ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА



E2 - 11939



A.S.Pak, A.V.Tarasov, Ch.Tseren, V.V.Uzhinsky



# THE PHASE FUNCTIONS IN THE NUCLEUS-NUCLEUS SCATTERING AT HIGH ENERGIES

1978

E2 - 11939

A.S.Pak.[1] A.V.Tarasov. Ch.Tseren. V.V.Uzhinsky[2]

# THE PHASE FUNCTIONS IN THE NUCLEUS-NUCLEUS SCATTERING AT HIGH ENERGIES

*Submitted to ЯФ*



---

[1] Institute of High Energy Physics of the Academy of Sciences of the Kaz.SSR, Alma-Ata.

[2] Institute of Nuclear Physics of the Academy of Sciences of the Uz.SSR, Tashkent.

Пак А.С. и др. E2 - 11939

Фазовая функция в ядерно-ядерном рассеянии

Развит метод суммирования эйконального ряда как для амплитуд упругого $A_1A_2$-рассеяния, так и для важных физических характеристик некоторых неупругих процессов $A_1A_2$-взаимодействия. Получены замкнутые выражения для этих амплитуд, удобные для проведения численных расчетов. В качестве примера приводится расчет для фазы и поправки к ней.

Работа выполнена в Лаборатории ядерных проблем ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1978

Pak A.S. et al. E2 - 11939

The Phase Functions in the Nucleus-Nucleus Scattering at High Energies

The method is developed for the summation of eikonal series for amplitudes of elastic $A_1A_2$-scattering as well as for important characteristics of some inelastic processes in $A_1A_2$ interactions. The close expressions, convenient for numerical calculations, are derived for these amplitudes. As an example, we present a calculation of the phase function and corrections to it.

The investigation has been performed at the Laboratory of Nuclear Problems, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1978

## I. INTRODUCTION

Many papers have been devoted to the theoretical analysis of the nucleus-nucleus (below $A_1A_2$) elastic scattering processes in the eikonal approximation /1/ . The paper of I.V.Andreev /2/, where more or less completed and closed expressions have been obtained for the numerical calculation of the $A_1A_2$ elastic scattering amplitude, pleasantly differs from others. However inelastic $A_1A_2$ reactions have not so far been considered in the rigorous eikonal theory.

The aim of this paper and the following publications is to develop the eikonal expansion summation method, which allows one to obtain completed expressions both for the $A_1A_2$ elastic scattering amplitude and for the important physical characteristics of some $A_1A_2$ inelastic processes. In particular, the excitation processes of one and both incident nuclei, the elastic and the quasielastic charge exchange reactions and the quasielastic scattering are considered.

The Coulomb scattering problem in the $A_1A_2$ interaction is described. The main results of ref. /2/ as intermediate calculation are presented.

## II. RELATION BETWEEN THE AMPLITUDES OF THE NUCLEUS-NUCLEUS SCATTERING WITH CORRELATED AND UNCORRELATED NUCLEON DISTRIBUTIONS

We use well-known relation of the eikonal theory, which connects the $A_1A_2$ scattering amplitude with the nucleon-nucleon scattering one and the ground state wave functions $\Psi_{A_1}$, $\Psi_{A_2}$ of the colliding nuclei:

$$F_{A_1A_2}(q) = \frac{ip}{2\pi} \int d^2b \exp(iqb)\Gamma_{A_1A_2}(b)$$

$$\Gamma_{A_1A_2}(b) = 1 - << \prod_{i=1}^{A_1}\prod_{k=1}^{A_2}[1-\gamma(b-s_i-\tilde{s}_k)]>_{A_1}>_{A_2}$$

$$<<0(s_i, \tilde{s}_k)>_{A_1}>_{A_2} = \int \prod_{i=1}^{A_1} dr_i \, |\Psi_{A_1}(r_1,...,r_{A_1})|^2 *$$

$$* \int \prod_{k=1}^{A_2} d\tilde{r}_k \, |\Psi_{A_2}(\tilde{r}_1,...,\tilde{r}_{A_2})|^2 0(s_i, \tilde{s}_k) \qquad (1)$$

$$r_i = \{s_i, z_i\}, \quad \tilde{r}_k = \{\tilde{s}_k, \tilde{z}_k\}$$

$$\gamma(b) = \frac{1}{2\pi ip}\int f_{NN}(q)\, e^{-iqb}\, d^2q \quad .$$

We use the usual representation for the squared nucleus wave function $A_{1(2)}$ in the expansion form

$$|\Psi_{A_{1(2)}}(r_1,...,r_{A_{1(2)}})|^2 = \prod_{i=1}^{A_{1(2)}} \rho_{1(2)}(r_i) +$$

$$+ \sum_{i\neq k}^{A_{1(2)}} C^{(2)}_{1(2)}(r_i, r_k) \prod_{\ell\neq i,k}^{A_{1(2)}} \rho_{1(2)}(r_\ell) +$$

$$+ \sum_{i\neq k\neq \ell}^{A_{1(2)}} C^{(3)}_{1(2)}(r_i, r_k, r_\ell) \prod_{n\neq i,k,\ell}^{A_{1(2)}} \rho_{1(2)}(r_n) + ... \qquad (2)$$

where $\rho(r)$ is the so-called one-body density, $C^{(2)}_{1(2)}$, $C^{(3)}_{1(2)}$ and so on are the second, third and higher order correlation functions*.

