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A-SUPERQUANTIZATION

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A-SUPERQUANTIZATION

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А-суперквантование

Изучается квантование тензорных полей с помощью операторов рождения и уничтожения, принадлежащих нечетной части специальной линейной супералгебры и порождающих всю алгебру. Исследуются свойства низшего нетривиального пространства Фока. Для этого случая сформулирован принцип Паули: заряд произвольного ансамбля частиц равен либо 0, либо 1. Построен ортонормированный базис в пространстве Фока. Найдены матричные элементы операторов рождения и уничтожения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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A-Superquantization

A quantization of tensor fields is studied with creation and annihilation operators that belong to the odd part of the special linear superalgebra and generate the whole algebra. The properties of the lowest nontrivial Fock space are investigated. For this case the Pauli principle has been formulated. The charge of every ensemble of particles is either 0 or 1. An orthonormalized basis in the Fock space is constructed. The matrix elements of the creation and annihilation operators are found.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The commutation relations between the creation and annihilation operators in the Lagrangian quantum field theory are usually derived from the Heisenberg equation of motion of the fields [1],

$$[P_\mu, \phi(x)] = -i\partial_\mu \phi(x) \qquad \mu = 0,1,2,3 \tag{1}$$

Here P_μ is the 4-momentum operator which is a certain integral combination of the fields and their space-time derivatives. Therefore (1) is a nonlinear integro-differential equation for the operator valued distribution $\phi(x)$. To quantize the field means to find solutions of the equation (1).

In the free field case at least some of the solutions of this equation are well known. Consider for definiteness a charged scalar field

$$\phi(x) = \phi^{+}(x, -) + \phi^{-}(x, +),$$

$$\phi^{*}(x) = \phi^{+}(x, +) + \phi^{-}(x, -). \tag{2}$$

In terms of the Fourier amplitudes $a^{\xi}(\mathbf{p}, \eta)$, introduced by the relation *

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* Throughout the paper $\xi, \eta, \epsilon = \pm$ or ± 1; $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$.

$$\phi^{\xi}(x,\eta) = \frac{1}{(2\pi)^{3/2}} \int d\underline{p}\, e^{i\xi px}\, a^{\xi}(\underline{p},\eta)$$

the Heisenberg equation of motion (1) reads as

$$[P_n, a^{\xi}(\underline{k},\eta)] = \xi k_n a^{\xi}(\underline{k},\eta). \tag{3}$$

In this case the gauge invariance gives another relation,

$$[Q, a^{\xi}(\underline{p},\eta)] = \xi\eta\, a^{\xi}(\underline{p},\eta). \tag{4}$$

The classical expression of the energy-momentum vector P_n can be written as

$$P_n = \frac{1}{2}\sum_{\sigma}\int d\underline{p}\, p_n \{a^{+}(\underline{p},\sigma), a^{-}(\underline{p},\sigma)\}. \tag{5}$$

In the quantized case the ordering of the operators $a^{\xi}(\underline{p},\sigma)$ in (5) seems to be the most appropriate for generalizations of the statistics. Inserting (5) in (3) one obtains the main quantization equation

$$\frac{1}{2}\sum_{\eta}\int d\underline{p}\, p_n [\{a^{+}(\underline{p},\eta), a^{-}(\underline{p},\eta)\}, a^{\xi}(\underline{k},\epsilon)] = \xi k_n a^{\xi}(\underline{k},\epsilon). \tag{6}$$

Remark that the equations (3) and (4) contain almost no information about the commutation relations between the operators $a^{\xi}(\underline{p},\eta)$. The relativistic and gauge invariance impose restrictions only on the commutation relations between certain integral combinations of the operators $\{a^{+}(\underline{p},\eta), a^{-}(\underline{p},\eta)\}$ and $a(\underline{k},\epsilon)$. Otherwise the commutation relations are quite arbitrary. Whatever the structure relations will be, from (3) and (4) it follows that $a^{+}(\underline{p},\eta)$ and $a^{-}(\underline{p},\eta)$ are creation and annihilation operators of particles with momentum $\underline{p}$ and charge η. From now on we call the operator $a^{\xi}(\underline{p},\eta)$ creation ($\xi = +$) or annihilation ($\xi = -$) operator.

