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TAUBERIAN THEOREM IN QUANTUM FIELD THEORY

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Тауберовы теоремы в квантовой теории поля

Обсуждаются некоторые приложения тауберовых теорем в квантовой теории поля. Установлено однозначное соответствие между асимптотическим поведением формфакторов в бьерковской области и их фурье-образов вблизи светового конуса. Другим применением является исследование асимптотических свойств двухточечных функций Вайтмене, которое основано на многомерном обобщении теорем Харда-Литтлвуда.

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Tauberian Theorem in Quantum Field Theory

Some applications of tauberian theorems in quantum field theory are discussed. One to one correspondence between the asymptotic behaviour of form factors in the Björken domain and of their Fourier transforms near the light cone is shown. Another application is the investigation of asymptotical properties of two point Wightman functions based on the many dimensional generalization of the Hardy-Littlewood theorem.

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1. Introduction

Tauberian theorems are usually assumed to connect the asymptotic behaviour of a (generalized) function in the neighbourhood of zero with that of its Fourier transform (or Laplace transform or some others integral transformations) at infinity. Theorems inverse to the tauberian ones are called abelian.

For the case of one variable the tauberian theory is rather advanced and it has many applications in number theory, in theory of differential equations, in harmonic analysis, and in mathematical physics.

For several variables problems of the tauberian theory are considerably complicated. In this case, only partial results have been obtained. One of the results relates to a many-dimensional generalization of the tauberian theorem by Hardy and Littlewood (see /7/).

A number of important results in the manydimensional tauberian theory has been obtained within the theoretical explanation of the experimental data on the automodel behaviour of some quantities of quantum field theory at high energies in particular of the form factor for deep inelastic lepton-hadron scattering*). One of the main problems of the theory is to establish whether this phenomenon does not contradict the general principles of local quantum field theory; the second problem is to find out what restrictions are imposed by these principles upon possible asymptotices. The treated problem is closely connected with the singular structure of the Fourier transform of form factors in the neighbourhood of the light cone. This problem has been extensively discussed in literature starting from the Björken paper /1/ 1967; in mathematical aspects-after paper /2/ 1972 by Bogolubov, Tavkhelidze and Vladimirov.

The problems under consideration are tightly related to the dimensional analysis of the deep inelastic scattering processes /3/ and to the study of the asymptotic behaviour of inclusive processes /4,5/.

*) Such a behaviour was first anticipated in the work by M.A.Markov /36/.

A detailed list of relevant references is contained in the survey [6].

As explicit forms of form factors are unknown, some information about them follows from the general principles of the local quantum field theory: the Lorentz covariance, spectrality, locality, unitarity and various symmetries [7-9].

Let $F(q)$, $q=(q_0,\vec{q})$, $\vec{q}=(q_1,q_2,q_3)$ be a form factor in the rest system with the hadron mass equal to 1. The distribution $F(q)$ obeys the following properties;

$$0)\ F\in\mathcal{S}'(\mathbb{R}^4),\quad 1)\ F(q)=-F(-q),\quad 2)\ F(q)=0\ \text{ if } \frac{-q^2}{|2q_0|}>1,$$

$$3)\ \tilde{F}(x)=0\ \text{ if } x^2<0,\quad 4)\ F(q_0,\vec{q})=F(q_0,S\vec{q}),\ \forall S\in SO(3)$$

Here $q^2=q_0^2-|\vec{q}|^2=q_0^2-q_1^2-q_2^2-q_3^2$ is the Lorentz square, $\tilde{F}(x)$ is the Fourier transform of $F(q)$. On test functions φ from $\mathcal{S}(\mathbb{R}^4)$ the Fourier transformation $\varphi\to\tilde{\varphi}$ has the form

$$\tilde{\varphi}(x)=\int e^{iqx}\varphi(q)\,dq, \tag{1.1}$$

where $qx=q_0x_0-(\vec{q},\vec{x})$ and $(\vec{q},\vec{x})=q_1x_1+q_2x_2+q_3x_3$ are the Lorentz and the Euclidean scalar products, respectively.

We shall investigate the asymptotic behaviour of the form factor $F(q)$ in the Björken limit

$$\xi=-\frac{q^2}{2q_0}=\text{const}\quad,\ \nu=2q_0\sim+\infty \tag{1.2}$$

and its Fourier transform $\tilde{F}(x)$ in the neighbourhood of the (boundary) light cone $x^2=0$.

In ref. [2] conditions were obtained for the weight function in the Jost-Lehmann representation of the form factor $F(q)$ (see §3.2) which provide the asymptotic behaviour of $F(q)$ in the Björken domain (1.2) and the corresponding asymptotics of $\tilde{F}(x)$ in the neighborhood of the light cone. In papers [10-12] an important notion of the power quasiasymptotics and also a more general scale of the so-called automodel quasiasymptotics of distributions were introduced (see § 2.3 and § 2.4). The equivalence of a power quasiasymptotics of the weight functions at infinity and a canonical quasiasymptotics in the neighbourhood of the light cone was also proved, and these conditions were shown to be

sufficient for the power asymptotics of the form factor in the Björken limit, and under some additional assumptions, to be necessary as well.

In the present paper, without any additional assumptions, we prove the equivalence between a quasiasymptotic behaviour in the neighbourhood of the light cone and an asymptotic behaviour in the Björken domain with respect to any automodel function. Namely, the following statement is valid (the exact formulation is given in §4.2).

Let a form factor $F(q)$ satisfies conditions 0)-4) and let ρ be an automodel function of order α . Then in order that

$$F(q)\sim\rho(\nu)\,f(\xi)\quad\text{in (1.2)} \tag{1.3}$$

is valid it is necessary and sufficient that

$$\tilde{F}(x)\sim\frac{\mathcal{E}(x_0)}{\Gamma(-\alpha-1)}\,\frac{1}{(x_+^2)^2}\,\rho\Big(\frac{1}{x_+^2}\Big)\,\hat{g}(|\vec{x}|),\quad x^2\sim0,\ x_+^2=\theta(x^2)\,x^2 \tag{1.4}$$

is valid. In addition the functions f and $\hat{g}$ are connected by the equation

$$\tilde{f}(\tau)=-\frac{e^{-i\pi/2\,\alpha}}{4^{\alpha+2}\pi^2}\,\frac{\hat{g}(\tau)}{(\tau-i0)^{\alpha+1}} \tag{1.5}$$

This result makes it possible to use the information about the behaviour of form factors in a neighbourhood of the light cone which follows in the framework of some models, for example, from the generalized Wilson type expansion in perturbation theory (see [37], and references therein).

There exist other definitions of the asymptotic behaviour in the Björken domain which differ from our one (see (4.22)). One of those deals with the asymptotics of the form factor [14]

$$F(q+tn)\ \text{as}\ t\to\infty\ \text{in}\ \mathcal{S}'_q(\mathbb{R}^4)\ (\text{if}\ n^2=0\ ). \tag{1.6}$$

Under some additional assumptions it was shown in [15] that a canonical asymptotics in the neighbourhood of the light cone implies the existence of a power asymptotics in the Björken limit in the sense (1.6). In [16,17] without any additional assumptions the equivalence was proved between these two asymptotical behaviours. Thus both the definitions (4.22) and (1.6) of the asymptotical behaviour of a form factor in the Björken domain are equivalent at least for the power case. For other definitions of the asymptotical behaviour in the Björken domain and for related results we refer to papers/18,19/. Various applications of asymptotical properties to other physical problems have been discussed in papers/13,20-22, 31/.

The problem to describe admissible asymptotics which do not contradict the axioms of local quantum field theory was formulated by A.N.Tavkhelidze in 1976. Recently , under some "technical" assumptions the problem on forbidden asymptotics has been cleared up in/12/. The theorem stated in §6 does not contain any additional assumptions.

Finally, in §7 asymptotic properties of the two-point Wightman function are studied by a many-dimensional generalization of the tauberian theorem by Hardy and Littlewood/23/.

The authors are indebted to N.N.Bogolubov and A.N.Tavkhelidze for fruitful discussions.

2. Quasiasymptotics of distributions

1. Automodel functions. A natural scale for definition of quasiasymptotics of distributions is the so-called automodel*) functions/12/. Therefore, we shall start with definition. Let ρ be a positive continuous on $(0,\infty)$ function. We say that ρ is asymptotically automodel (or simply automodel) if for any $a>0$ there exists the limit

$$\lim_{t\to\infty}\frac{\rho(at)}{\rho(t)} = C(a)\neq 0$$

and also the convergence is uniform with respect to any compact set in $(0,\infty)$.

It is not difficult to see that $C(a)$ satisfies the functional equation

$$C(a)\,C(b) = C(ab)$$

from which it follows that $C(a)=a^{\alpha}$ for some real α . In this case we call the function ρ automodel of order α .

We indicate, without proof, some properties of automodel functions.

Lemma 2.1. Let ρ be an automodel function of order α . Then for any $\varepsilon>0$ there exists $t_0>0$ such that

$$\begin{aligned} &a^{\alpha-\varepsilon}\leq\frac{\rho(at)}{\rho(t)}\leq a^{\alpha+\varepsilon} \quad \text{if} \quad t>t_0,\ a>1+\varepsilon \\ &a^{\alpha+\varepsilon}\leq\frac{\rho(at)}{\rho(t)}\leq a^{\alpha-\varepsilon} \quad \text{if} \quad \frac{t_0}{t}<a<1-\varepsilon \end{aligned} \tag{2.1}$$

The next lemma gives a suitable criterion of automodelity.

Lemma 2.2. Let ρ be a positive and differentiable in $(0,\infty)$ function. In order that ρ is an automodel function it is sufficient that the limit

$$\lim_{t\to\infty}\frac{t\rho'(t)}{\rho(t)}=\alpha$$

exists; α being an order of automodeluty of ρ .

Examples. The functions t^{α} , $t^{\alpha}\ln(1+t)$, $t^{\alpha}(2+\sin\ln\ln t)$ are automodel of order α . On the other hand all of them are not mutually asymptotically equivalent in the sense that the limits of their ratios as $t\to\infty$ do not exist (or are equal to zero).

2. The space of distributions $\mathcal{S}'_+$. We recall some standard notions and facts from the theory of tempered distributions (see for example /25,26/). We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the spaces of test (C^∞- rapid decreasing) functions and tempered distributions on $\mathbb{R}^n$, respectively. By $\mathcal{S}'(A)$ where A is a closed set in $\mathbb{R}^n$ we denote the (closed) subspace of $\mathcal{S}'(\mathbb{R}^n)$ consisting of all tempered distributions with supports in A .