-------------------

* The correlation, connected with the c.m. motion, is taken into account in a usual way /1/.



One introduces the thickness functions $T_1(s)$, $T_2(\tilde{s})$ by the usual relation

$$T_{1(2)}(s) = A_{1(2)} \int \rho_{1(2)}(s,z)\, dz \, . \qquad (3)$$

It is obvious that the profile function $\Gamma^{(0)}_{A_1A_2}(b)$ is the functional only of the thickness functions $T_1$, $T_2$ and the nucleon-nucleon profile function $\gamma$ in the uncorrelated nucleon distribution ($C_{1(2)}=0$) approximation for the colliding nuclei

$$\Gamma^{(0)}_{A_1A_2}(b) = \Gamma^{(0)}_{A_1A_2}\{T_1, T_2, \gamma\}.$$

By executing accurate averaging over the uncorrelated nucleon distribution of nuclei $A_1$ in relation (1) we obtain

$$\Gamma^{(0)}_{A_1A_2}\{T_1,T_2,\gamma\} = 1 - <[1-\frac{1}{A_1}\int\Gamma_{A_2}(b-s,\{\tilde{s}\})T_1(s)]^{A_1}>_{A_2} \quad , \qquad (4)$$

where

$$<0\{\tilde{s}\}>_{A_2} = \int \prod_{k=1}^{A_2}(\frac{T_2(\tilde{s}_k)}{A_2} d\tilde{s}_k)\, 0\{\tilde{s}\}$$

$$\Gamma_{A_2}(b-s,\{\tilde{s}\}) = 1 - \prod_{k=1}^{A_2}[1-\gamma(b-s-\tilde{s}_k)].$$

The consideration of the pair correlations in $A_1$ reduces to the correction term in the profile function

$$\Delta^{(2)}_1\Gamma_{A_1A_2} = -\frac{(A_1-1)}{2A_2}\int C^{(2)}_1(r_1, r_2)\, dr_1 dr_2 \, *$$

$$* <\Gamma(b-s_1, \{\tilde{s}\})\Gamma(b-s_2, \{\tilde{s}\})[1-\frac{1}{A_2}\int\Gamma_{A_2}(b-s,\{\tilde{s}\}) \, * \qquad (5)$$

$$* T_{A_2}(\{\tilde{s}_2\})]^{A_2-2}>_{A_2} \equiv -\frac{A_1^2}{2!}\int C^{(2)}_1(r_1,r_2)\frac{\delta^2\Gamma^{(0)}_{A_1A_2}\{T_1,T_2,\gamma\}}{\delta T(s_1)\,\delta T(s_2)} dr_1 dr_2 \, .$$

Similarly, the effect of the three-body correlation in the nucleus may be presented in the form

$$\Delta_1^{(3)}\Gamma_{A_1A_2} = \frac{A_1^3}{3!}\int C_1^{(3)}(r_1,r_2,r_3)\frac{\delta^3\Gamma^{(0)}_{A_1A_2}\{T_1,T_2,\gamma\}}{\delta T(s_1)\,\delta T(s_2)\,\delta T(s_3)}\,dr_1 dr_2 dr_3 \tag{6}$$

and so on.

Thus, if the concrete functional dependence $\Gamma_{A_1A_2}$ on $T_1$ and $T_2$ is known, the consideration of the nucleon correlation effects in the nuclei leads to a simple convolution operation of correlation functions together with the functional differentiation $\Gamma^{(0)}_{A_1A_2}$ over the thickness functions $T_{1,2}$.

Such dependence is determined in the optical limit by the atomic number of both nuclei $(A_1, A_2 \to \infty)$/2/. However, the representation of this dependence in the double series form is not quite convenient for the practical employment. Below the closed formula for this series summation is obtained.

## *III. TRANSITION TO THE OPTICAL LIMIT AND THE CORRECTION TO IT*

So, the task to find $\Gamma^{(0)}(T_1,T_2,\gamma)$ comes to the averaging of the binomial function $(1+X/A)^A$. It is known that the average value of any function $\langle\phi(X)\rangle$ can be expressed through the derivatives of this function $\phi^n(0)$ and the $X_n$-central moments of the variable $X$

$$X_1 = \langle X\rangle$$

$$X_n = \langle (X-\langle X\rangle)^n\rangle \qquad n \geq 2 .$$

One of the simplest expressions is that for the average value of $\exp(X)$

$$\langle\exp(X)\rangle = \exp\{X_1 + \frac{1}{2!}X_2 + \frac{1}{3!}X_3 + \frac{1}{4!}(X_4 - 3X_2^2) + \frac{1}{5!}(X_5 - 5X_2X_3) + \ldots\}. \tag{7}$$

Since at $A \gg 1$

$$(1+\frac{X}{A})^A = \exp(X) + 0(\frac{1}{A})$$

the optical limit may be considered as the reasonable approximation in the case of heavy and intermediate nuclei and the atomic number must be taken into account as the correction of order $\frac{1}{A_{1(2)}}$, i.e.,

$$(1+\frac{X}{A})^A = (1 - \frac{X^2}{2A} + \frac{X^3}{3A^2} - \ldots)\exp(X). \tag{8}$$