One solution of eq. (6) is given with operators $a^{\xi}(\underline{p},\eta)$ obeying the commutation relations

$$[a^{\xi_1}(\underline{p}_1,\eta_1), a^{\xi_2}(\underline{p}_2,\eta_2)] = \frac{1}{2}(\xi_2-\xi_1)\delta_{\eta_1\eta_2}\delta(\underline{p}_1-\underline{p}_2), \tag{7}$$

i.e., with the well known Bose creation and annihilation

operators. If one imposes the conditions on the commutators of the fields to be c-numbers, the Bose solution is also unique. Otherwise the main quantization equation (6) has several other solutions. For instance, the operators that satisfy the relations

$$[\{a^{\xi_1}(\underline{p}_1,\eta_1), a^{\xi_2}(\underline{p}_2,\eta_2)\}, a^{\xi_3}(\underline{p}_3,\eta_3)] =$$
$$(\xi_3-\xi_1)\delta_{\eta_1\eta_3}\delta(\underline{p}_1-\underline{p}_3)a^{\xi_2}(\underline{p}_2,\eta_2) +$$
$$+(\xi_3-\xi_2)\delta_{\eta_2\eta_3}\delta(\underline{p}_2-\underline{p}_3)a^{\xi_1}(\underline{p}_1,\eta_1) \tag{8}$$

give also a solution to (6). The Bose operators (7) belong to the class of operators defined by (8). These operators have been introduced by Green [2] and are called paraBose operators. Any tensor field can be quantized according to the paraBose statistics. In a similar way Green generalized the Fermi statistics of the spinor fields to the more general paraFermi statistics [2]. The paraFermi operators satisfy threelinear relations similar to (8), however the anticommutators are replaced by commutators.

2. *A-SUPERQUANTIZATION*

In the present paper we wish to study some other generalizations of the statistics of the tensor fields. For simplicity we proceed with considering a charged scalar field. The generalizations we are looking for are not found by accident. The guiding idea comes from the observation that the ordinary Bose statistics has a well defined Lie superalgebraical meaning [3]. It turns out that the Bose (and more generally, the paraBose) creation and annihilation operators span a basis in the odd part of the orthosymplectic Lie superalgebra and at the same time generate the whole algebra. More exactly, the

Bose operators generate one particular irreducible representation of the orthosymplectic algebra. In a similar way the Fermi (and paraFermi) operators generate an irreducible representation of the orthogonal Lie algebra [4]. As Lie superalgebras, the orthosymplectic and the odd-orthogonal algebras are simple algebras. The observation that the ordinary creation and annihilation operators generate simple Lie superalgebras is important not only because it makes it possible to translate most of the problems of the statistics into an algebraical language but it also provides a basis for further generalizations. Indeed the Bose and Fermi quantization is actually a quantization according to certain irreducible representations of simple Lie superalgebra. Nowadays all such superalgebras are fully classified [5]. In view of this it seems natural to ask whether one can satisfy the main quantization equation (6) with new kinds of creation and annihilation operators the main feature of which will be that they generate some of the other simple Lie superalgebras. This question has been studied in detail in refs. [6,7] only from a Lie-algebraical point of view. There we have shown that to every classical series of simple Lie algebras there corresponds a quantization of the spinor fields that is logically compatible with the main quantization principles of quantum field theory.

In the present paper we wish to study a solution of the main quantization equation (6) with the property that any n pairs of creation and annihilation operators generate the classical simple Lie superalgebra $A(0, n-1)$. According to the notion of algebra we call the corresponding quantization (statistics) A-superquantization (A-superstatistics). In this terminology the Bose (and, more generally, the paraBose) quantization is B-superquantization since n pairs of Bose (or paraBose) operators generate the Lie superalgebra $B(0, n)$ ref. [3].

