ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 11905

T.D.Palev

A CAUSAL A-STATISTICS.
II. LOWEST ORDER REPRESENTATION

1978

E2 - 11905

T.D.Palev *

A CAUSAL A-STATISTICS.
II. LOWEST ORDER REPRESENTATION

Submitted to "Reports on Mathematical Physics"

---

* Address after October 12, 1978: Institute for Nuclear Research and Nuclear Energy, Boul. Lenin 72, 1113 Sofia, Bulgaria.

Палев Ч.Д.

E2 -11905

Причинная А-статистика. II. Представление низшего порядка

Изучается низшее нетривиальное представление введенной в[1] причинной А-статистики. В этом случае принцип Паули гласит, что заряд произвольного ансамбля частиц равен либо 0, либо 1, и что в каждом состоянии не может быть более одной частицы. Построен ортонормированный базис в пространстве Фока. В этом базисе найдены матричные элементы операторов рождения и уничтожения.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1978

Palev T.D.

E2 - 11905

A Causal A-Statistics. II. Lowest Order Representation

The lowest order nontrivial representation of the causal A-statistics introduced in[1] is studied. In this case the Pauli principle asserts that the charge of an arbitrary ensemble of particles is either 0 or 1 and that there cannot be more than one particle in a given state. An orthonormal basis in the Fock space is constructed. The matrix elements of the creation and annihilation operators are found in this basis.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1978

In ref.[1] (hereafter referred to as I) we have studied some of the properties of a new kind of creation and annihilation operators $a^{\xi}_{\eta i}$. These operators were introduced in ref.[2] as an alternative way for a quantization of the spinor field. The operator $a^{\xi}_{\eta i}$ is a creation $(\xi = +)$ or annihilation $(\xi = -)$ operator of a particle with charge $\eta = \pm 1$ and other characteristics i. The main feature of the operators is that they generate the Lie algebra of the unimodular group. In order to use proper Lie-algebraical language we let the index i run a finite set of values $i = 1, 2, \ldots, n$. This is only an intermediate step. The final results are valid also for $n \to \infty$. The operators*

$$a^{\xi}_{\eta 1}, \; a^{\xi}_{\eta 2}, \ldots, a^{\xi}_{\eta 2} \qquad (1)$$

generate the classical simple Lie algebra A_{2n} of the group SL(2n+1). Therefore there exists one to one correspondence between the representations of the creation and annihilation operators and the representations of the algebra A_{2n}. In I we have widely used this link in order to classify all Fock space representations of the creation and annihilation operators. We call the operators (1) a-creation and annihilation operators and the corresponding statistics - a causal A-statistics, or simply A-statistics.

In the present paper we study in more detail the properties of the lowest dimensional nontrivial Fock space,

---

* Throughout the paper $\xi, \eta = \pm$ or ± 1; $i, j = 1, 2, \ldots, n$.

which for the A-statistics plays the same role as the Fermi representation in the paraFermi statistics.

First, we shortly recall some of the general properties of the A-statistics *. The operators $a^{\xi}_{\eta i}$ satisfy the commutation relations

$$\begin{aligned}
&[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{\eta k}] = \delta_{-\xi j, \eta k} a^{\xi}_{\xi i} \\
&[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i, -\eta k} a^{\xi}_{-\xi j} \\
&[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{\eta k}] = \delta_{\xi j, \eta k} a^{\xi}_{\xi i} + \delta_{ij} a^{\eta}_{\eta k} \\
&[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i, -\eta k} a^{\eta}_{\eta j} - \delta_{ij} a^{\eta}_{-\eta k} \\
&[a^{\xi}_{\xi i}, a^{\eta}_{\eta j}] = [a^{\xi}_{-\xi i}, a^{\eta}_{-\eta j}] = 0 .
\end{aligned} \tag{2}$$

The basis in the Cartan subalgebra $\tilde{\mathcal{H}}$ of $gl(2n+1) \supset A_{2n}$

$$h_{-n}, h_{-n+1}, \ldots, h_0, \ldots, h_{n-1}, h_n \tag{3}$$

can be chosen such that the creation and annihilation operators are negative and positive root vectors, respectively. The correspondence with their roots is

$$a^{\pm}_{i} = \pm(h^{i} - h^{o}), \quad a^{\pm}_{-i} = \mp(h^{-i} - h^{o}), \quad i=1,\ldots,n. \tag{4}$$

Here

$$h^{-n}, h^{-n+1}, \ldots, h^{o}, \ldots, h^{n-1}, h^{n} \tag{5}$$

is the dual to (3) basis in $\tilde{\mathcal{H}}$ with respect to the metric

$$(h_{\alpha}, h_{\beta}) = 2(2n+1)\delta_{\alpha\beta}, \quad \alpha, \beta = -n, -n+1, \ldots, n . \tag{6}$$

---

* Everywhere in this paper we follow the terminology and the notation introduced in I. If we refer to some Theorem, Lemma, or formula from I, in front of it we write I.