Assuming that $X = -\int\Gamma(b-s,\{s\})\,T_1(s)\,ds$, we obtain

$$\langle X^n\exp(X)\rangle = \int T_1(s_1)\ldots T_1(s_n)ds_1\ldots ds_n \frac{\delta^n}{\delta T_1(s_1)\ldots\delta T_1(s_n)}\langle\exp(X)\rangle \tag{9}$$

Thus, the calculation of correction terms of any order leads, essentially, to the functional $\Gamma^{(0)}_{A_2A_1}\{T_1,T_2\}$

$$\Gamma^{(0)}_{A_2A_1(opt)}\{T_1,T_2\} = \lim_{A_2\to\infty}[\langle 1-\exp(-\int\Gamma_{A_2}(b-s,\{\tilde{s}\})\,T_1(b-s)ds)\rangle_{A_2}] = 1-\exp\chi_{opt}\{T_1,T_2\}. \tag{10}$$

The integration variable replacement $s \to b-s$ is produced in eq. (10).

Using the averaging rule of exponent (7) we obtain for the successive terms of expansion of the functional $\chi(T_1,T_2)$ over the degree $T_{1(2)}$ the following expressions

$$i\chi(T_1,T_2) = \sum_{n=1}^{\infty}\frac{1}{n!}\chi_n \tag{11a}$$

$$\chi_1 = \int d^2s\, T_1(b-s)(E_1(s)-1) \tag{11b}$$

$$\chi_2 = \int d^2s_1 d^2s_2 T_1(b-s_1)T_2(b-s_2)[E_2(s_1,s_2) - E_1(s_1)E_1(s_2)]. \tag{11c}$$

$$\chi_3 = \int d^2s_1 d^2s_2 d^2s_3 T_1(b-s_1)T_1(b-s_2)T_2(b-s_3) * [E_3(s_1,s_2,s_3) - E_2(s_1,s_2)E_1(s_3) - E_2(s_1,s_3)E_1(s_2) - E_2(s_2,s_3)E_1(s_1) + 2E_1(s_1)E_1(s_2)E_1(s_3)] \tag{11d}$$

and so on, where

$$E_n(s_1,\dots,s_n)=\exp\left(\int d\tilde{s}\, T_2(\tilde{s})\left\{\prod_{i=1}^{n}[1-\gamma(s_i-\tilde{s})]-1\right\}\right).$$

Since the values $\chi_n$ are formed of the combinations of expressions of type $\Gamma^{(0)}_{A_1A_2}$ $(A_1 \to \infty, A_2=1,2,..n)$, the problem seems to be reduced to the initial. However, this is not so.

The structure of those combinations is such that integrands in (11b), (11d), as functions of relative variables $s_i - s_\ell$, $s_i-\tilde{s}$, substantially differ from zero only in the limits of nucleon-nucleon interaction radius $r_0$. It is not difficult to be convinced of it expanding the values $E_k(s_1,\dots,s_k)$ in series by magnitudes of type $\int d\tilde{s}\, T_2(\tilde{s}) \prod_{i=1}^{k}\gamma(\tilde{s}-s_i)$ $k \geq 2$ and considering the $\delta$-like behaviour of the profile function $\gamma$ in comparison with smooth change of thickness functions $T_1$ and $T_2$, if $r_0^2/R^2(A_{1(2)}) \ll 1$ ( $R$ is the radius of nuclei), that is practically always satisfied. Taking into account the normalization

$$\int \gamma(s)ds = \frac{\tilde{\sigma}}{2}, \quad \tilde{\sigma} = \sigma^{tot}_{NN}\left(1-i\,\frac{\mathrm{Re}\,f(0)}{\mathrm{Im}\,f(0)}\right)$$

we write $\gamma(s)$ in the form

$$\gamma(s)=\frac{\tilde{\sigma}}{2}\tilde{\delta}(s)$$

$$\tilde{\delta}(s) = \frac{1}{\pi r_0^2}\Phi\left(\frac{s^2}{n^2}\right) \qquad \int_0^\infty \Phi(X)dX = 1 \tag{12}$$

$\Phi(X) \ll 1$ at $X \gg 1$.

The concrete form of $\Phi(X)$ is determined by the q-dependence of NN scattering amplitude $f(q)$. The $\tilde{\delta}(s)$ -function at $r_0^2/R^2 \to 0$ must be considered as the $\delta(s)$ -function in the integration using smoothly variable functions $T_1$, $T_2$. So, for example

$$\int \gamma(s-\tilde{s})\, T_2(\tilde{s})d\tilde{s} = \frac{\tilde{\sigma}}{2}T_2(s).$$



