}H_{\kappa\ell m}(x)$. It is obvious that $D^{s}H_{\kappa\ell m}(x)$ are also harmonical polynomials of degree $\kappa-|s|$ because the operators D^{s} and Δ commute. Therefore

$$D^{s}H_{\kappa\ell m}(x) = \sum_{\ell_1 m_1} a_{\ell_1 m_1}\, H_{\kappa-|s|\,\ell_1 m_1}(x) \quad \text{for } |s|\le\kappa$$

$$D^s H_{\kappa e m}(x) = 0 \qquad \text{for } |s| > \kappa \,.$$

With the help of the orthogonality relation (A.3) and the representation (A.5) for the harmonical polynomials the coefficients $a_{e_1 m_1}$ can be calculated

$$a_{e_1 m_1} = \frac{(\kappa+1)!\, 2^{|s|}}{(\kappa-|s|+1)!} \int_{S^3} d\omega_{(\eta)}\, \eta_1^{j_1} \cdots \eta_4^{j_4}\, \bar{Y}_{\kappa-|s|\, e_1 m_1}(\eta)\, Y_{\kappa e m}(\eta), \qquad j_1 + \cdots + j_4 = \kappa - |s| \,. \tag{A.16}$$

Using eq. (A.7) this implies

$$|a_{e_1 m_1}| \le \frac{(\kappa+1)!\, 2^{|s|}}{(\kappa-|s|+1)!} \sqrt{(\kappa-|s|+1)^2 (\kappa+1)^2} \le \frac{(\kappa+1)!\, 2^{|s|} (\kappa+1)^2}{(\kappa+1-|s|)!} \,. \tag{A.17}$$

Now it is possible to estimate each term of the series (A.1)

$$\| H_{\kappa e m}(x)\, \varphi_{\kappa e m}(x^2) \|_{pq} = \sup_{x \in R^4} (1+x^2)^p \Big| \sum_{0 \le s' \le |q|} C_s \left(D^{q-s'} H_{\kappa e m}(x)\right)\left(D^{s'} \varphi_{\kappa e m}\right) \Big|$$

$$\le \mathrm{const}(r,q)\, (k+1)^{|q|+4} \max_{s \le q} \| \varphi_{\kappa e m}(t) \|^{(\kappa)}_{p+|q|\, s\, |q|} \tag{A.18}$$

$$\le \mathrm{const}\; (k+1)^{2|q|+4} \max_{\substack{s_1 \le 3s + |q| \\ s \le |q|}} \| \varphi(x) \|_{p+|q|+5\, s_1} \,.$$

To get a better k independent estimate we do not use the last expression and study $(\kappa+1)^{2\alpha} \varphi_{\kappa e m}(x^2)$ using (A.4):

$$\varphi_{\kappa e m}(x^2)\, (\kappa+1)^{2\alpha} = r^{-\kappa} \int_{S^3} d\omega_{(\eta)} \left[(1-\Delta_\eta)^\alpha\, \bar{Y}_{\kappa e m}(\eta) \right] \varphi(r,\eta)$$

$$= r^{-\kappa} \int_{S^3} d\omega_{(\eta)}\, \bar{Y}_{\kappa e m}(\eta) \left[(1-\Delta_\eta)^\alpha \varphi(r,\eta) \right] \tag{A.19}$$

$$(1-\Delta_\eta)^\alpha \varphi(r,\eta) = \Big\{ \Big(1 + \sum_{i=1}^4 x_i \frac{\partial}{\partial x_i}\Big)^2 - x^2 \Delta \Big\}^\alpha \varphi(x) = \varphi_\alpha(x) \in S(R^4) \tag{A.20}$$

Now eq. (A.8) can be rewritten with the help of the last two relations. This leads to

$$\| H_{\kappa e m}\, \varphi_{\kappa e m}(x^2) \|_{pq} \le \mathrm{const}\, (\kappa+1)^{-2\beta} \max_{\substack{s_1 \le 3s + |q| \\ s \le |q|}} \| \varphi_{|q|+2+\beta} \|_{p+|q|+5\, s_1}$$

so that the series (A.1) can be estimated as

$$\Big\| \sum_{\kappa e m} H_{\kappa e m}\, \varphi_{\kappa e m} \Big\| \le \sum_{\kappa e m} \| H_{\kappa e m}(x)\, \varphi_{\kappa e m}(x^2) \|$$

$$\le C \sum_{\kappa=0}^{\infty} (\kappa+1)^2 (\kappa+1)^{-2\beta} \max_{\substack{|s_1| \le 3s + |q| \\ s \le |q|}} \| \varphi_{|q|+2+\beta}(x) \|_{p+|q|+5\, s_1}$$

$$\le C_1 \max_{\substack{|s_1| \le 3s + |q| \\ s \le |q|}} \| \varphi_{|q|+2+\beta} \|_{p+|q|+5\, s_1} \qquad \text{for } \beta \ge 2$$

This shows that the series (A.1) converges in $S(R_4)$. Eq. (A.1) is a uniformly converging continous function.[9] For fixed r and all spherical harmonics $Y_{\kappa e m}$

$$\int d\omega_{(\eta)}\, Y_{\kappa_1 e_1 m_1} \Big\{ \sum H_{\kappa e m}(x)\, \varphi_{\kappa e m}(x^2) - \varphi(x) \Big\} = 0$$

is true. From the continuity it follows

$$\sum H_{\kappa e m}(x)\, \varphi_{\kappa e m}(x^2) - \varphi(x) \equiv 0 \,, \qquad \text{for fixed } r,$$

so that the series $\sum H_{\kappa e m}(x)\, \varphi(x^2)$ converges in each point to the function $\varphi(x)$.

6. References

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