Restricted on the Cartan subalgebra $\mathcal{H}$ of A_{2n} the scalar product (6) coincides with the Cartan-Killing form. The representation space W (the A_{2n}-module) is said to be a Fock space of the a -operators if it fulfils the conditions

1. Hermiticity condition

$$(a^{+}_{\eta i})^{*} = a^{-}_{\eta i}, \quad i=1,\ldots,n . \tag{7}$$

Here * denotes hermitian conjugate operation.

2. Existence of vacuum. There exists a vector $|0\rangle \in W$ such that

$$a^{-}_{\eta i}|0\rangle = 0 \qquad i=1,\ldots,n \tag{8}$$

The vector $|0\rangle$ is called a vacuum.

3. Irreducibility. The representation space W is spanned on all possible vectors

$$a^{+}_{\eta_1 i_1} a^{+}_{\eta_2 i_2} \ldots a^{+}_{\eta_m i_m} |0\rangle , \tag{9}$$

where m runs over all non-negative integers.

It turns out (Theorem I.1, I.2; I.47) the A_{2n}-module W is a Fock space if and only if it is a finite-dimensional irreducible module such that

$$\begin{aligned}
&a^{-}_{i} a^{+}_{j} x_{\Lambda} = \delta_{ij} p x_{\Lambda} , \\
&a^{-}_{-i} a^{+}_{-j} x_{\Lambda} = \delta_{ij} q x_{\Lambda} .
\end{aligned} \tag{10}$$

The highest weight vector x_{Λ} coincides (up to a multiplicative constant) with the vacuum $|0\rangle$; p and q are non-negative integers. In terms of these numbers the highest weight Λ reads as

$$\Lambda = (p+q)\sum_{i=1}^{n} h^{-i} + h^{o} \equiv (\overset{-n}{p+q}, \overset{-n+1}{p+q}, \ldots, \overset{-1}{p+q}, \overset{0}{p}, \overset{1}{0}, \ldots, \overset{n}{0}). \tag{11}$$

The relations (10) together with the commutation relations (2) determine completely the A_{2n}-module. Therefore the Fock spaces are labelled by all pairs (p,q) of two non-negative integers. We call the pair (p,q) an order of the A-statistics and denote by $W(p,q)$ the corresponding Fock space.

The Pauli principle for the A-statistics of order (p,q) states that there cannot be more than $p+q$ particles in a given state. The charge Z of an arbitrary ensemble of particles cannot exceed p and be less than minus q.

In the remaining part of the paper we study the properties of the simplest Fock space $W(1,0)$. In this case we obtain the *Pauli principle for the A-statistics of order* (1,0). There cannot be more than one particle in a given state $(\eta\, i)$. The charge of any ensemble of particles is either zero or 1.

The highest weight Λ in this case is

$$\Lambda = h^{-n}+h^{-n+1}+\ldots+h^{-1}+h^{0} \equiv (\overset{-n}{1},\ldots,\overset{-1}{1},\overset{0}{1},0,\ldots,\overset{n}{0}). \qquad (12)$$

From (I.29) it follows that the vector

$$\lambda = \sum_{\alpha=-n}^{n} \lambda_\alpha h^\alpha \qquad (13)$$

is a weight if and only if n of its co-ordinates λ_α are equal to 1 and all the others are zero. This means that any weight λ is equivalent to the highest weight and therefore is simple, i.e., it has multiplicity 1.