The same concerns the integration of the products of $\delta$-function, arguments of which are not overlapped. When integrating the products of $\tilde{\delta}(s)$ functions with the overlapping values of the arguments, as for example, in the simplest case

$$I_2 = \int T_1(b-s_1)T_1(b-s_2)T_2(\tilde{s}_1)T_2(\tilde{s}_2)\tilde{\delta}(s_1-\tilde{s}_1)*$$
$$*\tilde{\delta}(s_1-\tilde{s}_2)\delta(s_2-\tilde{s}_1)\delta(s_2-\tilde{s}_2)ds_1 d\tilde{s}_2 d\tilde{s}_2 d\tilde{s}_1$$

$$I_3 = \int T_1(b-s_1)T_1(b-s_2)T(b-s_3)T_2(\tilde{s}_1)T_2(\tilde{s}_2)T_2(\tilde{s}_3)\tilde{\delta}(s_1-\tilde{s}_1)*$$
$$*\tilde{\delta}(s_1-\tilde{s}_2)\tilde{\delta}(s_2-\tilde{s}_3)\tilde{\delta}(s_2-\tilde{s}_2)\tilde{\delta}(s_3-\tilde{s}_3)\tilde{\delta}(s_3-\tilde{s}_1)ds_1 ds_2*$$
$$*ds_3 d\tilde{s}_1 d\tilde{s}_2 d\tilde{s}_3 \quad ,$$

in order to obtain final results one must use the concrete form of the function $\Phi(X)$. It is obvious that only with the accuracy of the term of the order of $r_0^2/R^2_{1,2}$

$$I_2 = \frac{N_2}{\pi r_0^2}\int [T_1(b-s)T_2(s)]^2 d^2s,$$

$$I_3 = \frac{N_3}{\pi r_0^2}\int [T_1(b-s)T_2(s)]^3 d^2s\,. \tag{14}$$

The exact values of the coefficients $N_2$, $N_3$, as well as the relation between them, depend on the form of the function $\Phi(X)$ or on the differential cross-section form for NN scattering. Using the above-mentioned properties of expressions (11b), (11d) and their structure and accounting for the dimension considerations we obtain

$$\chi_1(b) = \frac{2}{\tilde{\sigma}}\int d^2s\, x\,[\exp(-y)-1]$$

$$\chi_n(b) = \frac{2}{\tilde{\sigma}}\int d^2s[x\exp(-y)]^2\phi_n(y,r_0) \qquad n \geq 2, \tag{15}$$

and so on, where

$$E_n(s_1,\dots,s_n)=\exp(\int d\tilde{s}\, T_2(\tilde{s})\{\prod_{i=1}^{n}[1-\gamma(s_i-\tilde{s})]-1\}).$$

Since the values $\chi_n$ are formed of the combinations of expressions of type $\Gamma^{(0)}_{A_1A_2}(A_1\to\infty, A_2=1,2,..n)$, the problem seems to be reduced to the initial. However, this is not so.

The structure of those combinations is such that integrands in (11b), (11d), as functions of relative variables $s_i-s_\ell$, $s_i-\tilde{s}$, substantially differ from zero only in the limits of nucleon-nucleon interaction radius $r_0$. It is not difficult to be convinced of it expanding the values $E_k(s_1,\dots,s_k)$ in series by magnitudes of type $\int d\tilde{s}\, T_2(\tilde{s})\prod_{i=1}^{k}\gamma(\tilde{s}-s_i)$ $k\geq 2$ and considering the $\delta$-like behaviour of the profile function $\gamma$ in comparison with smooth change of thickness functions $T_1$ and $T_2$, if $r_0^2/R^2(A_{1(2)})\ll 1$ ( R is the radius of nuclei), that is practically always satisfied. Taking into account the normalization

$$\int\gamma(s)ds=\frac{\tilde{\sigma}}{2},\qquad \tilde{\sigma}=\sigma^{tot}_{NN}(1-i\,\frac{\mathrm{Re}\,f(0)}{\mathrm{Im}\,f(0)})$$

we write $\gamma(s)$ in the form

$$\gamma(s)=\frac{\tilde{\sigma}}{2}\tilde{\delta}(s)$$

$$\tilde{\delta}(s)=\frac{1}{\pi r_0^2}\Phi(\frac{s^2}{n^2})\qquad \int_0^\infty\Phi(X)dX=1 \tag{12}$$

$\Phi(X)\ll 1$ at $X\gg 1$.

The concrete form of $\Phi(X)$ is determined by the q-dependence of NN scattering amplitude $f(q)$. The $\tilde{\delta}(s)$ -function at $r_0^2/R^2\to 0$ must be considered as the $\delta(s)$ -function in the integration using smoothly variable functions $T_1$, $T_2$. So, for example

$$\int\gamma(s-\tilde{s})\,T_2(\tilde{s})d\tilde{s}=\frac{\tilde{\sigma}}{2}T_2(s).$$