As in ref. [7], we shall consider the field $\phi(x)$ with mass m to be locked in a cube with edge L. In this case one obtains for the eigenvalues k^m_n of the 4-momentum P^m, $m = 0, 1, 2, 3$ (ref. [1])

$$k^{\alpha}_n = \frac{2\pi}{L} n^{\alpha}, \quad k^0_n = \sqrt{m^2 + \left(\frac{2\pi}{L}\right)^2 [(n^1)^2 + (n^2)^2 + (n^3)^2]}, \tag{9}$$

where $n = (n^1, n^2, n^3)$, $\alpha = 1, 2, 3$ and n^{α} runs over all non-negative integers. In this notation the equation (6) reads as

$$\frac{1}{2} \sum_{\eta, i} p^m_i [\{a^+_{\eta i}, a^-_{\eta i}\}, a^{\xi}_{\epsilon j}] = \xi k^m_j a^{\xi}_{\epsilon j}, \quad m = 0, 2, 1, 3. \tag{10}$$

In the above equation $a^{\xi}_{\eta i}$ is a creation ($\xi = +$) or annihilation ($\xi = -$) operator of a particle with charge η and other characteristics i. Any set of operators $a^{\xi}_{\eta i}$ satisfying the relations

$$\sum_{\eta} [\{a^+_{\eta i}, a^-_{\eta i}\}, a^{\xi}_{\delta j}] = 2\xi \delta_{ij} a^{\xi}_{\delta j} \tag{11}$$

gives a solution of (10). We call (11) a main quantization condition.

We now proceed to find a solution of the main quantization condition with creation and annihilation operators that are root vectors in the odd part of the special linear superalgebra, i.e., a Lie superalgebra of the type $A(0, 2n-1)$*. In order to define it we introduce first the general linear superalgebra $gl(1, 2n)$. The latter is given with the set of all $(2n+1) \times (2n+1)$ matrices. Denote by $e_{\alpha\beta}$, $\alpha, \beta = -n, -n+1, \dots, n-1, n$, a $(2n+1)$-dimensional square matrix with 1 on the cross of α-th row and β-th column and zeros elsewhere. Then the even and odd parts G_0 and G_1 of

$$gl(1, n) = G_0 + G_1 \tag{12}$$

are defined as follows (lin. env. = linear envelope):

$$G_1 = \text{lin. env.} \{e_{0i}, e_{i0} \mid i = \pm 1, \pm 2, \dots, \pm n\}, \tag{13}$$

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* Most of the Lie superalgebraical terminology we use is defined in ref. [8]. For more details see, for instance, ref. [5].

$$G_0 = \text{lin.env.}\{e_{00},\ e_{ij} \mid i,j = \pm 1, \pm 2, \dots, \pm n\}.$$

The multiplication $[\![\,,]\!]$ in $gl(1, n)$ is

$$[\![a, b]\!] = ab - (-1)^{\alpha\beta}\, ba, \quad a \in G_\alpha,\ b \in G_\beta, \tag{14}$$

where ab is the usual matrix multipilucation of a and b. To obtain the product of arbitrary two elements, one has to extend the relation (14) by linearity. The Lie superalgebra $gl(1, 2n)$ is not simple since it contains as nontrivial ideal the unit matrix. The special linear Lie superalgebra $A(0, 2n-1)$ is singled out of $gl(1, 2n)$ in a natural way,

$$A(0, 2n-1) = A_0(0, 2n-1) + A_1(0, 2n-1), \tag{15}$$

$$A_1(0, 2n-1) = \text{lin.env.}\{e_{0i}, e_{i0} \mid i = \pm 1, \dots, \pm n\},$$

$$A_0(0, 2n-1) = \text{lin.env.}\{e_{00} + e_{kk}, e_{ij} \mid i \neq j, i, j, k = \pm 1, \dots, \pm n\}. \tag{16}$$

