ОБЪЕДИНЕННЫЙ
ИНСТИТУТ
ЯДЕРНЫХ
ИССЛЕДОВАНИЙ
ДУБНА

E2 - 11904

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A CAUSAL A-STATISTICS.
I. GENERAL PROPERTIES

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Submitted to Reports on Mathematical Physics

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Причинная А-статистика. I. Общие свойства

Показано, что аксиомы вторичного квантования могут в принципе удовлетворяться операторами рождения и уничтожения, порождающими алгебру Ли унимодулярной группы. Пространства Фока W(p, q) нумеруются двумя произвольными неотрицательными числами p и q. Сформулирован принцип Паули: в протранстве Фока W(p, q) в одном состоянии не может находиться более p + q частиц. Заряд произвольного ансамбля частиц не может превышать p и быть меньше −q.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1978

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A Causal A-Statistics. I. General Properties

It is shown that the second quantization axioms can, in principle, be satisfied with creation and annihilation operators generating the Lie algebra of the unimodular group. The Fock spaces W(p, q) are labelled with two arbitrary non-negative numbers p and q. The Pauli principles have been formulated. In the Fock space W(p, q) there cannot be more than p + q particles in a single state. The charge of an arbitrary ensemble of particles cannot exceed p and be less than q.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1978

1. INTRODUCTION

In ref. /1/ we have studied some possible generalizations of the statistics of the spinor fields from a Lie algebraical point of view. The guiding idea for such investigations came from the observation that the ordinary spinor field statistics, the Fermi one, has a well defined Lie algebraical meaning. It turns out that any finite number $f_1^{\pm}, \dots, f_n^{\pm}$ of Fermi creation and annihilation operators generates one particular irreducible representation, the spinor representation of the algebra B_n of the odd-orthogonal group $SO(2n+1)$ /2/. Since the transition to an infinite set of Fermi operators does not change the algebraical structure, one can say that Fermi quantization is actually a quantization according to an irreducible representation of the infinite dimensional simple Lie algebra of the odd-orthogonal group. It turns out that the other representations of the same algebra lead to the introduced by Green paraFermi statistics /3/.

The group theoretical formulation of the Fermi statistics rises in a natural way the question whether one can define new kinds of creation and annihilation operators that satisfy the second quantization axioms for spinor fields but generate some of the other simple Lie algebras. In ref. [1] we have answered this question. We have associated a quantization with every classical simple Lie algebra in such a way that the corresponding creation and annihilation operators belong to the algebra and generate the whole algebra. Depending on the classical algebra the creation and annihilation operators generate, we call the corresponding quantization A-, B-, C- or D-quantization. In this terminology the Fermi statistics is included in the B-quantization. Some of the properties of A-, B- and D-statistics were mentioned in ref. [5].

Unfortunately, the Bose statistics does not allow generalizations along a Lie algebraical line. It is known that the second order polynomials of these operators are isomorphic to the symplectic algebra [4]. The Bose operators themselves, however, are not elements of this algebra and cannot be considered as elements of any other simple Lie algebra in such a way that they generate the whole algebra. Elsewhere we shall show that an approach incorporating generalizations of Bose and Fermi statistics can be developed on the ground of Lie superalgebras (shortly this question is discussed in ref. [6]).

It is important to point out that even within a given (A-, B-, C- or D-) statistics the quantization does not determine uniquely the creation and annihilation operators as elements of the Lie algebra. One can associate the creation and annihilation operators with different generators and this in general leads to different physical properties of the corresponding particles.

In the present paper we continue the investigations (published in ref. [7]) of the properties of the A-statistics. In more detail we shall consider the Fock space representations of a new realization of the creation and annihilation operators, which turns out to lead to local currents for the spinor fields [8]. Another realization of the A-statistics, giving also local currents, will be considered in ref. [9].