The space $W(1,0)$ is an example of nonproper Fock space (/2/, Def. 9.2), i.e., space in which some of the creation operators annihilate the vacuum state. In our case

$$a^+_{-i}|0> = 0 \quad \forall \quad i=1,2,\ldots,n. \qquad (14)$$

Indeed if $a^+_{-i}|0> \neq 0$ it would have had a weight

$$\lambda = \Lambda + h^0 - h^{-i} = (1,\ldots,1,\overset{-i}{0},1,\ldots,1,\overset{0}{2},0,\ldots,0) \qquad (15)$$

which is impossible since $\lambda_0 > 1$. This property means that the Fock space does not contain 1-particle states with negative charge, i.e., there cannot exist a single antiparticle.

As a consequence of the simplicity of the weights we have

Corollary. In $W(1,0)$ there exists one to one correspondence between the weight vectors and the weights.

Therefore the number of the different possible ensembles of particles

$$a^+_{\eta_1 i_1} \ldots a^+_{\eta_k i_k} |0> \qquad (16)$$

is equal to the number of all weights. This number is the dimension of $W(1,0)$, which evidently is

$$\dim W(1,0) = \frac{(2n+1)!}{n!\,(n+1)!} = \binom{2n+1}{n}. \qquad (17)$$

Remark that in the case of $2n$ pairs of Fermi operators the corresponding Fock space W_F has 2^{2n} different states. Using for big n the Stirling formula

$$n! \approx \sqrt{2\pi n}\; n^n e^{-n} \qquad (18)$$

we obtain

$$\frac{\dim W_F}{\dim W(1,0)} \approx \sqrt{\frac{\pi n}{4}} > 1. \qquad (19)$$

Hence for the same number of operators the Fermi statistics allows bigger variety of particle ensembles. This is not surprising. The Fermi statistics restricts the number of the particles in a given state, whereas the A-statistics in addition puts limitations on the total charge of the system.

Lemma 1. In $W(1,0)$ the product of any two creation (annihilation) operators, corresponding to the same charge, is zero,

$$a^+_{\eta i}\, a^+_{\eta j} = 0, \qquad a^-_{\eta i}\, a^-_{\eta j} = 0 \qquad (20)$$

Proof.

Let

$$x_\lambda = a^+_{\xi_1 i_1} \dots a^+_{\xi_k i_k} |0> \qquad (21)$$

be an arbitrary weight vector from $W(1,0)$ with weight λ. Then the zeroth co-ordinate of λ is 0 or 1, $\lambda_0 = 0$ or 1. Therefore if the vector

$$x_{\lambda'} = a^{\xi}_{\eta i} a^{\xi}_{\eta j} x_\lambda \qquad (22)$$

is not zero, for the zeroth co-ordinate of its weight λ' we would have obtained

$$\lambda'_0 = \lambda_0 - 2\xi\eta \qquad (23)$$

which is neither 1 nor 0. Hence for every weight λ the vector $x_{\lambda'}$ in (22) is zero. Since the linear envelope of all vectors (21) gives $W(1,0)$ we conclude that (20) holds.

From Lemma 1 we conclude that an arbitrary weight vector can be written either as

$$a^+_{-j_1} a^+_{i_1} a^+_{-j_2} a^+_{-i_2} \dots a^+_{-j_k} a^+_{i_k} |0> \qquad (24)$$

or as

$$a^+_{i_1} a^+_{-j_2} a^+_{-i_2} \dots a^+_{-j_k} a^+_{i_k} |0> \qquad (25)$$

The vectors of the type (24) describe ensembles of particles with overall charge 0, whereas the state (25) carries a charge 1.

From the commutation relations (2) we find

$$[[a^+_i , a^+_{-j}], a^+_k] = [[a^+_i , a^+_{-j}], a^+_{-k}] = 0. \qquad (26)$$

Taking into account Lemma 1 we have

$$a^+_i a^+_{-j} a^+_k = [a^+_i , a^+_{-j}] a^+_k =$$

$$= a^+_k [a^+_i , a^+_{-j}] = - a^+_k a^+_{-j} a^+_i . \qquad (27)$$

Similarly

$$a^+_{-i} a^+_j a^+_{-k} = - a^+_{-k} a^+_j a^+_{-i} . \qquad (28)$$

Therefore

$$a^+_{-j_1} a^+_{i_1} a^+_{-j_2} a^+_{i_2} \dots a^+_{-j_k} a^+_{i_k} |0> =$$

$$= (-1)^{p+q} \; a^+_{-\ell_1} a^+_{m_1} a^+_{-\ell_2} a^+_{m_2} \dots a^+_{-\ell_k} a^+_{m_k} |0>, \qquad (29)$$

where p (resp. q) is the number of the transpositions needed to pass from the order

$$i_1, i_2, \dots, i_k \quad \text{to} \quad m_1, m_2, \dots, m_k$$

$$(j_1, j_2, \dots, j_k \quad \text{to} \quad \ell_1, \ell_2, \dots, \ell_k). \qquad (30)$$