Let A be a closed set in $\mathbb{R}^n$ which coincides with the closure of its interiority. We denote by $\mathcal{S}_A$ the space of infinitely differentiable on A functions equipped with a topology by means of the countable set of norms

$$P_m[\varphi]=\sup_{|\alpha|\leq m}\sup_{x\in A}\left(1+|x|^2\right)^{m/2}\left|D^{\alpha}\varphi(x)\right| , \quad m=0,1\ldots$$

We denote its conjugate space as usual by $\mathcal{S}'_A$.

*) Automodel functions are called sometimes proper varying functions (see, for instance, ref./24/).

For a wide class of sets A, in particular, for regular ones (see /35/) with piecewise infinitely differentiable boundaries there is a canonical isomorphism

$$\mathcal{S}'_A = \mathcal{S}'(A). \tag{2.2}$$

The isomorphism (2.2) allows one to define on the space $\mathcal{S}'_A$ all operations which were defined on $\mathcal{S}'(\mathbb{R}^n)$, in particular, differentiation, convolution, Fourier transform and so on. The space of distributions which consists of Fourier transforms of distributions from $\mathcal{S}'_A$ will be denoted by $\tilde{\mathcal{S}}'_A$.

If $A = \bar{\mathbb{R}}^1_+ = [x : x \geq 0]$ then we denote by $\mathcal{S}_+$ and $\mathcal{S}'_+$ the spaces $\mathcal{S}_{\bar{\mathbb{R}}^1_+}$ and $\mathcal{S}'_{\bar{\mathbb{R}}^1_+}$ respectively.

It is convenient to define the convolution operation in $\mathcal{S}'_+$ without referring to the space $\mathcal{S}'(\bar{\mathbb{R}}^1_+)$. Let f belong to $\mathcal{S}'_+$ and φ belong to $\mathcal{S}_+$. As convolution $f \circledast \varphi$ we call the function

$$\left(f \circledast \varphi\right)(t) = \left(f(\cdot), \varphi(t+\cdot)\right), \quad t \geq 0. \tag{2.3}$$

The operation $\varphi \to f \circledast \varphi$ is linear and continuous from $\mathcal{S}_+$ in $\mathcal{S}_+$ and the operation $f \to f \circledast \varphi$ is linear and continuous from $\mathcal{S}'_+$ in $\mathcal{S}'_+$.

Let f and g belong to $\mathcal{S}'_+$. As convolution $f * g$ we call the distribution which is defined by the left-hand of the equation

$$(f * g, \varphi) = (f, g \circledast \varphi), \quad \varphi \in \mathcal{S}_+. \tag{2.4}$$

The convolution $f * g \in \mathcal{S}'_+$; it is commutative and associative ~~algebra with a unit~~ (the unit in $\mathcal{S}'_+$ is δ-function).

We denote by $f_\alpha(t)$, $-\infty < \alpha < \infty$, the standard family of distributions from $\mathcal{S}'_+$ (the canonical kernel of fractional differentiation and integration):

$$f_\alpha(t) = \begin{cases} \theta(t)\dfrac{t^{\alpha-1}}{\Gamma(\alpha)} & \text{if} \quad \alpha > 0 \\ f'_{\alpha-1}(t) & \text{if} \quad \alpha \leq 0 \end{cases}$$

(see for example /25/, s. 147). It is easy to verify that

$$f_\alpha * f_\beta = f_{\alpha+\beta}.$$

If $f \in \mathcal{S}'_+$, then its primitive of order α (its derivative of order $-\alpha$ if $\alpha < 0$) $f^{(-\alpha)}$ is defined by the equality

$$f^{(-\alpha)} = f_\alpha * f.$$

3. Quasiasymptotics at infinity of distributions from $\mathcal{S}'_+$

Now we shall give a definition of quasiasymptotics of distributions from $\mathcal{S}'_+$ /10/.

Definition. Let $f \in \mathcal{S}'_+$ and ρ be some automodel function. We say that the distribution f has a quasiasymptotics at infinity with respect to ρ if there exists the limit

$$\lim_{k\to\infty} \frac{f(kt)}{\rho(k)} = g(t) \not\equiv 0 \quad \text{in} \quad \mathcal{S}'_+.$$

Let us check that if ρ is automodel of order α, then g is homogeneous of degree α:

$$g(at) = \lim_{k\to\infty} \frac{f(kat)}{\rho(k)} = \lim_{k\to\infty} \frac{f(kat)}{\rho(ka)} \lim_{k\to\infty} \frac{\rho(ka)}{\rho(k)} = a^\alpha g(t), \quad a > 0.$$

Hence $g(t) = C f_{\alpha+1}(t)$.

Examples. The distributions $t\,\theta(t)$, $\theta(t)\ln t$, $\theta(t)\,e^{it}$, $\delta'(t)$ have quasiasymptotics at ∞ with respect to the automodel functions t, $\ln(1+t)$, t^{-1} and t^{-2}, respectively as by $k \to \infty$

$$\frac{1}{k}\theta(kt)\,kt \to \theta(t)\,t, \quad \frac{\theta(kt)}{\ln(1+k)}\ln(kt) \to \theta(t)$$

$$k\,\theta(kt)\,e^{ikt} \to i\,\delta(t), \quad k^2\delta'(kt) = \delta'(t).$$

Remark. Let $f \in \mathcal{S}'_+$ be of a compact support and $\varphi \in \mathcal{S}_+$. Then the quantity $\left(f(kt), \varphi(t)\right)$ decreases at infinity, at leasts as $1/k$. This fact shows that quasiasymptotic properties of f depend on its global properties rather than on its behaviour at infinity.

Note that the scale of the automodel functions is a maximal scale with respect to which it is possible to consider a quasiasymptotics of distributions. This is shown by the following

Theorem 2.1. Let $f \in \mathcal{S}'_+$ and ρ be some positive continuous on $(0, \infty)$ function. Further let there exist $\varphi \in \mathcal{S}_+$ such that

$$\lim_{k\to\infty} \frac{1}{\rho(k)} \left(f(kt), \varphi(t) \right) = C > 0$$

and the limit

$$\lim_{k\to\infty} \frac{1}{\rho(k)} \left(f(kt), t\varphi'(t) \right)$$

exists. Then the function ρ is automodel.

Proof. Denoting $\rho_1(k) = (f(kt), \varphi(t))$ we have

$$\frac{k\rho_1'(k)}{\rho_1(k)} = -\frac{(f(kt), \varphi(t) + t\varphi'(t))}{(f(kt), \varphi(t))} \to$$

$$\to -1 + \frac{1}{C} \lim_{k\to\infty} \frac{1}{\rho(k)} \left(f(kt), t\varphi'(t) \right)$$

if $k\to\infty$. By the lemma 2.2. the function ρ_1 is automodel. But $\rho_1(k)/\rho(k) \to C > 0$, so the function ρ is also automodel. The theorem is proved.

Note some properties of quasiasymptotics of distributions.

Lemma 2.3. Let ρ be an automodel function. In order that f from $\mathcal{S}_+'$ has a quasiasymptotics with respect to ρ it is necessary and sufficient that $f^{(-\alpha)}$ has a quasiasymptotics with respect to $t^\alpha \rho(t)$.

The proof of this lemma is completely analogous to the case of power quasiasymptotics.

Lemma 2.4. Let $f \in \mathcal{S}_+'$ and ρ be some automodel function of order $\alpha > -1$. Further let η from $\mathcal{S}_+$ has a compact support. Then there exists $\beta > 0$ which does not depend on η such that for any continuous on $[1,\infty)$ function γ satisfying the conditions

$$0 < \gamma(t) \le t^\beta, \quad t \ge 1 \tag{2.5}$$

the limit relation

$$\frac{1}{\rho(k)} \eta\left(\frac{kt}{\gamma(k)} \right) f(kt) \to 0, \quad k\to\infty \text{ in } \mathcal{S}_+'$$

holds.

Proof. For any $\varphi \in \mathcal{S}_+$ we have

$$\left| \frac{1}{\rho(k)} \left(\eta\left(\frac{kt}{\gamma(k)}\right) f(kt), \varphi(t) \right) \right| =$$

$$= \left| \left(f(t), \frac{1}{k\rho(k)} \eta\left(\frac{t}{\gamma(k)}\right) \varphi(t/k) \right) \right| \le \frac{C}{k\rho(k)} \gamma^N(k), \quad k \ge 1$$

To derive the last inequality, one has to take into account that the order of f is finite (equal to N) and supp η is bounded. As $\alpha > -1$ from extimates (2.1) it follows that

$$\frac{k\rho(k)}{k^\delta} \to 0, \quad k\to\infty$$

if $\delta > 0$ is sufficient small. Choosing now $\beta = \frac{\delta}{N}$ we complete the proof of the Lemma 2.4.

Lemma 2.5. Let ρ be an automodel function of order $\alpha > -1$. Then the function $\theta(t)\rho(t)$ has a quasiasymptotics at ∞ with respect to ρ .

Proof. We choose $\varepsilon > 0$ such that $\alpha - \varepsilon > -1$. Then according to the Lemma 2.1. we choose $t_0 = t_0(\varepsilon)$ such that the estimates (2.1) are valid. Let η be C^∞-function with the properties: $0 \le \eta \le 1$, $\eta(t) \equiv 1$, if $t > t_0 + 1$ and $\eta(t) = 0$ if $t < t_0$. Then owing to the Lemma 2.4 for $\gamma = 1$ we have

$$\lim_{k\to\infty} \theta(kt) \frac{\rho(kt)}{\rho(k)} = \lim_{k\to\infty} \eta(kt) \frac{\rho(kt)}{\rho(k)} \quad \text{in } \mathcal{S}_+' . \tag{2.6}$$

On the other hand for any $\varphi \in \mathcal{S}_+$ the limit relation

$$\frac{1}{\rho(k)} \left(\eta(kt)\rho(kt), \varphi(t) \right) = \int \frac{\eta(kt)\rho(kt)}{\rho(k)} \varphi(t)\,dt \to$$

$$\to \int_0^\infty t^\alpha \varphi(t)\,dt, \quad k\to\infty$$

holds as

$$\eta(kt) \frac{\rho(kt)}{\rho(k)} \to t^\alpha, \quad k\to\infty$$

and moreover from the estimates (2.1) it follows for all $k > t_0$ that

$$0 \le \frac{\eta(kt)\rho(kt)}{\rho(k)} \le \begin{cases} t^{\alpha-\varepsilon}, & 0 \le t \le 1-\varepsilon \\ t^{\alpha+\varepsilon}, & t \ge 1+\varepsilon \end{cases}$$

The Lemma 2.5 immediately follows from the equality (2.6).