The Cartan superalgebras $\tilde{\mathcal{H}}$ and $\mathcal{H}$ of $gl(1,2n)$ and $A(0, 2n)$ can be chosen to be the commutative subalgebras

$$\tilde{\mathcal{H}} = \text{lin.env.}\{e_{\alpha\alpha} \mid \alpha = -n, -n+1, \dots, n-1, n\},$$

$$\mathcal{H} = \text{lin.env.}\{e_{00} + e_{kk} \mid k = \pm 1, \dots, \pm n\}. \tag{17}$$

We are now ready to introduce the creation and annihilation operators. Define

$$a^{\xi}_{\xi i} = e_{\xi i, 0} \qquad i = 1, 2, \dots, n$$

$$a^{\xi}_{-\xi i} = -\xi e_{0, -\xi i} \qquad \xi = \pm. \tag{18}$$

One can easily verify that the operators (18) satisfy the main quantization condition (11)[x]. The operators

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[x] For the subsequent algebraical consideration, the factor 2 in the right-hand side of (11) is irrelevant and we neglect it. The operators that satisfy (11) are $\sqrt{2}\, a^{\xi}_{\eta i}$.

$a^{\xi}_{\eta i}$ **span a basis in the odd part $A_1(0,2n-1)$. Clearly they generate the whole algebra since**

$$\{a^{\xi}_{\xi i},\ a^{\eta}_{-\eta j}\} = -\eta\, e_{\xi i, -\eta j} \quad ; \xi, \eta = \pm;\ i, j = 1, 2, \dots, n. \tag{19}$$

The operators $a^{\xi}_{\eta i}$ are root vectors of the algebra since they are eigenvectors of the Cartan subalgebra with respect to the adjoint representation,

$$[h, a^{\xi}_{\eta i}] \sim a^{\xi}_{\eta i}, \quad a \in \mathcal{H}. \tag{20}$$

The multiplication in $A(0, 2n \mp 1)$ can be expressed only in terms of the operators $a^{\xi}_{\eta i}$. Using (18) we obtain

$$[\{a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}\}, a^{\eta}_{\eta k}] = -\xi\delta_{-\xi j, \eta k}\, a^{\xi}_{\xi i},$$

$$[\{a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}\}, a^{\eta}_{-\eta k}] = \eta\delta_{\xi i, -\eta k}\, a^{\xi}_{-\xi j},$$

$$[\{a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}\}, a^{\eta}_{\eta k}] = \xi\delta_{\xi j, \eta k}\, a^{\xi}_{\xi i} - \xi\delta_{ij}\, a^{\eta}_{\eta k},$$

$$[\{a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}\}, a^{\eta}_{-\eta k}] = \eta\delta_{\xi i, -\eta k}\, a^{-\xi}_{\xi j} + \xi\delta_{ij}\, a^{\eta}_{-\eta k},$$

$$\{a^{\xi}_{\xi i}, a^{\eta}_{\eta j}\} = \{a^{\xi}_{-\xi i}, a^{\eta}_{-\eta j}\} = 0. \tag{21}$$

For the Lie superalgebra $A(0, 2n-1)$, $i, j, k = 1, 2, \dots, n$. Any infinite set of operators $a^{\xi}_{\eta i}$, $i = 1, 2, \dots$, however, obeying the structure relations (21) satisfy the main quantization condition (11) and hence the quantization equation (10).

Definition. The operators $a^{\xi}_{\eta i}$, $i = 1, 2, \dots;\ \xi, \eta = \pm$ satisfying the relations (21) are called a-creation ($\xi = +$) and annihilation ($\xi = -$) operators.

Clearly any finite set $a^{\xi}_{\eta i_1}, \dots, a^{\xi}_{\eta i_n}$, $i \neq \dots \neq i$ of a-creation and annihilation operators generates the Lie superalgebra $A(0, 2n-1)$. Therefore the quantization (statistics) corresponding to the a-operators is A-superquantization (superstatistics).