This property allows one to order always the indices in (24) or in (25) in a way that

$$i_1 > i_2 > \dots > i_k ,$$

$$j_1 > j_2 > \dots > j_k . \qquad (31)$$

With this agreement the set of all vectors (24-25) constitutes a basis in $W(1,0)$ since all such vectors correspond to different weights and therefore are linearly independent.

Let now

$$|p ; p_1, \dots, p_n, q_1, \dots, q_n> \qquad (32)$$

denote a weight vector (24-25) with charge p, $p_i (q_i)$ particles in a state i (resp. $-i$) and ordering of all operators according to (31). The Pauli principle requires that all p_i and q_i are either 0 or 1.

We now proceed to calculate the matrix elements of the creation and annihilation operators in the basis (32). Let

$$x_\lambda = |p ; p_1, \dots, p_n, q_1, \dots, q_n> \qquad (33)$$

be an arbitrary vector from the basis (32). Consider first the operator a_j^+. Clearly

$$a_j^+ x_\lambda = 0 \qquad \text{for} \quad p=1 \quad \text{or} \quad p_j = 1 \tag{34}$$

since otherwise we would obtain either a state with a charge 2, or two particles in the state j. Suppose $p=0$ and $p_j=0$. Let $j_1<\dots<j_k<j<j_{k+1}<\dots$

$$a_j^+ x_\lambda = a_j^+ a_{-i_1}^+ a_{j_1}^+ \dots a_{-i_k}^+ a_{j_k}^+ a_{-i_{k+1}}^+ a_{j_{k+1}}^+ \dots |0\rangle =$$
$$= a_j^+ [a_{-i_1}^+, a_{j_1}^+] \dots [a_{-i_k}^+, a_{j_k}^+][a_{-i_{k+1}}^+, a_{j_{k+1}}^+] \dots |0\rangle. \tag{35}$$

Since a_j^+ commutes with any operator $[a_{-i}^+, a_k^+]$, we have

$$a_j^+ x_\lambda = [a_{-i_1}^+, a_{j_1}^+][a_{-i_2}^+, a_{j_2}^+] \dots [a_{-i_k}^+, a_{j_k}^+] \times$$
$$\times a_j^+ [a_{-i_{k+1}}^+, a_{j_{k+1}}^+] \dots |0\rangle = \tag{36}$$
$$= (-1)^k a_{j_1}^+ a_{-i_1}^+ \dots a_{j_k}^+ a_{-i_k}^+ a_j^+ a_{-i_{k+1}}^+ \dots |0\rangle .$$

Here $k = p_1+\dots+p_{j-1}$. Therefore

$$a_j^+ |0; p_1, \dots, p_{j-1}, 0, p_{j+1} \dots \rangle =$$
$$= (-1)^{p_1+\dots+p_{j-1}} |1; p_1, \dots, p_{j-1}, 1, p_{j+1}, \dots\rangle . \tag{37}$$

For any p and p_i the result is

$$a_j^+ |p; \dots, p_j, \dots\rangle = (1-p)(1-p_j)(-1)^{p_1+..+p_{j-1}} |p+1; \dots, p_{j+1}, \dots\rangle \tag{38}$$

Similarly for a_{-i}^+ we have

$$a_{-i}^+ |p; \dots, q_i, \dots\rangle = p(1-q_i)(-1)^{q_1+..+q_{i-1}} |p-1; \dots, q_i+1, \dots\rangle . \tag{39}$$

The operator a_{-i}^- annihilates x_λ for $p=1$ or $q_i=0$. Let $p=1$ and $q_i=1$; $i_1<\dots<i_k<i<i_{k+1}<\dots$.

$$a_{-i}^- x_\lambda = a_{-i}^- a_{-i_1}^+ a_{j_1}^+ \dots a_{-i_k}^+ a_{j_k}^+ a_{-i}^+ a_j^+ \dots |0\rangle =$$
$$= [a_{-i_1}^+, a_{j_1}^+] \dots [a_{-i_k}^+, a_{j_k}^+][a_{-i}^-, [a_{-i}^+, a_j^+]] \dots |0\rangle . \tag{40}$$