Now we formulate a criterion for a distribution to have a quasiasymptotics.

Theorem 2.2. In order that f from $\mathcal{S}'_+$ has a quasiasymptotics at ∞ with respect to an automodel function ρ of order α it is necessary and sufficient that some of its continuous on $[0,\infty)$ primitives $f^{(-\delta)}(t)$ $(\alpha+\delta>-1)$ satisfies the condition

$$\lim_{t\to\infty}\frac{f^{(-\delta)}(t)}{t^{\delta}\rho(t)} = C \neq 0 .$$

The sufficiency follows easily from the Lemmas 2.5 and 2.3. The necessity is proved similar to the power case (see /10/).

In what follows we shall need also the following .

Lemma 2.6. Let ρ_1 and ρ_2 be positive continuous on $[0,\infty)$ functions satisfying $\rho_1(t)/\rho_2(t)\to 1$, $t\to\infty$. Further let ρ_1 has a quasiasymptotics at ∞ with respect to an automodel function ρ of order $\alpha>-1$. Then ρ_2 has also a quasiasymptotics at ∞ with respect to ρ .

Proof. Owing to the theorem 2.2, for the proof of the Lemma 2.6 it is sufficient to establish that for some $\delta\geq 0$

$$\frac{\rho_2^{(-\delta)}(k)}{k^{\delta}\rho(k)} = \frac{1}{\rho(k)}\left(\rho_2(kt)\right)^{(-\delta)}(1)\longrightarrow C\neq 0\,,\ k\to\infty\,. \tag{2.7}$$

To prove (2.7) it is sufficient to show that

$$\frac{1}{\rho(k)}\left(\rho_1(kt)\,\varepsilon(kt)\right)^{(-\delta)}(1)\longrightarrow 0,\ k\to\infty\,, \tag{2.8}$$

where $\varepsilon(t)=\rho_2(t)/\rho_1(t)-1$ ($\varepsilon(t)\to 0$, $t\to\infty$). In turn, (2.8) will be established by the Lemma 2.4 for $\gamma(t)=t^{\beta}$ if we show that

$$\frac{1}{\rho(k)}\left(\eta(k^{1-\beta}t)\,\rho_1(kt)\,\varepsilon(kt)\right)^{(-\delta)}(1)\to 0\,,\ k\to\infty\,,$$

where number β is from the Lemma 2.4 and function $\eta(t)$ is from the proof of the Lemma 2.5. By the mean value theorem we have

$$\left|\frac{1}{\rho(k)}\left(\eta(k^{1-\beta}t)\,\rho_1(kt)\,\varepsilon(kt)\right)^{(-\delta)}(1)\right| =$$

$$\frac{1}{\rho(k)}\left|\int_0^{\infty}\eta(k^{1-\beta}t)\,\rho_1(kt)\,\varepsilon(kt)\,f_{\delta}(1-t)\,dt\right|\leq$$

$$\leq\frac{|\varepsilon(kt_k^*)|}{\rho(k)}\int_0^{\infty}\rho_1(kt)\,f_{\delta}(1-t)\,dt = |\varepsilon(kt_k^*)|\,\frac{\rho_1^{(-\delta)}(k)}{k^{\delta}\rho(k)}\,, \tag{2.9}$$

where t_k^* satisfies the inequalities

$$t_0\,k^{\beta-1}\leq t_k^*\leq 1\,.$$

From here it follows that $\varepsilon(kt_k^*)\to 0$ if $k\to\infty$. On the other hand from the Theorem 2.2. it follows that there exists a number $\delta\geq 0$ such that

$$\frac{1}{k^{\delta}\rho(k)}\,\rho_1^{(-\delta)}(k)\longrightarrow C\,,\ k\to\infty\,.$$

From here and from (2.9) the limit relation follows (2.8) which completes the proof of the Lemma.

4. Quasiasymptotics at zero of distributions

Now we introduce the notion of quasiasymptotics at zero of distributions /10/.

Definition. A distribution f from $\mathcal{S}'(R^1)$ has a quasiasymptotics at zero with respect to an automodel function ρ if there exists the limit

$$\lim_{k\to\infty}\frac{1}{\rho(k)}\,f(x/k) = g(x)\neq 0\quad\text{in } \mathcal{S}'(R^1)\,. \tag{2.10}$$

It should be noted that the quasiasymptotics at zero reflects the pure asymptotic behaviour of a distribution unlike the quasiasymptotics at infinity (see §2.3) which depends on global properties of a distribution. For instance, $\delta(t)$ has a quasiasymptotics at ∞ with respect to $1/t$; on the other hand if a tempered distribution f vanishes in a neighbourhood of zero then for any $N>0$ $k^{N}f(x/k)\to 0$, $k\to\infty$ in $\mathcal{S}'$.

Quasiasymptotics at infinity and at zero are connected by the following simple

Lemma 2.7. A distribution $f\in\mathcal{S}'_+$ has a quasiasymptotics at infinity with respect to an automodel function ρ if and only if its Fourier transform $\tilde{f}$ has a quasiasymptotics at zero with respect to $\rho(x)/x$.

<u>Remark</u>. For the class $\mathcal{S}'_+$ like in the case of a quasiasymptotics at ∞, we have in (2.10) $g(x) = C f_{-\alpha+1}(x)$ and by the Lemma 2.7 the most part of statements for quasiasymptotics at ∞ can be obviously carried over quasiasymptotics at 0.

3. <u>The Jost-Lehmann representation</u>

Here we derive the Jost-Lehmann representation [2] which we shall use later.

1. <u>$\mathcal{B}$ -transformation</u>. For functions φ from $\mathcal{S}_+$ we introduce a modified Fourie-Bessel transformation (we call it $\mathcal{B}$ -transformation [10]) by the formula

$$\mathcal{B}[\varphi](t) = -\int_0^\infty J_0(\sqrt{t\tau})\,\varphi'(\tau)\,d\tau = \varphi(0) - \frac{1}{2}\int_0^\infty \sqrt{\frac{t}{\tau}}\, J_1(\sqrt{t\tau})\,\varphi(\tau)\,d\tau \qquad (3.1)$$

It follows from the properties of the Bessel functions that the transformation $\varphi \to \mathcal{B}[\varphi]$ is continuous (and linear) from $\mathcal{S}_+$ in $\mathcal{S}_+$.

We state the formula

$$\mathcal{B}[\mathcal{B}[\varphi]] = \varphi,\ \varphi \in \mathcal{S}_+, \text{ i.e., } \mathcal{B}^2 = I \text{ on } \mathcal{S}_+. \qquad (3.2)$$

In fact, denoting $\mathcal{B}[\varphi]$ by $\check{\varphi}$ we have

$$\mathcal{B}[\check{\varphi}](\tau) = \int_0^\infty J_0(\sqrt{\tau t}) \int_0^\infty \frac{\partial}{\partial t} J_0(\sqrt{t\tau'})\,\varphi'(\tau')\,d\tau'\,dt =$$

$$= -\frac{1}{2}\int_0^\infty J_0(\sqrt{\tau t}) \int_0^\infty \sqrt{\frac{\tau'}{t}}\, J_1(\sqrt{t\tau'})\,\varphi'(\tau)\,d\tau\,dt\,.$$

Integrating by parts over τ and taking into account that

$$\frac{\partial}{\partial \tau'}\left[\sqrt{\frac{\tau'}{\tau}}\, J_1(\sqrt{t\tau'})\right] = \frac{1}{2} J_0(\sqrt{t\tau'}),\quad J_1(0) = 0$$

we obtain (3.2)

$$\mathcal{B}[\check{\varphi}](\tau) = \frac{1}{4}\int_0^\infty J_0(\sqrt{\tau t}) \int_0^\infty J_0(\sqrt{t\tau'})\,\varphi(\tau')\,d\tau'\,dt = \varphi(\tau)$$

by virtue of the classical inversion formula for the Fourier-Bessel transformation.

It follows from formula (3.2) that $\mathcal{B}$ -transformation is an (linear) isomorphism of $\mathcal{S}_+$ on $\mathcal{S}_+$.

Now in the usual way we extend the $\mathcal{B}$ -transformation on to all distributions f from $\mathcal{S}'_+$ namely

$$(\mathcal{B}[f], \varphi) = (f, \mathcal{B}[\varphi]),\quad f \in \mathcal{S}'_+,\quad \varphi \in \mathcal{S}_+ \qquad (3.3)$$

It follows from (3.3) and (3.2) that

$$\mathcal{B}[\mathcal{B}[f]] = f,\quad f \in \mathcal{S}'_+, \text{ i.e. } \mathcal{B}^2 = I \text{ on } \mathcal{S}'_+ \qquad (3.4)$$

Therefore $\mathcal{B}$-transformation is an (linear) isomorphism of $\mathcal{S}'$ on $\mathcal{S}'$.

Now we shall prove the equality

$$\mathcal{B}[f * g] = \mathcal{B}[f] * \mathcal{B}[g],\quad f \in \mathcal{S}'_+,\ g \in \mathcal{S}'_+. \qquad (3.5)$$

The formula (3.5) means that $\mathcal{B}$ -transformation is an automorphism of the convolution algebra $\mathcal{S}'_+$.

In order to prove the equality (3.5), we note first the validity of the following formula

$$\mathcal{B}[f \circledast \varphi] = (f(\tau), \mathcal{B}_t[\varphi(t+\tau)]),\quad f \in \mathcal{S}'_+,\ \varphi \in \mathcal{S}_+.$$

Then, taking into account the formula (2.3) we are left to prove the equality

$$\mathcal{B}_{t'}[\mathcal{B}_t[\varphi(t'+t)]] = \mathcal{B}[\varphi](\tau'+\tau),\quad \varphi \in \mathcal{S}_+, \qquad (3.6)$$

where $\mathcal{B}_{t'}$ and $\mathcal{B}_t$ are $\mathcal{B}$-transformations with respect to t' and t depending on τ' and τ respectively.

