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LIE-SUPERALGEBRAICAL ASPECTS OF QUANTUM STATISTICS

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Квантовая статистика в аспектах суперaлгебры Ли

Обсуждаются свойства ортодоксальной квантовой статистики с использованием метода суперaлгебр Ли с целью возможных обобщений в квантовой теории и в теоретической физике вообще. Показано, что алгебра, порожденная двойками ферми-или парaферми-операторов изоморфна классической алгебре Ли B_n ортогональной группы $SO(2n+1)$, в то время как n двоек бозе- или парабозе-операторов порождают простую ортосимплектическую суперaлгебру $B(0,n)$. Переход к бесконечному числу операторов рождения и уничтожения ($n\to\infty$) не меняет суперaлгебраическую структуру. Таким образом, обычное бозе- и ферми-квантование можно рассматривать как квантование по определенным неприводимым представлениям двух простых суперaлгебр Ли. Дана идея о том, как можно определить операторы рождения и уничтожения, которые удовлетворяют постулатам вторичного квантования и порождают некоторые другие классические простые суперaлгебры Ли.

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Lie-Superalgebraical Aspects of Quantum Statistics

The Lie-superalgebraical properties of the ordinary quantum statistics are discussed. It is indicated that the algebra, generated by n pairs of Fermi operators, is isomorphic to the classical simple Lie algebra B_n, whereas n pairs of Bose operators generate the simple Lie superalgebra $B(0,n)$. The idea of how one can introduce creation and annihilation operators that satisfy the second quantization postulates and generate other simple Lie superalgebras is given.

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In the present paper we emphasize certain Lie-superalgebraical properties of the ordinary quantum statistics which seem to present a natural background for a search of possible new quantum statistics. The main observation is that the Bose quantization appears as quantization according to a certain representation of the orthosymplectic Lie superalgebra, whereas the Fermi quantization is closely connected to the odd-orthogonal Lie algebra. Both the Lie superalgebras are simple. The creation and annihilation operators are root vectors generating the whole algebra.

Nowadays all simple Lie superalgebras are fully classified /1/. In view of the above mentioned properties of the Bose and Fermi operators it seems natural to ask whether one can satisfy the second quantization axioms, say, in the Lagrangian quantum field theory with new kinds of creation and annihilation operators the main feature of which is that they are root vectors generating some of the others simple Lie superalgebras. This question has been studied in detail in ref. /2/ only from a Lie-algebraical point of view. There we have shown that to every classical simple Lie algebra there corresponds a quantization that is logically compatible with the main quantization principles. In this paper we shall give an example of creation and annihilation operators that generate a proper Lie superalgebra.

We first recall a few basic definitions (for more details see, for instance ref. /1/). Demote by Z_2 the ring with two elements (0,1) and multiplication

$$0+0=0,\quad 0+1=1,\quad 1+1=0. \tag{1}$$

The algebra G with multiplication denoted as $[\![\ ,\ ]\!]$ is said to be a Lie superalgebra if it satisfies the following axioms:

a) G is Z_2-graded algebra, i.e., the linear space G can be represented as a direct sum of two subspaces,

$$G = G_0 + G_1 \tag{2}$$

so that if $a \in G_\alpha$, $b \in G_\beta$ then

$$[\![a,b]\!] \in G_{\alpha+\beta}\ , \qquad \alpha, \beta \in Z_2\ ; \tag{3}$$

b)

$$[\![a,b]\!] = -(-1)^{\alpha\beta}[\![b,a]\!] \quad \text{for } a \in G_\alpha,\ b \in G_\beta\ ; \tag{4}$$

c)

$$[\![a,[\![b,c]\!]]\!] = [\![[\![a,b]\!]c]\!] + (-1)^{\alpha\beta}[\![b,[\![a,c]\!]]\!],\ a \in G_\alpha,\ b \in G_\beta \tag{5}$$

The elements from G_0 and G_1 are called even and odd elements, respectively.