3. *A FOCK SPACE FOR THE* a*-OPERATORS*

In this paragraph we wish to study properties of the simplest nontrivial representation of the a-operators, that possesses all main features of the Fock space in an ordinary quantum mechanics. From a purely technical point of view it is convenient to consider from time to time a finite set $a^{\xi}_{\eta 1}, \ldots, a^{\xi}_{\eta n}$ of operators. The generalization of the results to the case of infinitely many operators is always evident.

By a representation of the a-operators we understand a mapping θ: $a^{\xi}_{\eta i} \to \tilde{a}^{\xi}_{\eta i}$ of the operators $a^{\xi}_{\eta i}$ onto a set of linear operators $\tilde{a}^{\xi}_{\eta i}$ that preserves the relations (21). Since 2n pairs of a-operators generate A(0, 2n − 1), to every (irreducible) representation of $a^{\xi}_{\eta i}$ there corresponds an (irreducible) representation of A(0, 2n − 1), and vice versa.

The Fock space of 2 pairs of a-operators was studied in ref. /9/. Here we shall generalize the results to the case of arbitrary number of operators.

Let W be the representation space. We assume that the space contains a vector $|0\rangle \in W$ called a vacuum such that

$$a^{-}_{\eta i}|0\rangle = 0 \qquad \eta = \pm,\ i = 1, 2, \ldots \tag{22}$$

In order to obtain a space generated out of the vacuum by means of creation operators we postulate

$$a^{-}_{\eta i} a^{+}_{\eta j}|0\rangle = \delta_{ij}\, c_{\eta}|0\rangle, \quad \eta = \pm,\ i = 1, 2, \ldots . \tag{23}$$

The scalar product in W is defined in the usual for the Fock space way:

$$\langle 0|0\rangle = 1; \quad \langle 0|a^{+}_{\eta i} = 0$$

$$(a^{+}_{\eta_1 i_1} a^{+}_{\eta_2 i_2} \ldots a^{+}_{\eta_m i_m}|0\rangle,\ a^{+}_{\xi_1 j_1} \ldots a^{+}_{\xi_n j_n}|0\rangle) =$$

$$= \langle 0|a^{-}_{\eta_m i_m} \ldots a^{-}_{\eta_2 i_2} a^{-}_{\eta_1 i_1} a^{+}_{\xi_1 j_1} \ldots a^{+}_{\xi_n j_n}|0\rangle. \tag{24}$$

The requirement $(a,a) > 0$ for all $0 \neq a \in W$ puts restrictions on possible values of the constant c_{ξ}. Here we shall consider the simplest case $c_{+} = 1$, $c_{-} = 0$. More exactly we require $(i, j = 1, 2, \ldots)$

$$a^{-}_{j}|0\rangle = a^{-}_{-i}|0\rangle = a^{+}_{-i}|0\rangle = 0, \quad a^{-}_{i} a^{+}_{j}|0\rangle = \delta_{ij}|0\rangle. \tag{25}$$

Lemma 1. The representation space W is a linear envelope of all vectors

$$\{a^{+}_{-i_1}, a^{+}_{j_1}\}^{m_{i_1 j_1}} \ldots \{a^{+}_{-i_r}, a^{+}_{j_r}\}^{m_{i_r j_r}}\ a^{+}_{\eta_1 k_1} \ldots a^{+}_{\eta_s k_s}|0\rangle \tag{26}$$

for $r, s = 0, 1, \ldots$ $i, j = 1, 2, \ldots$

Proof. The proof is a generalization of the one given in ref. /9/ to the case of several a-operators. It is based on the Poincare-Birkhoff-Witt theorem /5/ stating in our case that the basis in the universal enveloping basis of A(0, n − 1) is given by all possible monomials

$$P(m_{ij}, \theta_k, r_i)\cdot Q(r_{ij}, s_{ij}, \theta_{\xi i}, \theta'_i), \tag{27}$$