Indeed, the left-hand side of (3.6) is equal to

$$\int_0^\infty\int_0^\infty \varphi''(t'+t)\,J_0(\sqrt{\tau' t'})\,J_0(\sqrt{\tau t})\,dt\,dt'.$$

By putting $t = z\sin^2\theta$, $t' = z\cos^2\theta$ we convert the last integral to the form

$$2\int_0^\infty z\,\varphi''(z) \int_0^{\pi/2} J_0(\sqrt{z\tau'}\cos\theta)\,J_0(\sqrt{z\tau}\sin\theta)\,\sin\theta\cos\theta\,d\theta\,dz =$$

$$= 2\int_0^\infty z\varphi''(z)\,\frac{J_1[\sqrt{z(\tau'+\tau)}]}{\sqrt{z(\tau'+\tau)}}dz =$$

$$= -2\int_0^\infty \varphi'(z)\,\frac{\partial}{\partial z}\left\{\frac{z^{1/2}}{\sqrt{\tau'+\tau}}\,J_1(\sqrt{z(\tau'+\tau)})\right\}dz =$$

$$= -\int_0^\infty J_0(\sqrt{z(\tau+\tau')})\,\varphi'(z)\,dz = \mathcal{B}[\varphi](\tau'+\tau)$$

which proves the equality (3.6). Here we have used the formulas 6.683.2 from /30/ and (3.2).

The definitions (3.1) and (3.4) result immediately in the equality

$$\mathcal{B}[f(\kappa t)] = \frac{1}{\kappa^2}\mathcal{B}[f](\tau/\kappa),\quad f\in\mathcal{S}'_+,\ \kappa>0. \tag{3.7}$$

We prove now the equality

$$\mathcal{B}[f_\alpha] = 4^\alpha f_{-\alpha}. \tag{3.8}$$

Indeed, using (3.7) we are easily convinced of that $\mathcal{B}[f_\alpha]$ is a homogeneous distribution from $\mathcal{S}'_+$ of degree $-\alpha-1$. Therefore

$$\mathcal{B}[f_\alpha] = C_\alpha f_{-\alpha}. \tag{3.9}$$

On the other hand, by (3.5) we have $C_\alpha = C^\alpha$. To calculate the constant C it is sufficient to verify the equation (3.9) for $\alpha=-1$. We have:

$$(\mathcal{B}[f_{-1}],\varphi) = (f_{-1},\mathcal{B}[\varphi]) = \left(\delta',-\int_0^\infty J_0(\sqrt{t\tau})\,\varphi'(\tau)\,d\tau\right) =$$

$$= -\frac{1}{4}\int_0^\infty \tau\,\varphi'(\tau)\,d\tau = \frac{1}{4}\int_0^\infty \varphi(\tau)\,d\tau = \left(\frac{f_1}{4},\varphi\right)$$

so that $C=4$.

2. <u>The Jost-Lehmann representation</u>. Denote by $V^+=\{x:\ x_0>|\vec{x}|\}$ the future light cone in $\mathbb{R}^4$, by $V^-=-V^+$ the past light cone and by $V=V^+\cup V^-=[x:\ x^2>0]$ the light cone. We denote also by $\mathcal{S}'_a(V)$ the set of distributions f from $\mathcal{S}'(\bar{V})$ which are odd in the argument x_0. $\mathcal{S}'_a(\bar{V})$ is a closed subspace of the space $\mathcal{S}'(\mathbb{R}^4)$.

We define a transformation $f\to\phi$ of $\mathcal{S}'_a(\bar{V})$ in $\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3)\sim$ $\sim\mathcal{S}'(\bar{\mathbb{R}}^1_+\times\mathbb{R}^3)$ by the formula

$$(\phi,\varphi) = (f(x_0,\vec{x}),\ x_0\varphi(x_0^2,\vec{x})),\quad f\in\mathcal{S}'_a(\bar{V}),\ \varphi\in\mathcal{S}_+\otimes\mathcal{S}(\mathbb{R}_3) \tag{3.10}$$

The transformation $f\to\phi$ is linear, one-to-one, and continuous from $\mathcal{S}'_a(\bar{V})$ onto $\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3)$. Its inverse transformation $\phi\to f$ is given by the formula

$$(f,\varphi) = \left(\phi(\varkappa,\vec{x}),\ \frac{\varphi(x_0,\vec{x})-\varphi(-x_0,\vec{x})}{2x_0}\Big/_{x_0^2=\varkappa+|\vec{x}|^2}\right) \tag{3.11}$$

$$\phi\in\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3),\quad \varphi\in\mathcal{S}(\mathbb{R}^4).$$

We note that the transformation

$$\varphi\longrightarrow \frac{\varphi(x_0,\vec{x})-\varphi(-x_0,\vec{x})}{2x_0}\Big/_{x_0^2=\varkappa+|\vec{x}|^2}$$

which is present in (3.11) is continuous from $\mathcal{S}(\mathbb{R}^4)$ onto $\mathcal{S}_+\otimes\mathcal{S}(\mathbb{R}^3)$.

Formally, the transformations (3.10) and (3.11) can be written in the form:

$$\phi(\varkappa,\vec{x}) = \varepsilon(x_0)\,f(x_0,\vec{x})\Big/_{x_0^2=\varkappa+|\vec{x}|^2}$$

$$f(x_0,\vec{x}) = \varepsilon(x_0)\,\phi(x^2,\vec{x}),\quad \varepsilon(x_0)=\operatorname{sign} x_0.$$

Now we introduce the transformation

$$\Psi(\lambda,\vec{u}) = \frac{1}{4\pi^2 i}\,F^{-1}_{\vec{x}}\left[\mathcal{B}_\varkappa[\phi(\varkappa,\vec{x})]\right],\quad \phi\in\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3) \tag{3.12}$$

where $\mathcal{B}_\varkappa$ is the $\mathcal{B}$-transformation with respect to $\varkappa$ and $F^{-1}_{\vec{x}}$ is the inverse Fourier transformation with respect to $\vec{x}$ defined according to the formula (1.1):

$$F^{-1}_{\vec{x}}[\varphi](\vec{u}) = \frac{1}{(2\pi)^3}\int e^{-i(\vec{u},\vec{x})}\varphi(\vec{x})\,d\vec{x}.$$

The transformation $\phi\to\Psi$ defined by the formula (3.12) is an isomorphism of $\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3)$ on $\mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3)$. Hence the transformation $f \to \phi \to \psi$ defined by the formulas (3.10) and (3.12) is an isomorphism of $\mathcal{S}'_a(\bar{V})$ on $\mathcal{S}'_+ \otimes \mathcal{S}'(\mathbb{R}^3)$.

Now we are going to prove the equality

$$F^{-1}[f](q_0,\vec{q}) = \int \varepsilon(q_0)\,\delta[q_0^2 - |\vec{q}-\vec{u}|^2 - \lambda]\,\psi(\lambda,\vec{u})\,d\vec{u}\,d\lambda, \qquad (3.13)$$

$$f \in \mathcal{S}'(\bar{V})$$

The formula (3.13) is a formal version of the following explicit formula

$$\left(F^{-1}[f],\varphi\right) = \left(\psi(\lambda,\vec{u}),\ \int \frac{\varphi(q_0,\vec{q}) - \varphi(-q_0,\vec{q})}{2q_0}\Bigg|_{q_0^2=|\vec{q}-\vec{u}|^2+\lambda} d\vec{q}\right) \qquad (3.13')$$

$$\varphi \in \mathcal{S}(\mathbb{R}^4)$$

The equality (3.13) (and respectively, (3.13') is called the Jost-Lehmann representation of a distribution $F^{-1}[f]$ and ψ is called its weight function. We note that the transformation

$$\varphi \to \int \frac{\varphi(q_0,\vec{q}) - \varphi(-q_0,\vec{q})}{2q_0}\Bigg|_{q_0^2=|\vec{q}-\vec{u}|^2+\lambda} d\vec{q}$$

appeared in the formula (3.13') is linear and continuous from $\mathcal{S}(\mathbb{R}^4)$ in $\mathcal{S}_+ \otimes \mathcal{S}(\mathbb{R}^3)$.

It is sufficient to prove the representation (3.13') for test functions φ from $\mathcal{S}(\mathbb{R}^4)$ which are odd in q_0 and, therefore, can be represented in the form

$$\varphi(q) = q_0\,\chi(q_0^2,\vec{q}), \quad \chi \in \mathcal{S}_+ \otimes \mathcal{S}(\mathbb{R}^3). \qquad (3.14)$$

Taking into account the equality (/7/, s. 464)

$$F\left[\varepsilon(x_0)\,\delta(x^2) - \frac{1}{2}\sqrt{\frac{\lambda}{x^2}}\,\varepsilon(x_0)\,\theta(x^2)\,J_1(\sqrt{\lambda x^2})\right] = 4\pi^2 i\,\varepsilon(q_0)\,\delta(q^2-\lambda)$$

for any test functions

$$q_0\,\chi(q_0^2,\vec{q}) = F[x_0\,\hat{\chi}(x_0^2,\vec{x})], \quad \chi \in \mathcal{S}_+ \otimes \mathcal{S}(\mathbb{R}^3)$$

and any $\vec{u} \in \mathbb{R}^3$ we obtain the equality

$$\int e^{i(\vec{u},\vec{x})} \int_0^\infty J_0(\sqrt{\lambda\varkappa})\,\frac{\partial}{\partial\varkappa}\,\hat{\chi}(\varkappa+|\vec{x}|^2,\vec{x})\,d\varkappa\,d\vec{x} = \frac{1}{4\pi^3}\int \chi(\lambda+|\vec{q}-\vec{u}|^2,\vec{q})\,d\vec{q}. \qquad (3.15)$$

From the equations (3.11), (3.12), (3.4) and (3.1) for all test functions φ of the form (3.14) it follows that

$$\left(F^{-1}[f],\varphi\right) = \left(f,\ F^{-1}[\varphi]\right) = \left(\phi(\varkappa,\vec{x}),\ \hat{\chi}(\varkappa+|x|^2,\vec{x})\right) =$$

$$= -4\pi^2 i\left(\psi(\lambda,\vec{u}),\ \int e^{i(\vec{u},\vec{x})}\int_0^\infty J_0(\sqrt{\lambda\varkappa})\,\frac{\partial}{\partial\varkappa}\,\hat{\chi}(\varkappa+|\vec{x}|^2,\vec{x})\,d\varkappa\,d\vec{x}\right).$$

Now the formula (3.13') follows from the equality (3.15).

