СООБЩЕНИЯ
ОБЪЕДИНЕННОГО
ИНСТИТУТА
ЯДЕРНЫХ
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ДУБНА



E2 - 11974



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# STATISTICAL SYSTEMS WITH WELL-DEFINED ENTROPY

1978

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# STATISTICAL SYSTEMS WITH WELL-DEFINED ENTROPY

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Статистические системы с хорошо определенной энтропией

Исследованы математические свойства локальных наблюдаемых, относительно которых энтропия хорошо определена. Сформулирована концепция энтропии для состояний статистических систем, если состояния не даны посредством матриц плотности. Показана конечность и непрерывность локальной энтропии решеточного бозе-газа для каждого строго положительного состояния.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Сообщение Объединенного института ядерных исследований. Дубна 1978

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Statistical Systems with Well-Defined Entropy

In this paper mathematical properties of local observables leading to well-defined entropy are discussed. Especially for the Bose-lattice-gas the finiteness and continuity of the local entropy is shown for every strong positive state.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Communication of the Joint Institute for Nuclear Research. Dubna 1978



## 1. INTRODUCTION

As one of the most fundamental physical quantities the entropy has recently become a topic of intense investigations. The classical quantum-mechanical definition of the entropy $S(\rho) = -\mathrm{Tr}\,\rho \log \rho$ is closely connected with the description of states by density matrices $\rho$. Today voluminous surveys about the fundamental properties of the concave functional $S(\rho)$ as a measure of disorder on the density matrices exist [8,14,17]. It is an actual problem of mathematical physics to reveal such properties of the entropy $S(\rho)$ and their connections with other physical quantities (energy, etc.), which allow one to define the conception of entropy also for states of statistical systems in the thermodynamical limit, which are not given by density matrices in general.

In what follows we want to state which of the mathematical properties of the local structure of statistical systems ensure a correct definition of the entropy as a set-function and what its continuity properties are.

## 2. DEFINITION OF THE ENTROPY ON LOCAL SYSTEMS

The basic object to describe states of an infinite system in the algebraic approach to statistics is the $*$-algebra $\mathfrak{A} = \bigcup_V \mathfrak{A}_V$ of local observables, where $\mathfrak{A}_V$ is the observable $*$-algebra related to the bounded region (box) $V$. Since $\mathfrak{A}_V \subset \mathfrak{A}_{V'}$ for $V \subset V'$, the $*$-algebra is well-defined. For the sake of definiteness we regard only lattice systems on a lattice Z. V is always a finite subset of Z. A state $\omega$ is a linear, positive and normed functional on $\mathfrak{A}$, i.e., $\omega(\lambda A + \mu B) = \lambda\omega(A) + \mu\omega(B)$, $\omega(A^*A) \geq 1$ and $\omega(1) = 1$, where $A, B \in \mathfrak{A}$. Now it may happen, and this is in fact so for physically important models as we shall see below, that $\mathfrak{A}_V$ can be realized as a $*$-algebra of operators on a Hilbert space in such a way that a state $\omega$ restricted to $\mathfrak{A}_V$ is given by a density matrix $\rho_V$ in the form

$$\omega(A) = \mathrm{Tr}\, A\rho_V \qquad \text{for all } A \in \mathfrak{A}_V. \tag{2.1}$$

Then we can define the local entropy $S_V(\omega)$ of the state $\omega$ in the volume V by