2. A CAUSAL A-STATISTICS

Let $\psi(x)$ be a free spinor field and $\bar{\psi}(x)$ its Dirac conjugate field. Representing the fields with their frequency parts, we write

$$\begin{aligned} \psi(x) &= \psi^{+}(x,-) + \psi^{-}(x,+), \\ \bar{\psi}(x) &= \psi^{+}(x,+) + \psi^{-}(x,-), \end{aligned} \tag{1}$$

where*

$$\psi^{\xi}(x,\eta) = \frac{1}{(2\pi)^{3/2}} \int dp\, e^{i\xi p x}\, v^{\mu,\xi}(p,\eta)\, a_{\mu}^{\xi}(p,\eta). \tag{2}$$

Instead of choosing the normal product form for the operators, we always assume that the dynamical variables are antisymmetrized with respect to the operators $a_{\mu}^{\xi}(p,\eta)$. In this notation the 4-momentum operator P^{n} reads as

*Throughout the paper $\xi, \eta = \pm$ or ± 1.

$$P^n = \sum_{\eta,\mu} \int dp\, p^n [a^+_\mu(p, \eta),\ a^-_\mu(p, \eta)], \qquad (3)$$

The translation invariance imposes the first restrictions on the operators $a^\xi_\mu(p, \eta)$, namely

$$[P^n,\ a^\xi_\mu(p, \eta)] = \xi p^n a^\xi_\mu(p, \eta). \qquad (4)$$

The gauge invariance gives another relation,

$$[Q,\ a^\xi_\mu(p, \eta\xi)] = \eta\, a^\xi_\mu(p, \eta\xi), \qquad (5)$$

where Q is the charge operator.

Remark that up to now almost nothing is known about the commutation relations between the operators $a^\xi_\mu(p, \eta)$. The relativistic and gauge invariance impose restrictions only on the commutation relations between certain integral combinations of the operators $[a^+_\mu(p, \eta), a^-_\mu(p, \eta)]$ and $a^\xi_\nu(q, \eta)$. Otherwise the commutation relations are quite arbitrary. Whatever the structure relations will be from (4) and (5), it follows that $a^+_\mu(p, \eta)$ (resp. $a^-_\mu(p, \eta)$) is a creation (annihilation) operator of a particle with a momentum p and a charge η. In the ordinary theory the requirement $a^\xi_\mu(p, \eta)$ to be Fermi operators actually means that the rest of the commutation relations (apart from those fixed from (4) and (5)) are defined in such a way that the creation and annihilation operators are part of the generators of the orthogonal algebra, generating through commutations the spinor representation of the total algebra. If one allows another representations of the same algebra, then $a^\xi_\mu(p, \eta)$ become paraFermi operators of a certain order p, where p != 1,2,... is a label for the different irreducible representations of the algebra of the orthogonal group.

In the present paper we wish to choose the operators $a^\xi_\mu(p, \eta)$ to be generators of the algebra of the unimodular group and to study the Fock space representations of these operators. From a purely technical point of view it is more convenient to pass to discrete notation in momentum space, considering the field $\psi(x)$ with mass m to be locked in a cube with edge L. For the eigenvalues k^m_n of the 4-momentum P^m, $m = 0, 1, 2, 3$, one obtains

$$k^\alpha_n = \frac{2\pi}{L} n^\alpha,\quad k^0_n = \sqrt{m^2 + \left(\frac{2\pi}{L}\right)^2 [(n^1)^2 + (n^2)^2 + (n^3)^2]}, \qquad (6)$$

where $n = (n^1, n^2, n^3)$, $\alpha = 1, 2, 3$ and n^α runs over all nonnegative integers. In momentum space relation (4) reads as

$$[P^m,\ a^\xi_{\eta n}] = \xi k^m_n a^\xi_{\eta n}, \qquad (7)$$

where $a^\xi_{\eta n}$ is a creation ($\xi = +$) or annihilation ($\xi = -$) operator of a particle with charge η and other characteristic n; the index n replaces (p, μ), i.e., $a^\xi_\mu(p, \eta) \to a^\xi_{\eta n}$.

In order to study the representations of the creation and annihilation operators in a proper Lie-algebraical language, it is convenient to approximate the momentum space with a finite number of points, i.e., to consider a finite set of operators $a^{\xi}_{\eta i}$, $i=1,2,\dots,n$. This is only an intermediate step. The final results are easily generalized to the case $n\to\infty$.