From the formula (3.13') it follows that the transformations

$$f \to \phi \to \psi \to F^{-1}[f] \to f, \quad f \in \mathcal{S}'_a(\bar{V})$$

are one-to-one and continuous, so in the Jost-Lehmann representation (3.13) a weight function ψ from $\mathcal{S}'_+ \otimes \mathcal{S}'(\mathbb{R}^3)$ is unique and continuously dependent on $f \in \mathcal{S}'_a(\bar{V})$.

The following theorem is valid.

<u>Theorem 3.1</u>. (Jost-Lehmann/28/). Let a distribution $f(q)$ from $\mathcal{S}'(\mathbb{R}^4)$ vanishe in some neighbourhood of the interval

$$I(a) = [q : |q_0| < a_0,\ \vec{q} = \vec{a}]$$

and let its Fourier transform $\tilde{f}(x)$ belong to $\mathcal{S}'_a(\bar{V})$. Then the weight function $\psi(\lambda,\vec{u})$ of f vanishes in an open set $O_{I(a)}$ which consits of such points $(\lambda,\vec{u})$ for which the hyperboloid $q_0^2 - |\vec{q}-\vec{u}|^2 = \lambda$ intersects the interval $I(a)$, i.e.,

$$O_{I(a)} = [(\lambda,\vec{u}) : |\vec{u}-\vec{a}|^2+\lambda < a_0].$$

The proof was given in the oroginal paper by Jost and Lehmann/28/.

<u>Corollary</u>. If $F(q)$ satisfies the conditions 0)-4) §1, then its weight function $\psi(\lambda,\vec{u})$ is radial-symmetric in $\vec{u}$ with a support contained in the set (Fig.1)

$$\bigcup_{a\in \mathbb{R}_3} \sigma_I\left(1+\sqrt{1+\vec a^{\,2}},\, a\right) = \left[(\lambda,\vec u):\ \lambda \geq \left(1-\sqrt{1-|\vec u|^2}\right)^2,\ |\vec u|\leq 1\right] \quad (3.16)$$

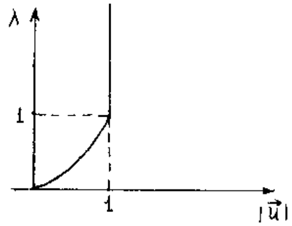

Fig. 1.

4. Formulation of the Main Theorem

1. Rotation invariant distributions. A distribution f from $\mathcal{S}'(\mathbb{R}^n)$ is said to be rotation invariant (or SO(n)-invariant) if it is invariant under all transformations of the SO(n) group:

$$(f,\varphi) = (f(x),\ \varphi(Sx)),\quad \varphi\in \mathcal{S}(\mathbb{R}^{(n)}),\quad S\in SO(n).$$

The space $\mathcal{S}'_{inv}(\mathbb{R}^n)$ of rotation invariant distributions is the closed subspace of the $\mathcal{S}'(\mathbb{R}^n)$ space. The theory of such distributions is wellknown/32,33/. Intuitively it is clear that distributions from $\mathcal{S}'_{inv}(\mathbb{R}^n)$ can be reduced to distributions from $\mathcal{S}'_+$ depending only on the radius $\tau = |\vec x|$. Here we'll present the elements of this theory for the case $n=3$.

The transformation

$$\varphi(x) \to \varphi_{inv}(x) = \frac{1}{4\pi}\int_{|s|=1} \varphi(|x|, s)\, ds = \int_{SO(3)} \varphi(Sx)\,\mu(dS), \quad (4.1)$$

where μ is the normalized Haare measure on the group SO(3), is a continuous and linear mapping from $\mathcal{S}(\mathbb{R}^3)$ onto $\mathcal{S}_{inv}(\mathbb{R}^3)$, $\mathcal{S}_{inv}(\mathbb{R}_3)$ is the space of the rotation invariant test functions from $\mathcal{S}(\mathbb{R}^3_+)$. The space conjugated to $\mathcal{S}_{inv}(\mathbb{R}^3)$ is canonically identified with the $\mathcal{S}'_{inv}(\mathbb{R}^3)$ space.

The transformation

$$\varphi(t) \longrightarrow \varphi(|x|^2) \quad (4.2)$$

is linear and continuous mapping from $\mathcal{S}_+$ into $\mathcal{S}_{inv}(\mathbb{R}^3)$. Let f be from $\mathcal{S}'(\mathbb{R}^3)$. We introduce the distribution f_+ from $\mathcal{S}'_+$ by the formula

$$(f_+,\varphi) = \frac{1}{2\pi}\left(f(x),\ \varphi(|x|^2)\right),\quad \varphi\in\mathcal{S}_+. \quad (4.3)$$

The mapping $f\to f_+$ is (linear) isomorphism of $\mathcal{S}'_{inv}(\mathbb{R}^3)$ on $\mathcal{S}'_+$.

Let us denote by $\mathcal{S}'_a$ the closed subspace which consists of all antisymmetric distributions from $\mathcal{S}'(\mathbb{R}_1)$. For a distribution $f_+\in\mathcal{S}_+$ we introduce the antisymmetrical distribution $f^a\in\mathcal{S}'_a$ by the formula (cf. (3.11))

$$(f^a,\varphi) = \left(f_+(t),\ \frac{\varphi(\sqrt t)-\varphi(-\sqrt t)}{2\sqrt t}\right),\quad \varphi\in\mathcal{S}_a. \quad (4.4)$$

The mapping $f_+\to f^a$ is an (linear) isomorphism of $\mathcal{S}'_+$ on $\mathcal{S}'_a$.

Thus the mapping $f\to f_+\to f^a$ is an (linear) isomorphism of $\mathcal{S}'_{inv}(\mathbb{R}^3)$ on $\mathcal{S}'_a$, the corresponding inverse mappings according to (4.1)-(4.4) are given by the following formulas

$$(f,\varphi) = 2\pi(f_+,\hat\varphi) = 2\pi\left(f(t),\ t\,\hat\varphi(t^2)\right),\quad \varphi\in\mathcal{S}(\mathbb{R}^3), \quad (4.5)$$

where $\quad \hat\varphi(|x|^2) = \varphi_{inv}(x)$

Examples. 1) $\delta\in\mathcal{S}'(\mathbb{R}^3)$

$$(\delta_+,\varphi) = \frac{1}{2\pi}\left(\delta,\ \varphi(|x|^2)\right) = \frac{1}{2\pi}\varphi(0) \text{ i.e. } \delta_+(t) = \frac{1}{2\pi}\delta(t)$$

$$(\delta^{(a)},\varphi) = \left(\frac{\delta(t)}{2\pi},\ \frac{\varphi(\sqrt t)-\varphi(-\sqrt t)}{2\sqrt t}\right) = \frac{\varphi'(0)}{2\pi} \text{ i.e. } \delta^a(t) = -\frac{\delta'(t)}{2\pi}$$

2) Let $f(|x|)$ be a locally integrable tempered function.

Then

$$(f_+,\varphi) = 2\int_0^\infty f(\tau)\,\varphi(\tau^2)\,\tau^2\,d\tau = \int_0^\infty f(\sqrt t)\sqrt t\ \varphi(t)\,dt,$$

$$(f^a,\varphi) = \int_0^\infty f(\sqrt t)\sqrt t\ \frac{\varphi(\sqrt t)-\varphi(-\sqrt t)}{2\sqrt t}\,dt,$$

i.e., $f_+(t) = \sqrt t\, f(\sqrt t)$;

$$f^a(t) = t\,f(t),\ t>0,\quad f^a(t) = t\,f(-t),\ t<0. \quad (4.6)$$

2. Automodel asymptotics in the Björken domain. Let $F(q)$ belong to $\mathcal{S}'(\mathbb{R}^4)$ and satisfy the conditions 1)-4) §1. Because of the rotational invariance of $F(q_0, \vec{q})$ with respect to $\vec{q}$ we can define, according to (4.3), the distribution $F_+(q_0, t)$ from $\mathcal{S}'\otimes\mathcal{S}'_+$:

$$\left(F_+, \varphi\right) = \frac{1}{2\pi}\left(F(q), \varphi(q_0, |\vec{q}|^2)\right), \quad \varphi \in \mathcal{S}\otimes\mathcal{S}_+ . \tag{4.7}$$

For the distribution F_+ in the half plane $q_0 > 0$ of variables (q_0, t) we introduce new variables by the formulas

$$\xi = \frac{t - q_0^2}{2q_0}, \quad \nu = 2q_0, \quad t = \frac{\nu^2}{4} + \nu\xi, \quad F_+(q_0, t) = \mathcal{F}(\nu, \xi), \tag{4.8}$$

i.e., for any $\varphi(\nu, \xi) \in \mathcal{S}(\mathbb{R}^2)$ with a support in $\nu > 0$ we have

$$\left(\mathcal{F}, \varphi\right) = \left(F_+(q_0, t), \frac{1}{q_0}\varphi\left(2q_0, \frac{t - q_0^2}{2q_0}\right)\right) \tag{4.8'}$$

so $\mathcal{F}$ is defined in $\nu > 0$. The last equation, owing to (4.7) can be written now as follows:

$$\left(\mathcal{F}, \varphi\right) = \frac{1}{2\pi}\left(F(q), \frac{1}{q_0}\varphi\left(2q_0, \frac{-q^2}{2q_0}\right)\right). \tag{4.9}$$

By the conditions 1) and 3) the Fourier transform of F belongs to $\mathcal{S}'_a(\bar{V})$. Hence in (4.9) one can use the Jost-Lehmann representation (3.13'):

$$\left(\mathcal{F}, \varphi\right) = \frac{1}{2\pi}\left(\Psi(\lambda, \vec{u}), \int \varphi\left(2\sqrt{|\vec{q}-\vec{u}|^2+\lambda}, \frac{|\vec{q}|^2 - |\vec{q}-\vec{u}|^2 - \lambda}{2\sqrt{|\vec{q}-\vec{u}|^2+\lambda^2}}\right)\frac{d\vec{q}}{2|\vec{q}-\vec{u}|^2+\lambda}\right) =$$

$$= \frac{1}{2}\left(\Psi(\lambda, \vec{u}), \int_0^\infty\int_{-1}^1 \varphi\left(2\sqrt{\tau^2+|\vec{u}|^2-2\tau|\vec{u}|\mu+\lambda},\right.\right. \tag{4.10}$$

$$\left.\left. \frac{-|\vec{u}|^2-\lambda+2\tau|\vec{u}|\mu}{2\sqrt{\tau^2+|\vec{u}|^2+2\tau|\vec{u}|\mu+\lambda}}\right)\frac{\tau^2 d\tau\, d\mu}{\tau^2+|\vec{u}|^2-2\tau|\vec{u}|\mu+\lambda}\right),$$

where Ψ is the weight function for F . As Ψ is rotation invariant with respect to $\vec{u}$ (see 3.2) and the test function on the right-hand side of (4.10), in fact, depends only on $|\vec{u}|^2$, then introducing an antisymmetrical in t distribution $\Psi^a(\lambda, t)$ according to the formulas (4.3) and (4.4), we obtain from (4.10)

$$\left(\Psi^a(\lambda, t), t\int_0^\infty\int_{-1}^1 \varphi\left(2\sqrt{\tau^2+t^2-2\tau t\mu+\lambda}, \frac{-t^2-\lambda+2\tau t\mu}{2\sqrt{\tau^2+t^2-2\tau t\mu+\lambda}}\right)\frac{\tau^2 d\tau\, d\mu}{\tau^2+t^2-2\tau t\mu+\lambda}\right)$$

$$= \pi^{-1}\left(\mathcal{F}, \varphi\right), \quad \varphi \in \mathcal{S}(\mathbb{R}^2), \ \operatorname{supp}\varphi \subset [(\nu, \xi) : \nu > 0]. \tag{4.11}$$