The algebra $A = A_0 + A_1$ is said to be a linear Lie superalgebra if its elements are linear operators and A is a Lie superalgebra with respect to the multiplication

$$[\![a,b]\!] = ab - (-1)^{\alpha\beta} ba \qquad a \in A_\alpha\ ,\ b \in A_\beta\ . \tag{6}$$

A representation ρ of the Lie superalgebra G is a linear map of G into a linear Lie superalgebra so that the multiplication is preserved,

$$\rho[\![a,b]\!] = [\![\rho(a),\rho(b)]\!], \qquad a,b \in G. \tag{7}$$

The concept of a subalgebra or ideal is defined in a natural way. The Lie superalgebra G is simple if it has no nontrivial ideals. The simple Lie superalgebras are known /1/. Since in our case the rank of the algebra is proportional to the number of the creation and annihilation operators, we list only those simple Lie superalgebras that can have an arbitrarily big rank. First of all, there are four well known classes of classical simple Lie algebras,

$$A_n, B_n, C_n, D_n \qquad n = 1,2,3,\ldots \tag{8}$$

Moreover there exist six series of classical Lie superalgebras that are not Lie algebras,

$$A(m,n), B(m,n), C(n), D(m,n), P(n), Q(n),\quad m,n = 1,2,\ldots \tag{9}$$

We now proceed to study the Lie algebraical properties of the ordinary quantum statistics.

FERMI STATISTICS

Let $f_1^\pm, f_2^\pm, \ldots, f_n^\pm$ be Fermi creation and annihilation operators, i.e., operators that fulfil the anticommutation relations *

$$\{f^{\xi}, f^{\eta}\} = \frac{1}{4}(\xi-\eta)^2 \delta_{ij}. \qquad (10)$$

We ask the question what is the Lie algebra these operators generate. Since the Fermi operators can be represented as finite matrices, it is clear that such an algebra exists and it is finite-dimensional. From the identity

$$[AB, C] = A\{B, C\} - \{A, C\}B$$

and the defining relations (10) one obtains

$$[[f_i^{\xi}, f_j^{\eta}], f_k^{\epsilon}] = \frac{1}{2}(\eta-\epsilon)^2 \delta_{jk} f_i^{\xi} - \frac{1}{2}(\xi-\epsilon)^2 \delta_{ik} f_j^{\eta}, \qquad (11)$$

$$i, j, k = 1, .., n.$$

Therefore the set of all operators

$$f_i^{\xi}, [f_j^{\eta}, f_k^{\epsilon}] \qquad i, j, k = 1, \dots, n \qquad (12)$$

is closed under arbitrary commutations and hence the linear envelope of the operators (12) is a Lie algebra we are looking for. A more detailed analysis shows that this is the algebra B_n of the odd-orthogonal group $SO(2n+1)$ and more exactly the representation with signature $(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2})$.

In quantum field theory the set of the Fermi operators is infinite. The Lie-algebraical structure is however preserved. Therefore one can view the Fermi quantization as a quantization according to a certain, the so-called spinor representation of the algebra of the (infinite parameter) orthogonal group.

The first question that naturally arises is why this particular representation is relevant for the quantization. Is it impossible to quantize according to some other representations of the same algebra? The answer is positive. It turns out the quantization by means of other representations leads to the paraFermi quantization. To show this we recall/3/ that the paraFermi operators $b_1^{\pm}, \dots, b_n^{\pm}$ satisfy by definition the same system of three-linear commutation relations (11), i.e.,

$$[[b_i^{\xi}, b_j^{\eta}], b_k^{\epsilon}] = \frac{1}{2}(\eta-\epsilon)^2 \delta_{jk} b_i^{\xi} - \frac{1}{2}(\xi-\epsilon)^2 \delta_{ik} b_j^{\eta}. \qquad (13)$$

Hence the linear envelope of the operators

$$b_i^{\xi}, [b_j^{\eta}, b_k^{\epsilon}] \qquad i, j, k = 1, \dots, n \qquad (14)$$

is isomorphic to the algebra B_n. The representations of the paraFermi operators in a space with single vacuum are labelled by one positive integer, the order of the statistics. It has well defined Lie-algebraical meaning. The corresponding representation of B_n is finite-dimensional irreducible representation with signature in an orthogonal basis of the Cartan subalgebra $(\frac{p}{2}, \frac{p}{2}, \dots, \frac{p}{2})$.