To proceed further we first introduce the notation for the algebra A_{2n} we shall make use of and recall some properties of it. We consider A_{2n} as a subalgebra of the algebra $gl(2n+1)$ of the general linear group $GL(2n+1)$. The algebra $gl(2n+1)$ may be determined as a linear envelope of the generators

$$e_{\alpha\beta},\ \alpha,\beta\in N\equiv(-n,-n+1,\dots,-1,0,1,\dots,n-1,n) \qquad (8)$$

with commutation relations

$$[e_{\alpha\beta}, e_{\gamma\epsilon}]=\delta_{\beta\gamma}e_{\alpha\epsilon}-\delta_{\alpha\epsilon}e_{\gamma\beta}. \qquad (9)$$

Let $\mathcal{H}$ and $\tilde{\mathcal{H}}$ be the Cartan subalgebras of A_{2n} and $gl(2n+1)$, resp. Denote by envX the linear envelope of an arbitrary set X. In terms of the $gl(2n+1)$ generators we have:

$$gl(2n+1)=\mathrm{env}\{e_{\alpha\beta}|\alpha,\beta\in N\},$$

$$A_{2n}=\mathrm{env}\{e_{\alpha\alpha}-e_{\beta\beta},\ e_{\alpha\beta}\ |\ \alpha\neq\beta\in N\},$$

$$\tilde{\mathcal{H}}=\mathrm{env}\{h_\alpha|h_\alpha=e_{\alpha\alpha},\ \alpha\in N\}, \qquad (10)$$

$$\mathcal{H}=\mathrm{env}\{h_\alpha-h_\beta\ |\ \alpha,\beta\in N\}.$$

For a covariant basis in $\tilde{\mathcal{H}}$ we choose the vectors

$$h_{-n},\dots,h_0,\dots,h_n. \qquad (11)$$

The algebra $gl(2n+1)$ is not semi-simple. Its Cartan-Killing form is degenerate. Therefore it is convenient to introduce a metric in $\tilde{\mathcal{H}}$ with the relation

$$(h_\alpha,h_\beta)=2(2n+1)\delta_{\alpha\beta}. \qquad (12)$$

Restricted on $\mathcal{H}$ this metric coincides with the Cartan-Killing form of A_{2n}.

From (9) and (12) one obtains

$$[h, e_{\alpha\beta}]=(h, h^\alpha-h^\beta)e_{\alpha\beta}\qquad h\in\mathcal{H},\ \alpha\neq\beta\in N, \qquad (13)$$

where $h^{-n},\dots,h^n$ is the contravariant (i.e., dual to $h_\alpha, \alpha\in N$) basis in $\tilde{\mathcal{H}}$. Hence, the generators $e_{\alpha\beta}, \alpha\neq\beta\in N$ are the root vectors of A_{2n}. The correspondence with their roots is

$$e_{\alpha\beta}\to h^\alpha-h^\beta,\ \alpha\neq\beta\in N. \qquad (14)$$

In the natural ordering of the basis (11) the generators

$$e_{\alpha\beta},\ \alpha<\beta\ (\alpha>\beta),\ \alpha,\beta\in N \qquad (15)$$

are the positive (negative) root vectors of A_{2n}.

We are now ready to introduce the creation and annihilation operators. We define

$$a^{\xi}_{\xi i} = e_{\xi i,0}\ , \quad i = 1,2,\ldots,n,$$

$$a^{\xi}_{-\xi i} = e_{0,-\xi i}\ , \quad \xi = \pm. \tag{16}$$

One can easily verify that the operators (16) satisfy the initial quantization equation (7) with 4-momentum written in discrete notation as

$$P^{m} = \sum_{i>0} k_i^m \{[a^+_i, a^-_i] + [a^+_{-i}, a^-_{-i}]\}. \tag{17}$$

Moreover it is evident that $a^{\xi}_{\eta i}$ generate the algebra A_{2n}, since any of the generators (10) either coincide with one of the operators (16) or it is a commutator of two such operators.

Let us consider the operator

$$\eta h_{\eta i} = [a^+_{\eta i}, a^-_{\eta i}] + \eta e_{0,0}\ , \quad \eta = \pm. \tag{18}$$

From (9) and (16) we have

$$[\eta h_{\eta i}, a^{\xi}_{\delta j}] = \xi \delta_{\eta\delta} \delta_{ij}\, a^{\xi}_{\delta j}\ . \tag{19}$$

Hence the Cartan element $\eta h_{\eta i}$ is the number operator of the particles in a state (ηi). In terms of the Cartan elements the 4-momentum operator P^m and the charge operator Q can be written as

$$P^{m} = \sum_{i>0} k_i^m (h_i - h_{-i}) + c_1,$$

$$Q = \sum_{i>0} (h_i + h_{-i}) m + c_2, \tag{20}$$

where c_1 and c_2 are constants depending on the representation. The operators $a^{\xi}_{\eta i}$ are root vectors of A_{2n}. The correspondence with their roots is

$$a^+_i \to -h^{\circ} + h^{i}, \qquad a^-_i \to h^{\circ} - h^{i},$$

$$a^+_{-i} \to -h^{-i} + h^{\circ}, \qquad a^-_{-i} \to h^{-i} - h^{\circ}, \tag{21}$$

and therefore the creation (annihilation) operators are negative (positive) root vectors.