From (3.18) one can conclude that

$$\operatorname{supp}\Psi^a \subset \left[(\lambda, t) : \lambda \geq 0, \ |t| \leq 1\right] = T . \tag{4.12}$$

Therefore distribution Ψ^a belongs to the space $\mathcal{S}'_{a,T} \simeq \mathcal{S}'_a(T)$ (see §2.2).

As in (4.8) $\xi \geq 0$, then from the condition 2) §1 it follows that (Fig.2)

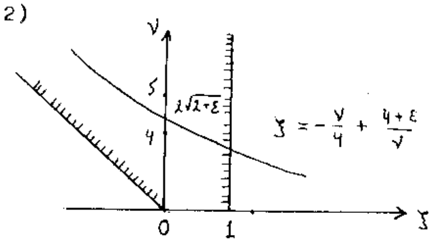

Fig.2.

$$\operatorname{supp}\mathcal{F} \subset \left[(\nu, \xi) : -\frac{\nu}{4} \leq \xi \leq 1, \ \nu > 0\right]. \tag{4.13}$$

Let now φ belong to $\mathcal{S}(\mathbb{R}^2)$ and satisfy the following condition (Fig.2):

$$\operatorname{supp} \subset \left[(\nu, \xi) : \xi \geq -\frac{\nu}{4} + \frac{4+\varepsilon}{\nu}, \ \nu > 0\right] \tag{4.14}$$

for some $\varepsilon > 0$. In the integral in (4.11) we change the variable of integration τ by the formula

$$\nu = 2\sqrt{t^2+\tau^2-2\tau t\mu+\lambda}. \tag{4.15}$$

Then taking into account (4.14) one can conclude that the integration over ν in (4.11) is actually restricted to the domain $\nu^2/4 \geq \lambda + t^2 + \delta$, where $\delta > 0$ depends only on $\mathcal{E}$. As a result, we obtain the following representation

$$\left(\mathcal{F}, \varphi\right) = \frac{\pi}{2}\left(\Psi^a(\lambda,t),\ t\int_{-1}^{1}\int_{2\sqrt{\lambda+t^2+\delta}}^{\infty} \varphi(\nu,\zeta)\,\omega\, d\nu\, d\mu\right), \tag{4.16}$$

where

$$\omega(\lambda,t;\nu,\mu) = 4\frac{-\lambda-t^2+2t^2\mu^2+\nu^2/4}{\nu^2\sqrt{1-\frac{4}{\nu^2}(\lambda+t^2-t^2\mu^2)}} + \frac{4t\mu}{\nu}$$

$$\zeta(\lambda,t;\nu,\mu) = -\frac{\lambda}{\nu} + t\mu\sqrt{1-\frac{4}{\nu^2}(\lambda+t^2-t^2\mu^2)} + \frac{2t^2\mu^2-t^2}{\nu}. \tag{4.17}$$

Let $\chi(\nu,\zeta)$ be a multiplier in $\mathcal{S}'(\mathbb{R}^2)$ (i.e., C^∞ a tempered function together with all its derivatives) which satisfies the condition (4.14). Let's apply the representation (4.16) to the function

$$\varphi(\nu,\zeta) = \chi(\nu,\zeta)\,\varphi_0(\zeta)\,\varphi_1(\nu). \tag{4.18}$$

$$\varphi_0 \in \mathcal{S},\quad \varphi_1 \in \mathcal{D},\quad \operatorname{supp}\varphi_1 \subset (0,\infty)$$

From (4.17) and (4.18) it follows that for any fixed $\nu > 0$ the function

$$t\int_{-1}^{1}\chi(\nu,\zeta)\,\varphi_0(\zeta)\,\omega\, d\mu$$

belongs to $\mathcal{S}_T$ together with all its derivatives and depends continuously on ν in the topology of $\mathcal{S}_T$. Moreover the integration over ν and action of functional Ψ^a in the right-hand side of (4.16) can be interchanged. So we obtain

$$\left(\mathcal{F}, \chi\,\varphi_0\varphi_1\right) = \int \varphi_1(\nu)\,\frac{\pi}{2}\left(\Psi^a(\lambda,t),\ t\int_{-1}^{1}\chi(\nu,\zeta)\,\varphi_0(\zeta)\,\omega\, d\mu\right)d\nu.$$

Thus the following propositions are valid:

1) The function $\left(\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot),\varphi(\cdot)\right)$ is infinitely differentiable for $\nu > 0$ and the following representation

$$\left(\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot),\ \varphi(\cdot)\right) =$$

$$= \frac{\pi}{2}\left(\Psi(\lambda,t),\ t\int_{-1}^{1}\chi(\nu,\zeta)\,\varphi(\zeta)\,\omega\, d\mu\right),\quad \varphi \in \mathcal{S}(\mathbb{R}^1) \tag{4.19}$$

holds if χ is a multiplier in $\mathcal{S}'(\mathbb{R}^2)$ which satisfies the condition (4.14).

2) If besides $\chi(\nu,\zeta) \xrightarrow[\nu\to\infty]{} 1(\zeta)$ in the space of multipliers in $\mathcal{S}'_\zeta(\mathbb{R}^1)$ then

$$\left(\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot),\ \varphi(\cdot)\right) = \frac{\pi}{2}\nu\left(\Psi^a(\nu\lambda,t),\ \varphi_\nu(\lambda,t)\right). \tag{4.20}$$

where

$$\varphi_\nu(\lambda,t) \to t\int_{-1}^{1}\varphi(-\lambda+t\mu)\,d\mu,\ \nu\to\infty \text{ in } \mathcal{S}_T. \tag{4.21}$$

The proposition 2) follows from the representation (4.19) after changing λ to $\nu\lambda$ and from the properties of the functions χ, ζ and ω. According to (4.17) the function $\varphi_\nu(\lambda,t)$ is defined by the formula

$$\varphi_\nu(\lambda,t) = t\int_{-1}^{1}\chi\left(\nu,-\lambda+t\mu\sqrt{1-\frac{4}{\nu^2}(\nu\lambda+t^2-t^2\mu^2)}+\frac{2t^2\mu^2-\mu^2}{\nu^2}\right)\times$$

$$\times\,\varphi\left(-\lambda+t\mu\sqrt{1-\frac{4}{\nu^2}(\nu\lambda+t^2-t^2\mu^2)}+\frac{2t^2\mu^2-\mu^2}{\nu^2}\right)\times$$

$$\times\left(\frac{-4\nu\lambda-4t^2+8t^2\mu^2+\nu^2}{\nu^2\sqrt{1-\frac{4}{\nu^2}(\nu\lambda+t^2-t^2\mu^2)}}+\frac{4t\mu}{\nu}\right)d\mu.$$

<u>Definition 1</u>. We say that a distribution $F(q)$ from $\mathcal{S}'(\mathbb{R}^4)$ satisfying the conditions 1)-4) §1 has an asymptotic in the Björken domain (1.2) with respect to a positive and continuous function ρ if for any function $\chi(\nu,\zeta)$ with the properties

a) χ is a multiplyer in $\mathcal{S}'(\mathbb{R}^2)$;

b) $\operatorname{supp}\chi \subset [(\nu,\zeta):\ \zeta \geqslant -\frac{\nu}{4} + \frac{4+\varepsilon}{\nu},\ \nu > 0]$

c) $\chi(\nu,\zeta) \to 1(\zeta),\ \nu\to\infty$ in the space of multiplyers in $\mathcal{S}'_\zeta(\mathbb{R}^1)$

the following relation

$$2\frac{\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot)}{\nu\rho(\nu)} \to f(\cdot) \not\equiv 0,\ \nu\to\infty \text{ in } \mathcal{S}'(\mathbb{R}^1) \tag{4.22}$$

holds where the mapping $F \sim \mathcal{F}$ is defined by the formula (4.9).

In this case we shall formally write (see (4.6))

$$F(q) \sim \rho(\nu)\,f(\zeta),\ \nu\to\infty. \tag{4.22'}$$

Definition 2. We say that a distribution $\widetilde{F}(x)$ from $\mathcal{S}'(\mathbb{R}^4)$ satisfying the conditions 1) - 4) § 1 has a quasiasymptotic model function ρ of order α if

$$\frac{\Phi(x/k, \vec{x})}{\rho(k)} \to f_{-\alpha+1}(x) \times g(\vec{x}) \not\equiv 0\,, \quad k \to \infty \quad \text{in } \mathcal{S}'_+ \otimes \mathcal{S}'(\mathbb{R}^3) \tag{4.23}$$

where the mapping $\widetilde{F} \to \Phi$ is defined by the formula (3.10). In this case we shall formally write

$$\widetilde{F}(x) \sim \frac{\varepsilon(x_0)}{\Gamma(1-\alpha)}\, \rho\Big(\frac{1}{x_+^2}\Big)\, g(\vec{x})\,, \quad x^2 \sim 0\,. \tag{4.23'}$$

The main theorem. Let a distribution $F(q)$ from $\mathcal{S}'(\mathbb{R}^4)$ satisfy the conditions 1)-4) § 1. The following statements are equivalent:

(i) $F(q)$ has an asymptotic $\rho(\nu) f(\xi)$ in the Björken domain in the sense (4.22), where ρ is some automodel function of order α.