where

$$P(\ldots) = \prod_{i,j=1}^{n} \{a^{+}_{-i}, a^{+}_{j}\}^{m_{ij}} \prod_{k=1}^{n} (a^{+}_{k})^{\theta_k} \prod_{i=1}^{n} \{a^{-}_{i}, a^{+}_{i}\}^{r_i};$$

$$Q(\ldots) = \prod_{i \neq j = 1}^{n} \{a^{-}_{i}, a^{+}_{j}\}^{r_{ij}} \prod_{i,j=1}^{n} \{a^{-}_{-i}, a^{+}_{j}\}^{s_{ij}} \times$$

$$\times \prod_{\substack{i=1 \\ \xi=\pm}}^{n} (a^{\xi}_{-\xi i})^{\theta_{\xi,i}} \prod_{i=1}^{n} (a^{-}_{-i})^{\theta'_i} . \tag{28}$$

Every element $x \in W$ is a linear combination of vectors $P(\ldots)Q(\ldots)|0\rangle$. The latter is different from zero only if all $r_{ij} = s_{ij} = \theta_{\xi i} = \theta'_i = 0$, i.e., if $Q = 1$. To complete the proof it remains to remark that

$$\prod_i \{a^{-}_{i}, a^{+}_{i}\}^{r_i}|0\rangle = |0\rangle. \tag{29}$$

Reformulating the lemma we can say that W is generated out of the vacuum by polynomials of creation operators. Since any finite set of a-operators $a^{\xi}_{\eta i_1}, \dots, a^{\xi}_{\eta i_n}$ generate $A(0, 2n-1)$, the lemma holds also for infinite number of operators.

Lemma 2. The product $a^+_{\eta i} a^+_{\eta j}$, $\eta = \pm$, $i, j = 1,2,\dots$ is a zero operator in W.

Proof. Denote by Φ_n the linear envelope of all states $P_n|0\rangle$, where P_n is an n-th order polynomial of creation operators. Then W is a union of all Φ, $n = 0,1,2,\dots$. From (21) and (25) we find

$$a^-_{-i} a^+_{-j} a^+_k|0\rangle = 0; \quad a^-_i a^+_j a^+_k|0\rangle = [\{a^-_i, a^+_j\}, a^+_k]|0\rangle + a^+_k\{a^-_i, a^+_j\}|0\rangle - a^+_j\{a^-_i, a^+_k\}|0\rangle = 0. \qquad (30)$$

Using the definition of the scalar product (24) we conclude that

$$(a^+_{\eta_1 i_1} \dots a^+_{\eta_m i_m}|0\rangle, a^+_j a^+_i|0\rangle) = \langle 0| a^-_{\eta_m i_m} \dots a^-_{\eta_1 i_1} a^+_j a^+_i|0\rangle = 0.$$

Thus for every $0 \neq x \in W$ $(x, a^+_j a^+_k|0\rangle) = 0$ and therefore

$$a^+_{\eta i} a^+_{\eta j}|0\rangle = 0 \Rightarrow a^+_{\eta i} a^+_{\eta j}\Phi_0 = 0. \qquad (31)$$

Taking into account that

$$a^+_{-i} a^+_{-j} a^+_k|0\rangle = a^+_{-i}\{a^+_{-j}, a^+_k\}|0\rangle = [a^+_{-i}, \{a^+_{-j}, a^+_k\}]|0\rangle = 0 \qquad (32)$$

we have

$$a^+_{\eta i} a^+_{\eta j}\Phi_1 = 0. \qquad (33)$$

The rest of the lemma is proved by induction. Suppose

$$a^+_{\eta i} a^+_{\eta j}\Phi_m = 0, \quad m = 0,1,2,\dots, n; \quad n > 1. \qquad (34)$$