The commutation relations of A_{2n} can be written also only in terms of $a^{\xi}_{\eta i}$.

From (9) and (16) we obtain $(i, j, k = 1,\ldots,n,$ $\xi, \eta = \pm)$:

$$[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{\eta k}] = \delta_{-\xi j, \eta k}\, a^{\xi}_{\xi i}\ ,$$

$$[[a^{\xi}_{\xi i}, a^{\xi}_{-\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i, -\eta k}\, a^{\xi}_{-\xi j}\ ,$$

$$[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{\eta k}] = \delta_{\xi j, \eta k}\, a^{\xi}_{\xi i} + \delta_{ij}\, a^{\eta}_{\eta k}\ , \tag{22}$$

$$[[a^{\xi}_{\xi i}, a^{-\xi}_{\xi j}], a^{\eta}_{-\eta k}] = -\delta_{\xi i, -\eta k}\, a^{\eta}_{-\eta j} - \delta_{ij}\, a^{\eta}_{-\eta k}\ ,$$

$$[a^{\xi}_{\xi i}, a^{\eta}_{\eta j}] = [a^{\xi}_{-\xi i}, a^{\eta}_{-\eta j}] = 0.$$

The generalization of the above commutation relations to the case of infinite and even continuum set of operators is evident. We have mentioned that for the spinor field the creation and annihilation operators (16) lead to local currents [8]. Because of this and in order to distinguish them from the operators we have introduced in [7], we call the operators satisfying the relations (22) causal a-operators and the corresponding quantization (statistics) - causal A-quantization (statistics). Unless otherwise stated in this paper by a-operators we mean causal a-operators.

3. FOCK SPACES FOR THE a-OPERATORS

We now proceed to study those representations of the a-operators that possess the main features of the Fock space representations in the ordinary quantum mechanics. We continue to consider a finite set of operators. The extension of the results to infinite number of a-operators will be evident. The following definition was already given in [7].

Definition. Let $a^{\xi}_{\eta 1}, \dots, a^{\xi}_{\eta n}$ be a-creation $(\xi = +)$ and annihilation $(\xi = -)$ operators. The A_{2n}-module W is said to be a Fock space of the a-operators if it fulfils the conditions:

1. Hermiticity condition

$$(a^{+}_{\eta i})^{*} = a^{-}_{\eta i}, \quad i = 1, \dots, n. \qquad (23)$$

Here * denotes hermitian conjugation operation.

2. Existence of vacuum. There exists a vacuum vector $|0\rangle \in W$ such that

$$a^{-}_{\eta i}|0\rangle = 0, \quad i = 1, \dots, n. \qquad (24)$$

3. Irreducibility. The representation space W is spanned over all possible vectors

$$a^{+}_{\eta_1 i_1} a^{+}_{\eta_2 i_2} \dots a^{+}_{\eta_m i_m} |0\rangle, \qquad (25)$$

where m runs over all non-integers.

The Fock space of the a-operators is called also A_{2n}-module of Fock, Fock module of the a-operators or simply Fock module. The following theorem is a straightforward generalization of the one proved in ref. [7].

Theorem 1. The A_{2n}-module W is a Fock space if and only if it is an irreducible finite-dimensional module such that

$$a^{+}_{\xi i} a^{-}_{\eta j} x_{\Lambda} = 0, \quad i \neq j = 1, \dots, n. \qquad (26)$$

The vacuum is unique (up to a multiplicative constant) and coincides with the highest weight vector x_{Λ}.

In order to classify the Fock space representations of the causal a-operators it remains to determine all irreducible A_{2n}-modules, satisfying the condition (26). For this purpose it is more convenient to use the irreducible $gl(2n+1)$-modules since they are A_{2n}-irreducible and give all A_{2n}-irreducible modules.