(ii) The weight function $\psi(\lambda, \vec{u})$ of $F(q)$ has a quasiasymptotic at infinity on λ with respect to ρ,

$$\frac{\psi(k\lambda, \vec{u})}{\rho(k)} \to f_{\alpha+1}(\lambda) \times \psi_0(\vec{u}) \not\equiv 0 \quad k \to \infty \quad \text{in } \mathcal{S}'_{T_0}, \tag{4.24}$$

where $T_0 = [(\lambda, \vec{u}) : \lambda \geq 0\,,\ |\vec{u}| \leq 1]$.

(iii) $\widetilde{F}(x)$ has a quasiasymptotic in the neighbourhood of the light cone with respect to the function $k^2 \rho(k)$:

$$\frac{\Phi(x/k, \vec{x})}{k^2 \rho(k)} \to f_{-\alpha-1}(x) \times g(\vec{x})_{k \to \infty} \ \text{in } \mathcal{S}'_+ \otimes \mathcal{S}'(\mathbb{R}^3)\,. \tag{4.25}$$

In addition the following relations

$$g(\vec{x}) = 4^{\alpha+2} \pi^2 i\, \widetilde{\psi}_0(\vec{x}),\quad \psi_0(\vec{u}) = 0\,,\ |\vec{u}| > 1\,, \tag{4.26}$$

$$f(\xi) = -2\pi \big(f_{\alpha+2} * \psi_0^a\big)(-\xi),\quad \psi_0^a(t) = 0\,,\ |t| > 1\,, \tag{4.27}$$

$$\widetilde{f}(\tau) = -\frac{e^{-i\pi/2\,\alpha}}{4^{\alpha+2}\pi^2}\ \frac{\hat{g}(\tau)}{(\tau - i0)^{\alpha+1}} \tag{4.28}$$

hold. Here $\psi_0^a(t)$ is defined by $\psi_0(\vec{u})$ according to the formula (4.5) and $\hat{g}(\tau)$ is defined by the equality $g(\vec{x}) = \hat{g}(|\vec{x}|) = \hat{g}(-|\vec{x}|)$.

Remark 1. The function $g(\vec{x})$ is the Fourier transform of rotation invariant distribution with a compact support, so the function $\hat{g}(\tau)$ can be even continued on $\tau < 0$ as a multiplyer in $\mathcal{S}'$. Therefore the right-hand side in (4.28) makes sense.

Remark 2. The limit relation (4.24) is equivalent to the following one

$$\frac{\psi^a(k\lambda, t)}{\rho(k)} \to f_{\alpha+1}(\lambda) \times \psi_0^a(t),\ k \to \infty \ \text{in } \mathcal{S}'_T\,. \tag{4.29}$$

Remark 3. The condition (i) in the main theorem can be weaken. Namely, one can require the existence of the limit (4.22) only for a single function χ of the form (5.16) (see the proof of the main theorem, § 5.3).

Remark 4. From the main theorem in follows that if $\varphi \in \mathcal{D}$ then for some $\nu_0 = \nu_0(\varphi)$

$$\big(\mathcal{F}(\nu, \cdot),\ \varphi(\cdot)\big) \in C^\infty(\nu > \nu_0)\,.$$

From this it follows if the condition (ii) holds then

$$2\,\frac{\mathcal{F}(\nu, \cdot)}{\nu\, \rho(\nu)} \to f(\cdot)\,,\quad \nu \to \infty \qquad \text{in } \mathcal{D}'\,. \tag{4.22''}$$

This result for the case $\rho(\nu) = \nu^\alpha$ has been proved in /2/.

Remark 5. In (4.23) it is sufficient to require only the existence of non-zero limit. In this case it will automatically be equal to $f_{-\alpha+1}(x) \times g(\vec{x})$ (see remark. § 2.4).

5. The proof of the Main Theorem

1. (ii) → (i). From (4.20) for any χ with the properties a), b), c) and $\varphi \in \mathcal{S}$ we have

$$2\left(\frac{\mathcal{F}(\nu, \cdot), \chi(\nu, \cdot)}{\nu\, \rho(\nu)},\ \varphi(\cdot)\right) = \frac{\pi}{2}\left(\frac{\psi^a(\nu\lambda, t)}{\rho(\nu)},\ \varphi_\nu(\lambda, t)\right).$$

From this taking into account the relations (4.21) and (4.29) (the last one is equivalent to (4.24)) when $\nu \to \infty$ we obtain

$$2\left(\frac{\mathcal{F}(\nu,\cdot)\,\mathcal{X}(\nu,\cdot)}{\nu\,\rho(\nu)},\ \varphi(\cdot)\right)\to\pi\left(f_{\lambda+1}(\lambda)\times\mathcal{Y}_0^a(t),\ t\int_{-1}^{1}\varphi(t\mu-\lambda)\,d\mu\right)=$$

$$=\pi\left(\mathcal{Y}_0^a(t),\ \int_{-1}^{1}\frac{\partial}{\partial\mu}\left(f_{\lambda+2}*\varphi\right)(t\mu)\,d\mu\right)=$$

$$=2\pi\left(\mathcal{Y}_0^a,\ f_{\lambda+2}*\varphi\right)=-2\pi\left(\mathcal{Y}_0^a*f_{\lambda+2},\ \varphi(-\xi)\right).$$

Therefore the limit f in (4.22) exists and the relation (4.27) holds.

2. (ii)$\Leftrightarrow$(iii). From the formulas (3.7) and (3.12) (see §3) we have

$$\frac{\Psi(\kappa\lambda,\vec{u})}{\rho(\kappa)}=\frac{1}{4\pi^2 i}F_{\vec{X}}^{-1}\left[\mathcal{B}_{\varkappa}\left[\frac{\Phi(\varkappa/\kappa,\ \vec{X})}{\kappa^2\rho(\kappa)}\right]\right].\tag{5.1}$$

From this we conclude that quasiasymptotic at ∞ of the weight function Ψ with respect to $\rho(\kappa)$ exists if and only if there exists a quasiasymptotic at 0 of the distribution Φ with respect to $\kappa^2\rho(\kappa)$.

If $\kappa\to\infty$ in (5.1) then using the relations (3.8), (4.24), (4.25) we obtain

$$\Psi_0(\vec{u})\times f_{\lambda+1}(\lambda)=\frac{1}{4\pi^2 i}F^{-1}[g]\times\mathcal{B}[f_{-\lambda-1}]=$$

$$=-\frac{i}{\pi^2}4^{-\lambda-2}F^{-1}[g]\times f_{\lambda+1}(\lambda).$$

It proves the equality (4.26).

The equality (4.28) follows from (4.26) and (4.9) if one uses the formulae

$$\tilde{f}_{\lambda}(\tau)=e^{-i\frac{\pi}{2}\lambda}(\tau+i0)^{-\lambda},\quad\tilde{\Psi}_0(\vec{x})=\frac{2\pi}{\tau}\tilde{\mathcal{Y}}_0^a(\tau),\quad\tau=|\vec{x}|.$$

3. (i)$\Rightarrow$(ii). Let the relation (4.22) hold. We prove the boundedness of the sequence

$$\frac{\Psi(\kappa\lambda,\vec{u})}{\rho(\kappa)},\ \kappa\to\infty\qquad\text{in } \mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3).\tag{5.2}$$

From (3.12) for any $\varphi\in\mathcal{S}_+\otimes\mathcal{S}(\mathbb{R}^3)$ we have

$$\left(\Psi(\kappa\lambda,\vec{u}),\ \varphi(\lambda,\vec{u})\right)=\left(\Phi(\varkappa,\vec{X}),\ \frac{1}{\kappa}\varphi_*(\kappa\varkappa,\vec{X})\right),\ \kappa>0,\tag{5.3}$$

where

$$\varphi_*(\varkappa,\vec{X})=\frac{1}{4\pi^2 i}\mathcal{B}_{\lambda}\left[F_{\vec{u}}^{-1}[\varphi]\right]\in\mathcal{S}_+\otimes\mathcal{S}(\mathbb{R}^3).$$

For any integer $N\geq 0$ let us introduce the function

$$\varphi_N(\varkappa,\vec{X})=\varphi_*(\varkappa,\vec{X})-\sum_{|\alpha|\leq N}\frac{X^{\alpha}}{\alpha!}\tilde{g}_N(\vec{X})\,D_{\vec{X}}^{\alpha}\varphi_*(\varkappa,\vec{0}),$$

where $g_N\in\mathcal{D}(\mathbb{R}^3)$, $g_N(\vec{u})=0$, $|\vec{u}|>2$; $D^{\alpha}\tilde{g}_N(\vec{0})=0$, $1\leq|\alpha|\leq N-1$, $\tilde{g}_N(\vec{0})=1$. By means of (3.12) and (3.18) one can rewrite the equality (5.3) in the following form

$$\left(\Psi(\kappa\lambda,\vec{u}),\ \varphi(\lambda,\vec{u})\right)=\left(\Phi(\varkappa,\vec{X}),\ \frac{1}{\kappa}\varphi_N(\kappa\varkappa,\vec{X})\right),\ \kappa>0,\tag{5.4}$$

where

$$\varphi_N\in\mathcal{S}_+\otimes\mathcal{S}(\mathbb{R}^3),\quad D_{\vec{X}}^{\alpha}\varphi_N(\varkappa,\vec{0})=0\qquad|\alpha|\leq N.\tag{5.5}$$

Here we assume that the function $\varphi_N(\varkappa,\vec{X})$ has been continued onto $\varkappa<0$ as a function from $\mathcal{S}(\mathbb{R}^4)$ in such a way that there exists a sufficient number of its primitives with respect to $\varkappa$.