The same concerns the integration of the products of $\delta$ -function, arguments of which are not overlapped. When integrating the products of $\tilde{\delta}(s)$ functions with the overlapping values of the arguments, as for example, in the simplest case

$$I_2=\int T_1(b-s_1)T_1(b-s_2)T_2(\tilde{s}_1)T_2(\tilde{s}_2)\tilde{\delta}(s_1-\tilde{s}_1)*$$
$$*\tilde{\delta}(s_1-\tilde{s}_2)\delta(s_2-\tilde{s}_1)\delta(s_2-\tilde{s}_2)ds_1d\tilde{s}_2d\tilde{s}_2d\tilde{s}_1$$

$$I_3=\int T_1(b-s_1)T_1(b-s_2)T(b-s_3)T_2(\tilde{s}_1)T_2(\tilde{s}_2)T_2(\tilde{s}_3)\tilde{\delta}(s_1-\tilde{s}_1)*$$
$$*\tilde{\delta}(s_1-\tilde{s}_2)\tilde{\delta}(s_2-\tilde{s}_3)\tilde{\delta}(s_2-\tilde{s}_2)\tilde{\delta}(s_3-\tilde{s}_3)\tilde{\delta}(s_3-\tilde{s}_1)ds_1ds_2*$$
$$*ds_3d\tilde{s}_1\,d\tilde{s}_2\,d\tilde{s}_3\quad,$$

in order to obtain final results one must use the concrete form of the function $\Phi(X)$. It is obvious that only with the accuracy of the term of the order of $r_0^2/R^2_{1,2}$

$$I_2=\frac{N_2}{\pi r_0^2}\int[T_1(b-s)T_2(s)]^2d^2s,$$

$$I_3=\frac{N_3}{\pi r_0^2}\int[T_1(b-s)T_2(s)]^3\,d^2s\;. \tag{14}$$

The exact values of the coefficients $N_2$, $N_3$, as well as the relation between them, depend on the form of the function $\Phi(X)$ or on the differential cross-section form for NN scattering. Using the above-mentioned properties of expressions (11b), (11d) and their structure and accounting for the dimension considerations we obtain

$$\chi_1(b)=\frac{2}{\tilde{\sigma}}\int d^2s\,x\,[\exp(-y)-1]$$

$$\chi_n(b)=\frac{2}{\tilde{\sigma}}\int d^2s[x\,\exp(-y)]^2\phi_n(y,r_0)\qquad n\geq 2, \tag{15}$$

where $x$ , $y$ , $r_0$ denote the dimensionless combination of the values

$$x = \frac{\tilde{\sigma}}{2} T_1(b-s), \quad y = \frac{\tilde{\sigma}}{2} T_2(s), \quad r_0 = \frac{\tilde{\sigma}}{4\pi r_0^2}. \tag{16}$$

The concrete form of $\phi_n$ is different for various parametrization $\Phi(X)$ because of the above-discussed dependence of coefficients $N_2$ and $N_3$ on the form of $\Phi(X)$.

Taking into account that the hadron-nucleus scattering amplitude depends only on the forward nucleon-nucleon scattering amplitude ($\gamma(b) = \frac{\tilde{\sigma}}{2}\delta(b)$) at $r_0^2/R^2(A_{1,2}) \to 0$, it is improbable that the appearing in the nucleus-nucleus scattering phase dependence on the NN amplitude behaviour at the different from zero transverse momentum affects essentially the $A_1A_2$ scattering characteristics.

This means that the dominant contribution to $\chi_n$ must introduce "the model-independent" part determined by $\phi_n(y,\gamma_0)|_{\gamma_0=0}$. We expand the function $\phi_n$ into a series by the parameter $\gamma_0$

$$\phi_n(y,\gamma_0) = \sum_{k=0}^{\infty} \phi_n^{(k)}(y)\gamma_0^{(k)} \tag{17}$$

and consider first of all, "model-independent" part.

It is not difficult to test that values $\phi_n^{(k)}(y)$ $(n \geq 2)$ are the polynomial of the degree $n+k-1$ by the variable $y$. In particular

$$\phi_n^{(0)} = \sum_{\ell=1}^{n-1} a_{n\ell}^{(0)} y^{\ell} \qquad n \geq 2 . \tag{18}$$

It is easy to obtain the concrete expressions for coefficients $a_{n\ell}^{(0)}$ by using direct calculation. However, it is not necessary to make, since the symmetry of the task relatively to the rearrangement $A_1 \leftrightarrow A_2$ , the structure of expression (18) and the condition $a_{\ell n}^{(0)} = 0$ at $\ell \geq n$ in (18) uniquely determine all coefficients $a_{\ell n}^{(0)}$ . Indeed, according to (15),representing $\chi(b)$ in the form

$$\chi(b) = \frac{2}{\tilde{\sigma}} \int f(x,y)\, d^2 s$$

$$f(x,y) = x(e^{-y} - 1) + \sum_{n=2}^{\infty} \frac{1}{n!} (xe^{-y})^n \phi_n(y) \tag{19}$$



in consequence of the symmetry ($A_1 \leftrightarrow A_2$) we have

$$f(x,y) = f(y,x)\ldots \tag{20}$$

We present the function $f(x,y)$ in the double series expansion:

$$f(x,y) = \sum_{m,n=1}^{\infty} g_{mn} x^n y^m , \tag{21}$$

where, according to (20),

$$g_{mn} = g_{nm} , \quad g_{mn} = g_{mn}(\gamma_0) \tag{22}$$

Then, taking into account (21) and (22) for the zeroth coefficients over $\gamma_0$ approximation, one obtains

$$g^{\circ}_{1m} = \frac{(-1)^m}{m!} ,$$

$$g^{\circ}_{nm} = \sum a^{\circ}_{n\ell} \frac{(-n)^{m-\ell}}{(m-\ell)!} . \tag{23}$$

Taking into consideration (22), we obtain successively

$$a^{(0)}_{21} = 2g^{\circ}_{21} = 2g^{\circ}_{12} = 1$$

$$a^{\circ}_{32} = 2g^{\circ}_{32} = g^{\circ}_{23} = 1 \tag{24}$$

. . . . . . . . . . . . . . .