The finite-dimensional irreducible $gl(2n+1)$-module is completely characterized by its highest weight $\Lambda \in \mathcal{H}$. The coordinates $(L_{-n}, L_{-n+1}, \dots, L_{n-1}, L_{n})$ of Λ in the contravariant basis (11) can be chosen to be integers. They always obey the inequalities

$$L_{-n} \geq L_{-n+1} \geq \dots \geq L_0 \geq L_1 \geq \dots \geq L_n . \tag{27}$$

The A_{2n} -modules are labelled by all different n -tuples $(L_{-n}, \dots, L_0, \dots, L_n)$ satisfying the inequalities (27).

Consider a representation with highest weight $\Lambda = (L_{-n}, \dots, L_n)$. The vector

$$\lambda = (\ell_{-n}, \dots, \ell_n) \equiv \sum_{\alpha=-n}^{n} \ell_\alpha h^\alpha \tag{28}$$

is a weight if and only if

$$\ell_{\alpha_0} + \dots + \ell_{\alpha_m} \leq L_{-n} + \dots + L_{-n+m}, m = 0, 1, 2, \dots, 2n-1,$$

$$\ell_{-n} + \dots + \ell_n = L_{-n} + \dots + L_n, \ \alpha_0 \neq \alpha_1 \neq \dots \neq \alpha_m \in N. \tag{29}$$

From (29) it follows that the co-ordinates of an arbitrary weight are non-negative integers. If $\lambda = (\ell_{-n}, \dots, \ell_n)$ is a weight then the vectors obtained from λ by permuting the co-ordinates $\ell_{-n}, \dots, \ell_n$ are also weights. All such weights are said to be equivalent. The equivalent weights have the same multiplicity. Another property we shall use is that any weight $\lambda < \Lambda$, i.e., the first non-zero covariant co-ordinate of the vector $\Lambda - \lambda$ is positive.

We now pass to the main problem of this section, classification of the Fock spaces. Unless otherwise stated, the roots and the weights are represented by their orthogonal co-ordinates in the contravariant basis $h^{-n}, \dots, h^0, \dots, h^n$.

Theorem 2. The irreducible A_{2n}-module W is a Fock space if and only if the co-ordinates of its highest weight Λ fulfil the conditions

$$L_{-n} = L_{-n+1} = \dots = L_{-1} = L,$$

$$L_1 = L_2 = \dots = L_n = 0, \tag{30}$$

i.e., Λ is of the form

$$\Lambda = (L, L, \dots, L, L_0, 0, \dots, 0), \tag{31}$$

where $L \geq L_0$ otherwise L and L_0 are arbitrary non-negative integers.

Proof

The operators $a^-_{\xi i}$ being positive root vectors always annihilate the highest weight vector, $a_{\xi i} x_\Lambda = 0$. Using this, commutation relations

$$[a_i^-, a_j^+] = -e_{ji}, [a_{-i}^-, a_{-j}^+] = e_{-i,-j}, \ i \neq j = 1, \dots, n \tag{32}$$

and expressing $[a^-_{\xi i}, a^+_{-\xi j}]$ in terms of the A_{2n} -generators, we can write (26) as

$$e_{ij} x_\Lambda = 0, \ e_{-i,-j} x_\Lambda = 0, \ i \neq j = 1, \dots, n. \tag{33}$$

The generators $e_{\xi i, \xi j}$ are root vectors of A_{2n}. The correspondence with their root is

$$e_{ij} \leftrightarrow h^i - h^j, \ e_{-j,-i} \leftrightarrow h^{-j} - h^{-i}, \ i \neq j = 1, \dots, n. \tag{34}$$

For $i < j$ e_{ij} and $e_{-j,-i}$ are positive root vectors and (33) holds. It remains to determine those A_{2n} -modules with highest weights

$$\Lambda = (L_{-n},\ldots,L_{-1},L_o,L_1,\ldots,L_n) \qquad (35)$$

for which the sums

$$\Lambda - h^j + h^i \quad \text{and} \quad \Lambda - h^{-i} + h^{-j},\ i > j = 1,\ldots,n \qquad (36)$$

are not weights.

As we know, for $j<i$ $L_j \geq L_i$. Suppose $L_j > L_i$. Then the vector

$$\lambda = \Lambda - h^j + h^i \quad j<i \qquad (37)$$

has co-ordinates

$$\lambda = (L_{-n},\ldots,\overset{j}{L_j - 1},\ldots,\overset{i}{L_i + 1},\ldots,L_n) \qquad (38)$$

that satisfy all inequalities (29), and λ is a weight. Therefore $e_{ij}\, x_\lambda \neq 0$ and the corresponding A_{2n} -module is not a Fock space.