Since

$$[a_{-i}^-, [a_{-i}^+, a_j^+]] = a_j^+ \tag{41}$$

and $k = q_1+\dots+q_{i-1}$, for arbitrary p and q_i we obtain

$$a_{-i}^- |p; \dots, q_i, \dots\rangle = (1-p) q_i (-1)^{q_1+..+q_{i-1}} |p-1; \dots, q_i-1, \dots\rangle . \tag{42}$$

Remark that

$$a_{-i_1}^- a_{-i_1}^+ a_{j_1}^+ \dots |0\rangle = a_{j_1}^+ \dots |0\rangle . \tag{43}$$

For a_j^- the result is

$$a_j^- |p; \dots, p_j, \dots\rangle = p p_j (-1)^{p_1+..+p_{j-1}} |p-1; \dots, p_j-1, \dots\rangle . \tag{44}$$

In particular

$$a_{j_1}^- a_{j_1}^+ a_{-i_1}^+ \dots |0\rangle = a_{-i_1}^+ \dots |0\rangle . \tag{45}$$

The metric in $W(1,0)$ is defined in a complete analogy with the scalar product in the Fock space of Bose (or Fermi) operators. Postulate

a) $\langle 0|0\rangle = 1$

b) $\langle 0| a_{\eta i}^+ = 0$

c) $$(a_{-i_1}^+ a_{j_1}^+ \dots a_{-i_r}^+ a_{j_r}^+ |0\rangle, a_{-k_1}^+ a_{-\ell_1}^+ \dots a_{-k_m}^+ a_{\ell_m}^+ |0\rangle) =$$
$$\langle 0| a_{j_r}^- a_{-i_r}^- \dots a_{j_1}^- a_{-i_1}^- a_{-k_1}^+ a_{\ell_1}^+ \dots a_{-k_m}^+ a_{\ell_m}^+ |0\rangle . \tag{46}$$

From (43) and (45) we immediately obtain

$$(|p; p_1,\dots,p_n, q_1,\dots,q_n>, |p; p_1,\dots,p_n, q_1,\dots,q_n>) = 1 . \qquad (47)$$

If in (46) at least one $i_s \neq k_s$ or $j_s \neq \ell_s$, then the vector

$$a^-_{j_r} a^-_{-i_r} \dots a^+_{-k_m} a^+_{\ell_m} |0> \qquad (48)$$

is different from $|0>$. This vector is either zero or as a vector with definite weight can be represented as

$$a^+_{-t_1} \dots a^+_{r_s} |0>. \qquad (49)$$

In both cases the scalar product (46) is zero. Therefore the basis (32) is an orthonormal basis in $W(1,0)$.

We see that the (1,0) order A-statistics has some of the properties of the Fermi statistics, namely in every state there can be at most 1 particle. Other properties only partially hold: the states (32) are antisymmetric only for transpositions of particles with the same charge. The restriction on the total charge of an arbitrary ensemble of particles is a new feature. As a consequence of it there follows that 1- or many-particle states of only negative particles cannot exist.

Since the Pauli principle and Lemma 1 are independent of the number of the operators most of the results hold for $n \to \infty$ or their generalization is evident. For instance instead of (32) we have to say that all ordered vectors

$$a^+_{-i_1} a^+_{j_1} a^+_{-i_2} a^+_{j_2} \dots a^+_{-i_k} a^+_{j_k} |0>, \qquad (50)$$

where

$$i_1 < i_2 < \dots < i_k$$
$$j_1 < j_2 < \dots < j_k \qquad (51)$$
$$k = 0,1,2,\dots$$

together with the ordered charge-one states constitute basis in the Fock space. In fact all results can be obtained on the basis of the independent on n relations (2), (7), (8), (10) and the requirement the metric in the Fock space to be positive definite. The Lie-algebraical approach gives only a good intuition and simplifies greatly the calculations.

As we have pointed out in I the charge of the particle need not necessarily be the electric charge. It can be any other charge. The generalization of the results in this case is straightforward.

REFERENCES

1. *Palev T. JINR, E2-11907, Dubna, 1978.*
2. *Palev T. Thesis. Institute for Nuclear Research and Nuclear Energy. Sofia, 1976.*

Received by Publishing Department on September 21 1978.