. . . . . . . . . . . . . . .

and so on. Hence, it follows

$$g_{2m} = (-1)^m \frac{2^{m-1}\, m}{m!\,2!} ,$$

$$g_{3m} = (-1)^{m+3} \frac{3^{m-1}\, m^2}{3!\, m!} , \tag{25}$$

. . . . . . . . . . . . . . .

$$g_{nm} = (-1)^{m+n} \frac{n^{m-1} m^{n-1}}{m!\,n!} .$$

This is the result obtained by Andreev in the framework of the generating function method which allows one to calculate the phase function $\chi(b)$ as the double series of the convolutions

$$\frac{2}{\tilde{\sigma}} \int \left(\frac{\tilde{\sigma}}{2} T_1(b-s)\right)^m \left(\frac{\tilde{\sigma}}{2} T_2(s)\right)^n d^2 s .$$

Since the values $(\tilde{\sigma}/2\ T_{1,2})$ are not small for a majority of nuclei and the coefficients $g_{mn}$ at large $m$ and $n$ $(m \to n)$ do not tend to zero, it is questionable to limit oneself to several lower terms of expansion (21) at the calculations of nucleus-nucleus scattering characteristics. Therefore, it is important to find the closed expression for the series sum.

## *IV. SERIES SUMMATION*

Assume that $x$ and $y$ are real values. This corresponds to the neglection of the NN scattering amplitude real part in comparison with the imaginary one. We write the values $m^{n-1}$ and $n^{m-1}$ formally so that

$$m^{n-1} = \int \delta^{(n-1)}(z) e^{-mz}\, dz, \quad n^{m-1} = \int \delta^{(m-1)}(u)\, e^{-nu}\, du \tag{26}$$

and substitute them to the series

$$f^{\circ}(x,y) = \sum_{m,n=1}^{\infty} (-1)^{m+n} \frac{x^m y^n m^{n-1} n^{m-1}}{m!\, n!} . \tag{27}$$

Taking into account, again purely formally,

$$\sum_{m=1}^{\infty} (-x e^{-u}) \frac{\delta^m(z)}{m!} = \delta(z - x e^{-u})$$

$$\begin{aligned} \sum_{m=1}^{\infty} (-x e^{-u}) \frac{\delta^{m-1}(z)}{m!} &= \int_0^x \delta(z - x e^{-u})\, dx' = \\ &= \theta(z - x e^{-u}) - \theta(z) \end{aligned} \tag{28}$$

$$\sum_{n=1}^{\infty} \frac{(-y e^{-z})^n \delta^{(n-1)}(u)}{n!} = \theta(u - y e^{-z}) - \theta(u)$$



we obtain for $f^{\circ}(x,y)$ the expression in the double integral form

$$f(x,y) = \int dz\, du\, [\theta(z - x e^{-u}) - \theta(z)][\theta(u - y e^{-z}) - \theta(u)].$$

This integral is expressed through solutions $u$ and $z$ of the system of equations

$$\begin{cases} z = x e^{-u} \\ u = y e^{-z} \end{cases} \tag{29}$$

so that

$$f(x,y) = x + y + \sum_{\rho=1}^{k} (-1)^{\rho} (u_\rho + z_\rho + u_\rho z_\rho)$$

if their numeration is taken in accordance with conditions

$$z_1 > z_2 > \ldots \quad \text{and} \quad u_1 < u_2 < u_3 \ \ldots$$

The analysis shows that the maximal possible number of real roots $u_\rho, z_\rho$ (for real $x$ and $y$) of the system of equations (29) is equal to 3 $(k = 3)$. However, this possibility is realized only at the $(x,y) > e$. If one of the values of $x$ (or $y$) is smaller than $e$, the system of equations has a single solution, which we denote by $u$, $z$. In this case

$$f^{\circ}(x,y) = x + y - u - z - uz = z(e^u - 1) + u(e^z - 1) - uz. \tag{30}$$

At $\sigma_{tot}^{NN} = 40$ *mb* and standard Fermi distribution parameters for the one-particle density

$$\rho = \rho_0 (1 + e^{\frac{r-c}{a}})^{-1}$$

with $c = 1.12 A^{1/3}$ and $a = 0.545$ the condition $x$ (or $y$) $< e$ is satisfied, if the atomic number of one of the colliding nuclei is smaller than 64, which covers sufficiently wide class of reactions.

Since the possibility of using formal relation (26) for the series summation of the generalized functions may be a subject to doubt of it is necessary an independent test of the accordance of analytical expression (30) for $f^{\circ}(x,y)$ and its form in series (27).

Performing accurate summation over m in (27), on the one hand, and expanding (30) in series by the degree, on the other hand, it is easy to infer that both expressions (27) and (30) may be presented in the form

$$f(x,y)=\sum_{n=1}^{\infty}\frac{x^n}{n!}\,\frac{1}{n}\,\left(y\frac{d}{dy}\right)^{n-1}(e^{-ny}-1)$$

and equal each other.