$$S_V(\omega) = -\mathrm{Tr}\,\rho_V \log \rho_V. \tag{2.2}$$

Let us first remark that in case of reducible realizations of $\mathfrak{A}_V$ on a Hilbert space the density matrix $\rho_V$ and therefore also the entropy of the state $\omega$ may be not uniquely determined. For example, if $\mathfrak{A}_V$ is the algebra of all 4x4-matrices of the form $A = \begin{pmatrix} A_0 & 0 \\ 0 & A_0 \end{pmatrix}$, where $A_0$ is a 2x2-matrix. The two density matrices $\rho_1 = \begin{pmatrix} \rho & 0 \\ 0 & 0 \end{pmatrix}$, $\rho_2 = \begin{pmatrix} \frac{1}{2}\rho & 0 \\ 0 & \frac{1}{2}\rho \end{pmatrix}$, $\rho$ a positive and normed 2x2-matrix, define the same state on $\mathfrak{A}_V$, i.e., $\mathrm{Tr}\,A\rho_1 = \mathrm{Tr}\,A\rho_2$. But $S(\rho_1) = -\mathrm{Tr}\,\rho\log\rho \neq S(\rho_2) = -2\,\mathrm{Tr}(\frac{1}{2}\rho)\log(\frac{1}{2}\rho)$.

For many lattice models (Ising model, Heisenberg model, etc.) $\mathfrak{A}_V$ is isomorphic to the $*$-algebra of all $n_V \times n_V$-matrices, where $n_V$ is finite and depends on V. Then we can apply the following theorem to define the entropy.

*Theorem 1*

i) If $\mathfrak{A}_V$ is isomorphic to the $*$-algebra of all $n \times n$-matrices, then every nontrivial irreducible representation $A \to \pi(A)$ of $\mathfrak{A}_V$ is finite dimensional and faithful. Two such representations are unitary equivalent.

ii) If $\omega$ is a state on $\mathfrak{A}_V$ and $\pi$ a nontrivial irreducible representation of $\mathfrak{A}_V$ on a finite dimensional Hilbert space $\mathcal{H}_V$, then there exists a unique positive operator $\rho_V$ on $\mathcal{H}_V$ such that $\omega(A) = \mathrm{Tr}\,\pi(A)\rho_V$ for all $A \in \mathfrak{A}_V$. The entropy

$$S_V(\omega) = -\mathrm{Tr}\,\rho_V \log \rho_V \tag{2.3}$$

is independent of the choice of $\pi$.

The statement i) is a well-known fact for the representations of $\mathcal{B}(\mathcal{H})$, $\dim \mathcal{H} < \infty$, /7,§22/. ii) is a straightforward consequence of i).

### 3. *THE ENTROPY OF A BOSE-LATTICE-GAS*

For a Bose-lattice-gas $\mathfrak{A}_V$ is the $*$-algebra of all polynomials of finite many pairs $P_j$, $Q_j$ satisfying the CCR, where $j$ runs over the lattice points of V. We shall see that also in this case the entropy of a state $\omega$ can be defined in the analogous way as for the case that $\mathfrak{A}_V$ are algebras of finite-dimensional matrices. But since now $\mathfrak{A}_V$ has only representations by unbounded operators, one needs some special facts about such representations, which we are going to describe first.

Let $\mathcal{D}$ be a dense domain in a Hilbert space $\mathcal{H}$ then we denote by $\mathcal{L}^+(\mathcal{D})$ the $*$-algebra of all (unbounded) operators A on $\mathcal{H}$ so that $A, A^*$ are defined on $\mathcal{D}$ and leave $\mathcal{D}$ invariant, i.e., $A, A^*\mathcal{D} \subset \mathcal{D}$. We denote $A^+ = A^*|_{\mathcal{D}}$. If R is a $*$-algebra, then a representation $\pi$ of R is a $*$-homomorphism of R into $\mathcal{L}^+(\mathcal{D})$. A representation is called self-adjoint if $\mathcal{D} = \bigcap_{A \in R} \mathcal{D}(\pi(A)^*)$, where $\mathcal{D}(\pi(A)^*)$ is the natural domain of the adjoint operator $\pi(A)^*$ /2,9,13,15/.

Let $\pi_0$ denote the Schrödinger representation of $\mathfrak{A}_V$ defined by $\pi_0(Q_j) = x_j$, $\pi_0(P_j) = \frac{1}{i}\frac{\partial}{\partial x_j}$ on the Schwartz space $\mathcal{S} \subset L_2$ of rapidly decreasing functions. $\pi_0$ is a self-adjoint representation. A state $\omega$ on $\mathfrak{A}_V$ is called strongly positive if $\omega(A) \geq 0$ for all $A \in \mathfrak{A}_V$ with $\pi_0(A) \geq 0$ on $\mathcal{S}$. A representation $\pi$ is called strongly positive, if $\pi_0(A) \geq 0$ implies $\pi(A) \geq 0$. After these preparations we can state the main theorem.