So we see that the Fermi quantization and its generalization, the paraFermi one, is a quantization according to a certain class of representation of the simple Lie algebra of the orthogonal group.

* Throughout the paper $\xi, \eta, \epsilon, \delta = \pm$ or ± 1; $[x, y] = xy - yx$ and $\{x, y\} = xy + yx$.

BOSE STATISTICS

Consider now n pairs $\beta_1^{\pm}, \dots, \beta_n^{\pm}$ of Bose creation and annihilation operators, i.e., operators defined with the commutation relations

$$[\beta_i^{\xi}, \beta_j^{\eta}] = \frac{1}{2}(\eta - \xi)\delta_{ij} . \qquad (15)$$

Clearly the space spanned on these operators is a Lie algebra, the so-called Heizenberg algebra. What is more interesting, however, is that Bose operators can also be considered as generators of a simple Lie superalgebra $G = G_0 + G_1$ and more exactly they are elements from the odd part of the algebra. Indeed suppose that $\beta_i^{\xi} \in G_1$ and ask what is the Lie superalgebra these operators generate. From (15) we obtain

$$[\{\beta_i^{\xi}, \beta_j^{\eta}\}, \beta_k^{\delta}] = (\delta-\eta)\delta_{jk}\beta_i^{\xi} + (\delta-\xi)\delta_{ik}\, \beta_j^{\eta} . \qquad (16)$$

Let us denote by G_0 the linear envelope of all operators $\{\beta_i^{\xi}, \beta_j^{\eta}\}$ and let G_1 be the space spanned on the creation and annihilation operators. It follows immediately from (16) that with respect to the multiplication (6) $G = G_0 + G_1$ is a linear Lie superalgebra. It turns out [5] G is simple and in the Kac notation [1] this is the algebra $B(0, n)$. The even part of $B(0, n)$ is isomorphic to a direct sum of the symplectic algebra G_n and one-dimensional centre.

In this case it also turns out that the other representations of $B(0, n)$ determine new kind of creation and annihilation operators, the so-called paraBose operators [3]. The representation of these operators in a space with single vacuum are labelled by one positive integer p, to order of the statistics. Putting this result in a Lie superalgebraical language we can say that a large class of infinite-dimensional representations of the superalgebra $B(0, n)$ was found by Green in 1953. We point out this result since at the present day there exists no satisfactory representation theory even for the finite-dimensional representations of the Lie superalgebras.

We summarize the results we have obtained so far.

Simple classical Lie superalgebra	Kind of the statistics (quantization) they determine
B_n	Fermi, paraFermi
C_n	Bose, paraBose
A_n, C_n, D_n, $A(m,n)$, $C(n)$, $D(m,n)$, $P(n)$, $Q(n)$.	?

Now we are ready to formulate the main problem we are going to discuss.

<u>Problem.</u> Is it possible to define creation and annihilation operators that satisfy the second quantization axioms and generate some of the Lie superalgebras from the third group in the above table ?

To answer this question we recall that the commutation relations between the creation and annihilation operators are usually derived from the infinitesimal form of the translation invariance of the field $\Psi(x)$, namely[6]

$$[P^m, \Psi(x)] = -i\partial^m \Psi(x) \qquad m = 0,1,2,3. \tag{17}$$

It is convenient to pass to discrete notation in momentum space considering the field $\Psi(x)$ with mass m to be locked in cube. In this case the relation (17) reads as

$$[P^m, x_i^{\pm}] = k_i^m x_i^{\pm}, \tag{18}$$

where i stands for all discrete indices; k_i^m is the 4-momentum of the particle in a state i. The solutions $x_i^{\pm}$ of the operator equation (18) are by definition creation and annihilation operators. Indeed in spite of the fact that the commutation relations of these operators are still unknown, it is clear that the state $x_i^{\pm}|q\rangle$ carries momentum $q \pm k_i$ if $|q\rangle$ is a state with momentum q and $x_i^{\pm}|q\rangle \neq 0$.