So, the problem of the double series summation is reduced, essentially, to the solution of the system of equations

$$z=xe^{-u}\,, \tag{31a}$$

$$u=ye^{-z} \tag{31b}$$

for two variables $u, z$ or of one equation $\ln\frac{z}{x}+ye^{-z}=0$

for the variable $z$ with the subsequent determination $u$ through (31) which is very simply performed by computer calculations.

## V. CORRECTION OF THE ORDER OF $\gamma_0$

We consider now the question about the "model-dependent" component of phase function $\chi(b)$.

The direct calculation [2] of the corrections of the order $\gamma$ with $k\leq 4$ to the coefficients $g_{mn}$ $m+n<9$ in the model with Gaussian parametrization of the NN scattering amplitude

$$\gamma(b)=\frac{\tilde{\sigma}}{4\pi r_0^2}\exp\left(-\frac{b^2}{2r_0^2}\right),\quad \gamma(0)\approx\gamma_0$$

shows that these corrections are small for the lower coefficients, but they increase with $(n+m)$. So, for example,

$$\text{if}\quad g_{22}^{(1)}/g_{22}^{(0)}=-0.063\,\gamma_0\,,\qquad\qquad \text{then}\quad g_{45}^{(1)}/g_{45}^{(0)}=-0.621\,\gamma_0\,, \tag{32}$$



i.e., for the larger coefficients $g_{mn}$ the corrections turn out to be comparable with the main effect. However, taking into account that in the calculation of $\chi^{(0)}(b)$ the contributions of the higher terms to the alternating series (27) are mutually cancelled, it is impossible to draw the definite conclusion about the quantities of the order $\gamma_0$ on the basis of (32).

It is desirable to obtain the closed expression similar to (30) at least for the contribution of the order $\gamma_0$ to $\chi(b)$ and to compare them with the contribution of the zeroth approximation $\chi^{(0)}(b)$.

Since values $\phi_n^{(1)}(y)$ are the polynomials of degree $n$ (see (18), the structure of the expression for $\chi_n$ in (15) and condition (20) do not define unambiguously all the coefficients $a_{n\ell}^{(1)}$, but allow one to express them through the magnitudes $b_n=\gamma_0 a_{nn}^{(1)}$. This allows one to connect the coefficients $g_{mn}^{(1)}$ with the quantities $b_n$

$$g_{mn}^{(1)}=(-1)^{m+n}\sum_{\ell=2}^{\infty}c_\ell\,\frac{m^{n-\ell}\,n^{m-\ell}}{(n-\ell)!\,(m-\ell)!}$$

$$c_\ell=b_\ell-\sum_{k=2}^{\ell-1}\frac{e^{\ell-k}}{(\ell-k)!}\,c_k\,,\qquad c_2=b_2$$

Employing the same formal approach of the double series summation as above it is not difficult to obtain

$$\sum_{m,n=\ell}^{\infty}(-1)^{m+n}\,\frac{x^m y^n\, m^{n-\ell}\,n^{m-\ell}}{(n-\ell)!\,(m-\ell)!}=\frac{(uz)^\ell}{1-uz}$$

and finally

$$f^{(1)}(x,y)=\sum g_{mn}^{(1)}x^n y^m=\sum_{\ell=2}^{\infty}d_\ell\,(uz)^\ell\,,\quad d_\ell=\sum_{k=2}^{\ell}c_k\,. \tag{33}$$

The direct and somewhat cumbersome calculation is reduced to the result

$$d_\ell=-\frac{1}{2\ell}\,\frac{\sigma}{4\pi^2}\int\left[f_{NN}(q)/f_{NN}(0)\right]^{2\ell}d^2q$$

from which one determines "$f(q)$-dependence" of the main correction term to the phase $\chi^{(0)}(b)$.