*Theorem 2*

i) Every nontrivial irreducible and strongly positive self-adjoint representation of $\mathfrak{A}_V$ is unitary equivalent to the Schrödinger representation $\pi_0$.

ii) If $\omega$ is a strongly positive state on $\mathfrak{A}_V$ and $\pi$ a nontrivial irreducible and strongly positive self-adjoint representation of $\mathfrak{A}_V$ on $\mathfrak{H}$ then there exists a unique positive nuclear operator $\rho_V \in \mathfrak{L}^+(\mathfrak{H})$, so that $\omega(A) = \mathrm{Tr}\,\pi(A)\rho_V$ for all $A \in \mathfrak{A}_V$. The entropy

$$S_V(\omega) = -\mathrm{Tr}\,\rho_V \log \rho_V \tag{3.1}$$

is finite and independent of the choice of $\pi$.

The statement i) is well-known uniqueness theorem for the CCR /6,7/ in a form proved in /9/. If $\omega$ is a strongly positive state on $\mathfrak{A}_V$, then it is given by a density matrix in the described sense, as it was proved in /12,16,3,11/. The finiteness of the entropy was proved in /4,5/. The independence of the entropy of the representation is a consequence of i).

## 4. *CONTINUITY PROPERTIES OF THE ENTROPY*

As it was outlined in the foregoing section every state $\omega$ has an entropy $S_V(\omega)$ as a set-function depending on V. In the case described by Theorem 1 the states $\omega$ are linear functionals on a normed $*$-algebra $\mathfrak{A}$ and therefore the norm $||\omega||$ of the states is well-defined. For every fixed V the mapping $\omega \to \rho_V$ is norm-continuous if we take the usual operator norm on the finite dimensional density matrices. Hence for fixed V the entropy $S_V(\omega)$ depends continuously on $\omega$ with respect to the norm on the states.

In the case of the Bose-lattice-gas regarded in the foregoing section the situation is much more difficult. Since the observable algebra $\mathfrak{A}$ contains unbounded elements the states $\omega$ are not normable. The strong topology $\beta^*$ on the states, which we call the physical topology /4, 5/, is a locally convex topology defined by the following system of seminorms

$$\beta^*: \quad ||\omega||_{\mathfrak{M}} = \sup_{A \in \mathfrak{M}} |\omega(A)| < \infty, \tag{4.1}$$

where $\mathfrak{M}$ runs over all weakly-bounded sets of $\mathfrak{A}$. For every bounded volume V the physical topology $\beta^*_V$ on the states on $\mathfrak{A}_V$ is given by the system of seminorms

$$\beta^*_V: \quad ||\rho_V||_k = ||(N_V + 1)^k \rho_V (N_V + 1)^k||, \tag{4.2}$$

$k = 0, 1, 2, \ldots$, on the corresponding density matrices /5/. $N_V$ is the number operator. Then we have the following theorem.

*Theorem 3*

i) The mapping $\omega \to \rho_V$ is continuous with respect to the topologies $\beta^*$ and $\beta^*_V$ for every V.

ii) For every V the entropy $S_V(\omega)$ is a continuous function of $\omega$ with respect to the physical topology $\beta^*$ on the states on $\mathfrak{A}$.

The statement i) can be proved by a modification of the proof of Theorem 1 in our paper /5/; ii) is then a consequence of Theorem 2 in the same paper.

Especially for translation invariant states it makes physical sense to define the entropy density $S(\omega) = \lim_{V \to \infty} \frac{S_V(\omega)}{V}$, which is finite for many lattice models. The entropy density has in some sense better continuity properties than the entropy. For quasi-free states the continuity of the entropy density was proved in /1/.

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*Received by Publishing Department on October 24 1978.*