If $L_{-i} > L_{-j}$, $i>j$ then

$$\lambda' = \Lambda - h^{-i} + h^{-j} \qquad (39)$$

is also a weight. Thus $e_{-j,-i}\, x_\Lambda \neq 0$ and the space is not a Fock space.

It remains to consider the A_{2n} -modules with

$$\Lambda = (L,\ldots,L,L_o,0,\ldots,0). \qquad (40)$$

Cleary the vector λ determined in (37) in this case is not a weight since it has one negative co-ordinate. Therefore $e_{ij}\, x_\Lambda = 0$. Suppose for $i>j$ that

$$\lambda' = \Lambda - h^{-i} + h^{-j} = (L,\ldots,\overset{-i}{L-1},\ldots,\overset{-j}{L+1},\ldots,L,L_o,0,\ldots,0) \qquad (41)$$

is a weight. Then

$$\lambda'' = (L+1,\ L-1,\ L,\ \ldots,\ L,L_o,0,\ldots,0) \qquad (42)$$

should be also a weight. This however is impossible since $\lambda'' > \Lambda$. Hence λ' is not a weight and $e_{-j,-i}\, x_\Lambda = 0$. We conclude that every A_{2n} -module with highest weight Λ of the form (4+) is a Fock space. This completes the proof.

Consider the Fock space $W(p,q)$ with highest weight

$$\Lambda = (p+q,\ p+q,\ldots,\ p+q,\ p,\ 0,\ldots,0). \qquad (43)$$

Let $x_\lambda \in W(p,q)$ be an arbitrary weight vector with weight λ. For any $h \in \mathcal{H}$

$$h x_\Lambda = (h,\lambda) x_\lambda. \qquad (44)$$

Since

$$[a_i^-, a_i^+] = h_o - h_i \quad \text{and} \quad [a_{-i}^-, a_{-i}^+] = h_{-i} - h_o \qquad (45)$$

are elements from the Cartan subalgebra, we have from (44)

$$a_i^- a_j^+ x_\Lambda = (h_o - h_i,\ \Lambda) =$$

$$= (h_o - h_i,\ ph^\circ + (p+q)\sum_{j=1}^{n} h^{-j})x_\Lambda = p\,x_\Lambda,$$

$$a_{-i}^- a_i^+ x_\Lambda = (h_{-i} - h_o,\ ph^\circ + (p+q)\sum_{j=1}^{n} h^{-j})x_\Lambda = q x_\Lambda. \qquad (46)$$

From now on instead of x_Λ we shall often write $|0\rangle$. Unifying (26) and (46) we obtain

$$a_i^- a_j^+ |0\rangle = \delta_{ij}\, p|0\rangle \qquad i, j = 1,\dots,n. \qquad (47)$$
$$a_{-i}^- a_{-j}^+ |0\rangle = \delta_{ij}\, q|0\rangle$$

We obtain similar relations as for the parastatistics of order p [3]. There is however one essential difference. For the parastatistics always p=q so that the Fock space representations are labelled by one positive integer. In our case we have two numbers that determine the statistics, one for the particles and another for the antiparticles. We call the pair (p, q) an order of the A-statistics.

The equations (47) together with the commutation relations (22), the hermiticity condition (23) and the requirement the metric in the Fock space to be positive definite determine completely the representation space and the representations of the creation and annihilation operators of order (p, q). The causal A-statistics can be defined by the relations (22), (23) and (47) only. In this case all calculations can be performed without using any Line algebraical properties of the a-operators and even without knowing them. This point of view is convenient for generalization to the case

of infinite number of operators. The knowledge of the Lie algebraical properties however helps a lot in all calculations. Therefore we shall continue to consider a finite number of creation and annihilation operators $a^{\xi}_{\eta 1}, \dots, a^{\xi}_{\eta n}$ and later on we shall let $n \to \infty$.

Remark that the fixed charge causal operators are a-operators in the sense defined in ref. [7]. They also satisfy the same Fock space conditions, for instance (46a) for $\eta = +$. This allows us to draw some immediate conclusions for the causal A-statistics (see Lemma 4 in ref. [7]).