As we mentioned, it is possible to define operators that satisfy (18) and generate any of the classical Lie algebras[2]. We know how to construct creation and annihilation operators corresponding to the Lie superalgebras $A(0,n)$ and $C(n)$. The quantization corresponding to the simple Lie algebras is called A-, B-, C- or D-quantization depending on the algebra the operators generate. The corresponding creation and annihilation operators are denoted as $a_i^{\pm}$, $b_i^{\pm}$, $c_i^{\pm}$ and $d_i^{\pm}$.

For all mentioned statistics the operators $x_i^{\pm}$ can be considered as root vectors.

Moreover the ordering in the Cartan subalgebra can be fixed in such a way that the creation (annihilation) operators are negative (positive) root vectors.

Definition. The quantization is said to be simple if the creation and annihilation operators $x_i^{\pm}$ satisfy the translation invariance (18) and any finite number of pairs $x_{i_1}^{\pm}, \dots, x_{i_n}^{\pm}$ generate a simple Lie superalgebra.

In all cases we known, the momentum tenzeror M^{mn} and the 4-vector P^n expressed in terms of the fields satisfy the commutation relations of the algebra of the Poincaré group.

In order to preserve the main properties of the ordinary theory one has to consider those representations, leading to a natural particle interpretation. Therefore we give the following definition.

DEFINITION OF A FOCK SPACE

Let $x_1^{\pm}, \dots, x_n^{\pm}, \dots$ be creation and annihilation operators. The representation space is said to be a Fock space of the x-operators if it fulfils the conditions

1. Hermiticity condition

$$(x_i^{+})^* = x_i^{-} \tag{19}$$

Here * denotes hermitian conjugation operation.

2. Existence of vacuum. There exists a vector $|0\rangle$ from the representation space such that for all i

$$x_i^- |0\rangle = 0. \qquad (20)$$

3. Irreducibility. The representation space is spanned on all positive vectors

$$x_{i_1}^+ x_{i_2}^+ \dots x_{i_m}^+ |0\rangle, \quad m = 0, 1, 2, \dots \qquad (21)$$

For the Lie-algebraical statistics the first condition is equivalent to the anti-hermiticity of the generators of the compact form. Therefore the Fock representations of finite number of operators are finite-dimensional irreducible representations, and they contain a highest weight. The Fock spaces are characterized by the following theorem/2/.

Theorem. Up to a multiplicative constant the vacuum is unique and coincides with the highest weight vector. The representation space is a Fock space if and only if

$$x_i^- x_j^+ |0\rangle = 0 \quad \forall \; i \neq j. \qquad (22)$$

We now proceed to give some examples.

A-STATISTICS

In this case any finite number of operators $a_{i_1}^{\pm}, \dots, a_{i_n}^{\pm}$ generate the Lie algebra A_n. The initial quantization equation (18) does not determine uniquely the a-operators as elements of the algebra. Here we shall mention two realizations which lead to quite different physical properties.

In the first realization the a-operators satisfy the double commutation relations/2/

$$\begin{aligned} [[a_i^+, a_j^-], a_k^+] &= \delta_{jk} a_i^+ + \delta_{ij} a_k^+ \\ [[a_i^+, a_j^-], a_k^-] &= -\delta_{ik} a_j^- - \delta_{ij} a_k^- \\ [a_i^+, a_j^+] &= [a_i^-, a_j^-] = 0. \end{aligned} \qquad (23)$$

The representation space is a Fock space if and only if

$$a_i^- a_j^+ |0\rangle = p\delta_{ij} |0\rangle, \quad p = 1, 2, 3, \dots \qquad (24)$$

The same relation holds for the parastatistics of order p. Therefore we call p an order of the A-statistics. The requirement (24) together with the commutation relations (23) determines the representation of the a-operators of order p. The A-statistics can be defined by the relations (23). The representations are determined by the equation (24). All calculations can be carried out without referring to any Lie-algebraical properties.