Because of (31) Φ_{n+1} is linearly spanned on all vectors $P_{n-1} a^+_{-p} a^+_q|0\rangle$, $p, q = 1,2,\dots$, where P_{n-1} is a polynomial of order $n-1$ in the creation operators. Since

$$[\{a^+_{-p}, a^+_q\}, a^+_{\eta i}] = 0 \quad \text{implies} \quad [\{a^+_{-p}, a^+_q\}, P_{n-1}] = 0 \qquad (35)$$

and $P_{n-1}|0\rangle \in \Phi_{n-1}$, from (34) we have

$$a^+_{\eta i} a^+_{\eta j} P_{n-1} a^+_{-p} a^+_q|0\rangle = a^+_{-p} a^+_q a^+_{\eta i} a^+_{\eta j} P_{n-1}|0\rangle = 0. \qquad (36)$$

Thus (34) holds for $n+1$. Therefore $a^+_{\eta i} a^+_{\eta j}$ annihilate every vector $x \in W$. Hence $a^+_{\eta i} a^+_{\eta j} = 0$.

As an immediate consequence of this lemma, we have the

Corollary. The representation space W is a linear envelope of all vectors $(p_j = 1,2,\dots)$

$$a^+_{\xi p_n} a^+_{-\xi p_{n-1}} \dots a^+_{-p_4} a^+_{p_3} a^+_{-p_2} a^+_{p_1}|0\rangle, \quad \xi = (-1)^{n+1}. \qquad (37)$$

Lemma 3. The state (37) is symmetric with respect to arbitrary permutations of creation operators corresponding to the same charge.

Proof.

$$\dots a^+_i a^+_{-j} a^+_k \dots|0\rangle = \dots\{a^+_i, a^+_{-j}\}a^+_k \dots|0\rangle = \dots a^+_k\{a^+_i, a^+_{-j}\}\dots|0\rangle = \dots a^+_k a^+_{-j} a^+_i \dots|0\rangle. \qquad (38)$$

Similarly

$$\dots a^+_{-i} a^+_j a^+_{-k} \dots|0\rangle = \dots a^+_{-k} a^+_j a^+_{-i} \dots|0\rangle. \qquad (39)$$

Because of this lemma every vector (37) is completely determined by the number of creation operators $a^{+}_{\eta i}$ in the state ηi. Therefore we introduce the notation

$$|q; p_{i_1} \dots p_{i_s}; n_{j_1} \dots n_{j_r}> \tag{40}$$

for every vector (37) generated out of the vacuum by means of a monomial which is a homogeneous function of order p_i (resp., n_j) of the operator $a^{+}_{i}(a^{+}_{-j})$, i.e., contains $p_i (n_j)$ operators $a^{+}_{i}(a^{+}_{-j})$. The number q is the charge of the ensemble of particles, i.e.,

$$q = \sum_i p_i - \sum_j n_j . \tag{41}$$

According to the Corollary, $q = 0$ or 1.

In the usual interpretation (40) corresponds to an ensemble containing $p_i(n_j)$ particles in the state $i(-j)$. The number of particles in every state is unlimited. The charge, however, should be always 0 or 1. This is actually

The Pauli principle of the A-superstatistics. The charge of every ensemble of particles is either 0 or 1. Otherwise the number of particles occupying any state is arbitrary.

It is not difficult to write down the transformation properties of the vectors (40) under left multiplications with creation and annihilation operators. For this purpose it is convenient to represent the space W as a direct sum of its zero-charge and one-charge subspaces, W_0 and W_1, resp., $W = W_0 + W_1$. Since the operator $a^{\xi}_{\eta i}$ carries a charge $\xi\eta$, from the Pauli principle we conclude

$$a^{+}_{-i} W_0 = a^{-}_{i} W_0 = a^{+}_{i} W_1 = a^{-}_{-i} W_1 = 0. \tag{42}$$