Lemma 1. Given A_{2n} -module of Fock W(p, q). The vector

$$(a^+_{\eta 1})^{\ell_1} (a^+_{\eta 2})^{\ell_2} \dots (a^+_{\eta n})^{\ell_n} |0\rangle \qquad (48)$$

is non-zero if and only if

$$\ell_1 + \ell_2 + \dots + \ell_n \le p \quad \text{for } \eta = +$$
$$\ell_1 + \ell_2 + \dots + \ell_n \le q \quad \text{for } \eta = -. \qquad (49)$$

Therefore there cannot be more than p (more than q) only positive (negative) charge particles. This does not mean that the total amount of particles is restricted from above. An ensemble however with an arbitrary big amount of particles should necessarily contain particles as well as antiparticles. The exact statement is contained in

The Pauli principle. In the Fock space W(p, q) there cannot be more than p+q particles in a single state. The charge Z of an arbitrary ensemble of particles cannot be more than p and less than minus q.

Proof

We have to find the conditions under which the vector

$$x = a^{+}_{\eta_k i_k} \dots a^{+}_{\eta_j i_j} \dots a^{+}_{\eta_1 i_1}|0> \qquad (50)$$

does not vanish. For this purpose we calculate the weight

$$\lambda = \sum_{i=1}^{n} \lambda_{-i} h^{-i} + \lambda_0 h^{\circ} + \sum_{i=1}^{n} \lambda_i h^{i} \qquad (51)$$

of x. Suppose the state x contains $\ell_{\eta i}$ particles in the state (ηi), i.e., $\ell_{\eta i}$ operators $a^{+}_{\eta i}$. The weight of x is a sum of the highest weight Λ and the roots of all creation operators appearing in (50). Taking into account that the root of

$$a^{+}_{\eta i} \quad \text{is} \quad \eta(h^{\eta i} - h^{\circ}) \qquad (52)$$

from (43) we obtain

$$\begin{aligned}\lambda &= \Lambda + \sum_{i=1}^{n} [\ell_{-i}(-h^{-i} + h^{\circ}) + \ell_i(-h^{\circ} + h^{i})] = \\ &= \sum_{i=1}^{n} (p + q - \ell_{-i}) h^{-i} + (p - Z) h^{\circ} + \sum_{i=1}^{n} \ell_i h^{i},\end{aligned} \qquad (53)$$

where

$$Z = \sum_{i=1}^{n} (\ell_i - \ell_{-i}) \qquad (54)$$

is the charge of the state (50). The vector λ is a weight only if

$$0 \le \lambda_{\alpha} \le p + q, \quad \alpha \in N. \qquad (55)$$

For $\alpha = 0$ this gives

$$-q \le Z \le p \qquad (56)$$

that is the total charge of an arbitrary state (50) cannot exceed p and be less than -q.

For the other co-ordinates (55) gives

$$\ell_{-i} \le p + q, \quad \ell_i \le p + q, \quad i = 1, \dots, n, \qquad (57)$$

i.e., in the arbitrary state ηi, $\eta = \pm$, there cannot be more than $p+q$ particles. This proves the Pauli principle.

We should point out that the inequalities (57) give only necessary conditions for x to be different from zero. In order to be sure that the vector (50) does not vanish, one has to show that for any $0 \le j \le k$ all vectors in the sequence

$$a^{+}_{\eta_j i_j} \dots a^{+}_{\eta_1 i_1} |0>, \quad j = 1, 2, \dots, k \qquad (58)$$

are different from zero.

Remark that the Paule principle depends on the order of the statistics (p, q), however it is independent of the number of creation and annihilation operators under consideration (i.e., of the rank 2n of the algebra A_{2n}). In fact the Pauli principle remains valid in the case $n \to \infty$.

In order to study the physical properties of the particles one has to fix the order (p, q) of the statistics and within W(p, q) to determine the basis in the representation space and calculate the matrix elements of the creation and annihilation operators. In a forthcoming paper [10] we shall study the lowest nontrivial representation $W(1,0)$ of the A-statistics, which is an analogue of the Fermi statistics in the paraFermi quantization.

In conclusion we should point out that the charge η need not necessarily be interpreted as an electrical charge. If one considers more than one field then η could be any charge. In the weak interactions for instance the Pauli principles put limitations on the leptonic charge.

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Received by Publishing Department
on September 21 1978.