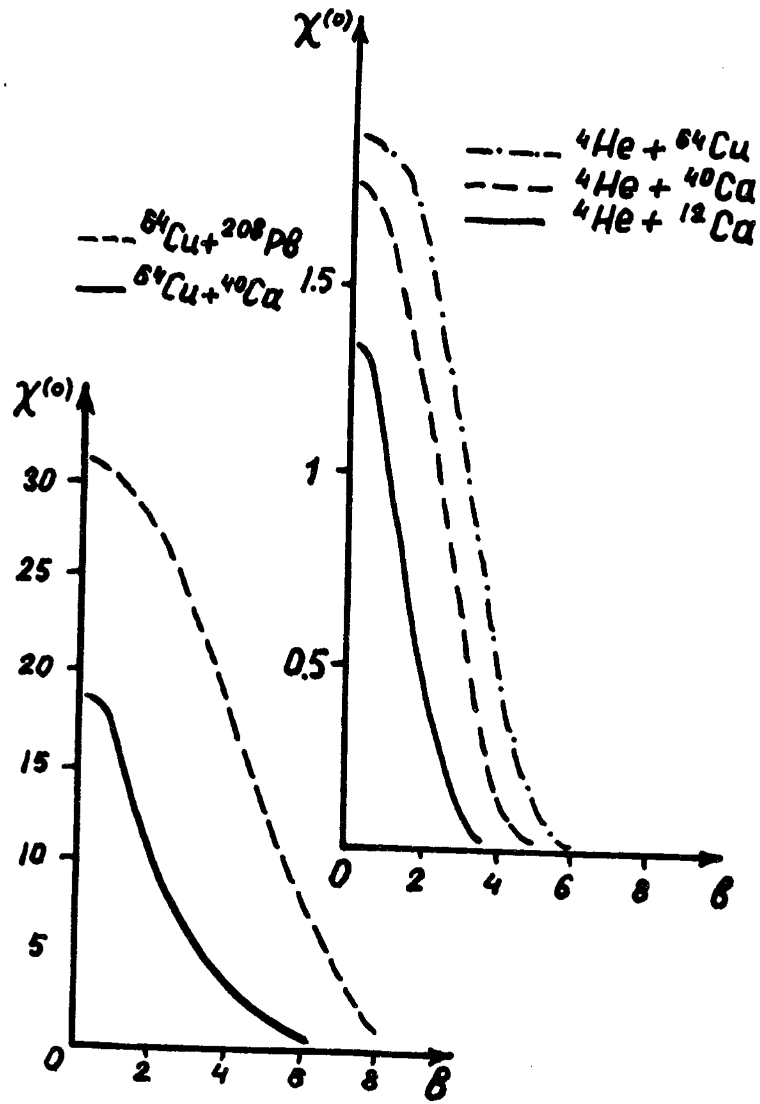


*Fig. 1*



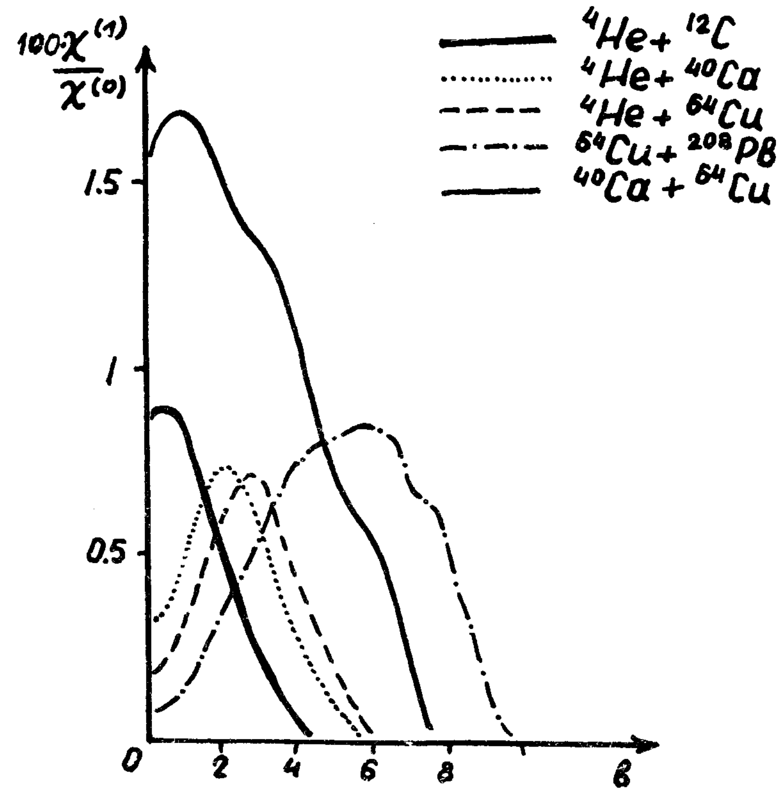


*Fig. 2*

Assuming for the estimate

$$f(q)/f(0) = e^{-\frac{r_0^2 q^2}{2}} , \tag{34}$$

we obtain

$$d_\ell = \gamma_0 / 4\ell^2 ,$$

$$f^{(1)}(x,y) = -\frac{\gamma_0}{4}\left[ uz + \int_0^1 \frac{dt}{t} \ln(1 - uzt) \right].$$

The results of the numerical calculations are shown in *Figs. 1,2*.

The values of $\chi^{(0)}(b)$ are given in *fig. 1* and $[\chi^{(1)}(b)/\chi^{(0)}(b)]\cdot 100$ is plotted in *fig. 2*. It is obvious that the value $\chi^{(1)}(b)$ indeed is the correction to $\chi(b)$ and this correction is so small that its consideration could exceed the precision, due to infiniteness in the initial magnitudes, in particular, in the nuclear density parameters. Thus, we would not consider the correction of the higher order of $\gamma_0$.

The authors are grateful to Prof. L.I.Lapidus and I.V.Andreev for useful discussions and interest to the present work.

*REFERENCES*

1. *Fald Pilkuhn Ann. of Phys., 1970, 58, No. 2. Czyz W., Maximon L.C. Ann. of Phys., 1969, 52, No. 1; Nucl.Phys., 1977, A278, No. 3. Alkhazov G.D. et al. Nucl.Phys., 1977, A280, No. 2. Chaumeaux A., et al. Nucl.Phys., 1976, A267, No. 3. Franco C., Varma G.K. Phys. Rev., 1977, C15, No. 4.*
2. *Andreev J.V., P.N.Lebedev Phys. Inst., Preprint No. 92, M., 1976.*

*Received by Publishing Department on October 6 1978.*