For the other left multiplications after some calculations one obtains:

$$a^{+}_{i} |0; \dots, p_i, \dots; \dots> = |1; \dots, p_i + 1, \dots; \dots> ,$$

$$a^{+}_{-j} |1; \dots; \dots, n_j \dots> = |0; \dots; \dots, n_j + 1, \dots>, \tag{43}$$

$$a^{-}_{i} |1; \dots, p_i, \dots; \dots> = p_i |0; \dots, p_i - 1, \dots; \dots>,$$

$$a^{-}_{-j} |0; \dots; \dots, n_j, \dots> = n_j |1; \dots; \dots, n_j - 1, \dots> .$$

In particular from the above relations it follows that

$$a^{-}_{\eta i} a^{-}_{\eta j} = 0,$$

$$\{a^{-}_{-j}, a^{-}_{i}\} |q; \dots, p_i, \dots; \dots, n_j, \dots> =$$

$$= p_i n_j | q; \dots, p_i - 1, \dots; \dots, n_j - 1, \dots> . \tag{44}$$

Since

$$[\{a^{-}_{i}, a^{-}_{-j}\}, a^{-}_{\eta k}] = 0 \tag{45}$$

(45)

from (42) it immediately follows that

$$(|1; \dots), |0; \dots) = 0. \tag{46}$$

The exact statement is contained in the

Theorem. The vectors (40) constitute an orthogonal basis in the Fock space W. The norm is

$$\| |q; p_{i_1}, \dots, p_{i_m}; n_{j_1}, \dots, n_{j_k}> \|^2 = p_{i_1}! \dots p_{i_m}! n_{j_1}! \dots n_{j_k}! . \tag{47}$$

The vectors

$$|q; p_{i_1}, \dots, p_{i_m}; n_{j_1}, \dots, n_{j_k}) = \frac{|q; p_{i_1}, \dots, p_{i_m}; n_{j_1}, \dots, n_{j_k}>}{\sqrt{p_{i_1}! \dots p_{i_m}! n_{j_1}! \dots n_{j_k}!}} \tag{48}$$

constitute an orthonormal basis in W.

We omit the proof. It follows from (42-45) and the structure relations (21).

Let Q be the charge operator, i.e.,

$$Q|q; \dots\rangle = q|q; \dots\rangle \qquad (49)$$

Then

$$[Q, a^{\xi}_{\eta i}] = \xi\eta\, a^{\xi}_{\eta i} \qquad (50)$$

so that (4) holds. In terms of Q and the creation and annihilation operators one can express the number operator $N_{\eta i}$ of particles in a state ηi:

$$N_i = \{a^+_i, a^-_i\} + Q - 1,$$
$$N_{-j} = \{a^+_{-j}, a^-_{-j}\} - Q. \qquad (51)$$

Therefore the 4-momentum operator P^n reads as

$$P^n = \sum_j p^n_j [\{a^+_j, a^-_j\} + \{a^+_{-j}, a^-_{-j}\} - 1] \qquad (52)$$

and the charge Q satisfies the identity

$$Q = \sum_j [\{a^+_j, a^-_j\} - \{a^+_{-j}, a^-_{-j}\} + 2Q - 1]. \qquad (53)$$

Putting together the results of ref. [7] and of the present investigation, we observe that the lowest Fock representations of the A-superstatistics of the tensor and spinor fields lead to the known restrictions of the B (=Bose and Fermi)-statistics. In addition, however, the A-superstatistics puts limitations on the possible charge of the ensemble to be 0 or 1.

The charge of the A-field should not necessary be associated with the particle-antiparticle charge. It could label different fields. For instance, if $\eta = +$ corresponds to A-electron field and $\eta = -$ to A-neutrino field (however $m_e = m_{\bar\nu}$), then the neutrino could appear only if it is accompanied by an electron.

The definition (18) of the a-operators is not the most general one. Within the same superalgebra one can define creation and annihilation operators in such a way that the statistics of the corresponding tensor field will allow no more than one particle in a given state ηi and, in addition, will put limitations on the total amount of particles in an ensemble [10]. The advantage of the realization of the A-superstatistics considered here is that it leads to a local current $J^n(x)$.

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