The order of the statistics p has a well defined physical meaning. It turns out that

$$(a_{i_1}^+)^{\ell_1} (a_{i_2}^+)^{\ell_2} \dots (a_{i_m}^+)^{\ell_m} |0\rangle \neq 0 \qquad (25)$$

if and only if $\ell_1 + \dots + \ell_m \leq p$. This is actually the Pauli principles for the A-statistics. If the order of the statistics is p then the number of any ensemble of particles cannot exceed p.

As an orthonormal basis one can choose the vectors

$$|p;\ell_{i_1},\ldots,\ell_{i_n}\rangle = \sqrt{\frac{(p-\Sigma\ell_i)!}{p!}}\,\frac{(a^+_{i_1})^{\ell_1}\ldots(a^+_{i_n})^{\ell_n}}{\sqrt{\ell_{i_1}!\ldots\ell_{i_n}!}}\,|0\rangle, \quad (26)$$

where for definiteness $0<i_1<i_2<\ldots<i_n$. For the matrix elements we have

$$a^+_i\,|p;..,\ell_i,..\rangle = \sqrt{(\ell_i+1)(p-\sum_j \ell_j)}\,|p;...,\ell_i+1,...\rangle,$$
$$a^-_i\,|p;..,\ell_i,...\rangle = \sqrt{\ell_i\,(p-\sum_j \ell_j+1)}\,|p;....,\ell_i-1,...\rangle. \quad (27)$$

To distinguish from the above realization, we call the second realization A_c-statistics[7]. The creation and annihilation operators in this case are labelled with three indices. The operator $a^{\xi}_{\eta i}$ creates $(\xi=+)$ or annihilates $(\xi=-)$ a particle with charge η and other characteristics i. The a_c-operators satisfy the commutation relations $(\xi,\eta=\pm,\ i,j,k=1,2,\ldots)$.

$$[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{\eta k}] = \delta_{-\xi j,\eta k}\,a^{\xi}_{\xi i}$$
$$[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i,-\eta k}\,a^{\xi}_{-\xi j}$$
$$[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{\eta k}] = \delta_{\xi j,\eta k}\,a^{\xi}_{\xi i} + \delta_{ij}\,a^{\eta}_{\eta k} \quad (28)$$
$$[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i,-\eta k}\,a^{\eta}_{-\eta j} - \delta_{ij}\,a^{\eta}_{-\eta k}$$
$$[a^{\xi}_{\xi i}, a^{\eta}_{\eta j}] = [a^{\xi}_{-\xi i}, a^{\eta}_{-\eta j}] = 0$$

The Fock spaces $W(p,q)$ in this case are labelled by two nonnegative integers.

<u>The Pauli principle.</u> In the Fock space $W(p,q)$ there cannot be more than $p+q$ particles in a single state. The charge of an arbitrary ensemble of particles cannot be more than p and less than $-q$.

Remark that the A_c-statistics does not put limitations on the total amount of particles in the ensemble. The current corresponding to the A_c-statistics is a local operator.

A-SUPERSTATISTICS

In this case the creation and annihilation operators $a^{\pm}_1, a^{\pm}_2,\ldots$ generate the Lie superalgebra $A(0,n)$. The structure relations of the operators read as follows[8]:

$$[\{a^+_i, a^-_j\}, a^+_k] = \delta_{kj}\,a^+_i - \delta_{ij}\,a^+_k$$
$$[\{a^+_i, a^-_j\}, a^-_k] = \delta_{ki}\,a^-_j + \delta_{ij}\,a^-_k \quad (29)$$
$$\{a^+_i, a^+_j\} = \{a^-_i, a^-_j\} = 0.$$

The Fock space of the a-superoperators puts limitations on the total amount of particles in an ensemble. The current corresponding to a charged field is not local operator. Elsewhere, we shall consider another realization of the a-superoperators, that leads to local currents and does not restrict the number of particles.

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