Taking into account (3.10) we obtain from (5.4)

$$\left(\Psi(\kappa\lambda,\vec{u}),\ \varphi(\lambda,\vec{u})\right)=\left(\tilde{F}(X),\ \frac{X_0}{\kappa}\varphi_N(\kappa X^2,\vec{X})\right),\ \kappa>0.\tag{5.6}$$

By the condition 4) $\tilde{F}(X)$ is rotation invariant of $\vec{X}$. Therefore one can replace the function $\varphi_N(\varkappa,\vec{X})$ in (5.6) by its average over group SO(3) (see 4.1), which by means of (5.6) can be represented in the form

$$|\vec{X}|^{2n}\varphi_n(\varkappa,|\vec{X}|^2),\quad\varphi_n\in\mathcal{S}\otimes\mathcal{S}_+,\quad n=[N/2].$$

Let $\eta(\xi)$ be a C^{∞}-function such that $\eta(\xi) \equiv 1$, $\xi \leq 2$, $\eta(\xi) \equiv 0$, $\xi \geq 3$. Taking into account the condition 3), for any integer $n \geq n_0$ (n_0 is an order of $\widetilde{F}$) we represent the equality (5.6) in the form

$$\left(\Psi(\kappa\lambda, \vec{u}), \varphi(\lambda, \vec{u})\right) = \left(\widetilde{F}(x), \frac{x_0}{\kappa} |\vec{x}|^{2n} \varphi_n(\kappa x^2, |\vec{x}|^2)\, \eta\left(\frac{|\vec{x}|^2}{x_0^2}\right)\right) \tag{5.7}$$

$$\kappa > 0 .$$

Expressing the right-hand side of (5.7) in terms of $F(q)$ (the Fourier transform of $\widetilde{F}(x)$) and then in terms of $F_+(q_0, t)$ (according to the formula (4.3)) after obvious manipulations we obtain

$$\left(\Psi(\kappa\lambda, \vec{u}), \varphi(\lambda, \vec{u})\right) =$$

$$= \left(F(q), \frac{1}{\kappa} \int x_0 |\vec{x}|^{2n} \varphi_n(\kappa x^2, |\vec{x}|^2)\, \eta\left(\frac{|\vec{x}|^2}{x_0^2}\right) e^{i x_0 q_0 - i(\vec{x}, \vec{q})} dx\right) =$$

$$= \left(F_+(q_0, t), \frac{i}{\kappa} \int_0^{\infty}\int_0^{\infty} \sin(q_0\sqrt{\xi})\, \frac{\sin\sqrt{t\tau}}{\sqrt{t}}\, \tau^n \varphi_n(\kappa(\xi-\tau), \tau)\, \eta\left(\frac{\tau}{\xi}\right) d\xi\, d\tau\right)$$

so

$$\left(\frac{\Psi(\kappa\lambda, \vec{u})}{\rho(\kappa)}, \varphi(\lambda, \vec{u})\right) = \left(F_+, g_\kappa\right), \quad \varphi \in \mathcal{S}_+ \otimes \mathcal{S}(\mathbb{R}^3), \ \kappa > 0, \tag{5.8}$$

where

$$g_n(q_0, t) =$$

$$= \frac{i}{\kappa\rho(\kappa)} \int_0^{\infty}\int_0^{\infty} \sin(q_0\sqrt{\xi})\, \frac{\sin\sqrt{t\tau}}{\sqrt{t}}\, \varphi_n(\kappa(\xi-\tau), \tau)\, f_n(\tau, \xi)\, d\xi\, d\tau . \tag{5.9}$$

Here $\varphi_n \in \mathcal{S} \otimes \mathcal{S}_+$ and f_n is a $C^{(n)}(\mathbb{R}^2)$ tempered function together with all its derivatives of an order no more than n and vanishes out of the positive quadrant.

To go further we need the following.

<u>Lemma</u>. Let the relation (4.22) holds. Then there exists such a number $C > 0$ and an integer $m > 0$ that for any $\varphi \in \mathcal{S} \otimes \mathcal{S}_+$ the inequality

$$\left|\left(F_+, \varphi\right)\right| \leq C \|\varphi\|_m = C\left(P_m[\varphi] + Q_m[\varphi]\right) \tag{5.10}$$

holds, where

$$P_m[\varphi] = \sup_{\substack{|q_0| \leq 5,\ \text{or}\ 0 \leq t \leq 5 \\ 0 \leq \alpha+\beta \leq m}} (1+q_0^2)^{m/2} (1+t)^m \left|\frac{\partial^{\alpha+\beta} \varphi(q_0, t)}{\partial q_0^{\alpha}\, \partial t^{\beta}}\right| \tag{5.11}$$

$$Q_m[\varphi] = \int_{\substack{|\nu| > 5 \\ 0 \leq \alpha \leq m}} \sup_{\frac{\nu^2}{4} + \nu\xi \geq 0} (1+\xi^2)^{m/2} \left|\frac{\partial^{\alpha}}{\partial \xi^{\alpha}} \varphi\left(\frac{\nu}{2}, \frac{\nu^2}{4} + \nu\xi\right)\right| \rho(|\nu|)\, \nu^2\, d\nu . \tag{5.12}$$

<u>Proof</u>. Let $\eta_1(t)$ and $\eta_2(t)$ be C^{∞}-functions with the properties

$$\eta_1(t) \equiv \begin{cases} 0, & t \leq 0 \\ 1, & t \geq \delta > 0 \end{cases} \qquad \eta_2(t) \equiv \begin{cases} 0, & |t| \leq 5 \\ 1, & |t| \geq 10 . \end{cases} \tag{5.13}$$

Assuming that $\delta + \varepsilon < 1$, $\varepsilon > 0$ we put

$$F_1(q_0, t) = \eta_1(t - 4 - \varepsilon)\, \eta_2(2q_0)\, F_+(q_0, t) \tag{5.14}$$

$$F_+ = F_1 + F_2 .$$

From (5.13) and (5.14) it follows that

$$\operatorname{supp} F_2 \subset \left[(q_0, t) : |q_0| \leq 5 \ \text{or} \ 0 \leq t \leq 5\right] .$$

Hence for some $C > 0$ and $m \geq 0$

$$\left|\left(F_2, \varphi\right)\right| \leq C\, P_m[\varphi] .$$

To prove the Lemma, we are left to establish the estimate

$$\left|\left(F_1, \varphi\right)\right| \leq C\, Q_m[\varphi] . \tag{5.15}$$

The distribution $F_+(q_0, t)$ is odd on $q_0 = \frac{\nu}{2}$. Therefore it is sufficient to consider the case $q_0 > 0$. So we shall assume that $\eta_2(t) \equiv 0$ when $t < 5$. Then the function

$$\chi(\nu, \xi) = \eta_1\left(\frac{\nu^2}{4} + \nu\xi - 4 - \varepsilon\right) \eta_2(\nu) \tag{5.16}$$

satisfies the conditions a), b) and c) of the definition 1 § 4.2 (see fig.2). From (5.14) and (4.8') we have:

$$
(F_1, \varphi) = \left(F_+(q_0,t),\ \eta_1(t-\psi-\varepsilon)\,\eta_2(2q_0)\,\varphi(q_0,t)\right) =
$$

$$
= \frac{1}{2}\left(\chi(\nu,\xi)\ \mathcal{F}(\nu,\xi),\ \nu\varphi\left(\frac{\nu}{2}, \frac{\nu^2}{4}+\nu\xi\right)\right). \tag{5.17}
$$

As we have proved in §4.2, the distribution $\frac{1}{\sqrt{\rho(\nu)}}\chi(\mathcal{F})(\nu,\xi)$ is continuous with respect to ν when $\nu > 0$, and from the existence of the limit (4.29) the uniform boundedness follows with respect to ν (of some m-th norm of its restriction $(\nu\rho(\nu))^{-1}\chi(\nu,\cdot)\,\mathcal{F}(\nu,\cdot))$ on $\mathcal{S}(\mathbb{R}^1)$. Therefore from (5.17) we obtain (5.15);

$$
|(F_1,\varphi)| = \frac{1}{2}\left|\int_5^\infty \left(\frac{1}{\sqrt{\rho(\nu)}}\chi(\nu,\xi)\,\mathcal{F}(\nu,\xi),\ \varphi\left(\nu/2, \frac{\nu^2}{4}+\nu\xi\right)\right)\rho(\nu)\,\nu^2 d\nu\right| \le
$$

$$
\le C\int_5^\infty \sup_{\substack{\xi \ge -\frac{\nu}{4} \\ 0\le\alpha\le m}} (1+\xi^2)^{m/2}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\varphi\left(\frac{\nu}{2}, \frac{\nu^2}{4}+\nu\xi\right)\right|\rho(\nu)\,\nu^2 d\nu .
$$

The Lemma is proved.

From the Lemma and the equality (5.8) we conclude that the sequence (5.2) is bounded if the numerical sequences $P_m[g_k]$ and $Q_m[g_k]$, $k\to\infty$ are bounded. Denoting the primitive of $\varphi_n(x,\tau)$ on x as $\varphi_n^{(-1)}(x,\tau)$ (from $\mathcal{S}\otimes\mathcal{S}_+$) we rewrite the expression (5.9) for g_k as follows:

$$
g_k(q_0,t) = \frac{i}{k^2\rho(k)}\int_0^\infty\int_0^\infty \sin(q_0\sqrt{s})\Big[\frac{\sin\sqrt{ts}}{\sqrt{t}}\,\varphi_n^{(-1)}(k(s-\tau),\tau)\frac{\partial f_n(\tau,s)}{\partial\tau} +
$$

$$
+ \frac{\sin\sqrt{ts}}{\sqrt{t}}\, f_n(\tau,s)\,\frac{\partial}{\partial\xi}\varphi_n^{(-1)}(k(s-\tau),\xi)\Big|_{\xi=\tau} + \tag{5.18}
$$

$$
+ \cos\sqrt{ts}\ \frac{f_n(\tau,s)}{2\sqrt{\tau}}\,\varphi_n^{(-1)}(k(s-\tau),\tau)\Big]\,ds\,d\tau .
$$

Let us prove that for any integers $l \ge 0$ and $\tau \ge 0$ there exists such an integer n that the expressions

$$
\rho(k)k^3\left(1+\frac{\nu^2}{k^2}\right)^l\left(\frac{k}{\nu}\right)^\tau(1+\xi^2)^{m/2}\frac{\partial^\alpha}{\partial\xi^\alpha}g_k\left(\frac{\nu}{2}, \frac{\nu^2}{4}+\nu\xi\right) =
$$

$$
= \rho(k)k^3\left(1+\frac{4q_0^2}{k^2}\right)\left(\frac{k}{2q_0}\right)^\tau\left[1+\left(\frac{t-q_0^2}{2q_0}\right)^2\right]^{m/2}\left(2q_0\frac{\partial}{\partial t}\right)^\alpha g_k(q_0,t),\quad |\alpha|\le m \tag{5.19}
$$

are uniformly bounded on (k,ν,ξ) when $k\to\infty$, $\nu>5$, $t=\frac{\nu^2}{4}+\xi\nu\ge 0$.

Let $0\le t\le 1$. Substituting the expression (5.18) into (5,19) integrating by parts and using the properties of the function f_n we represent (5.19) as a finite sum of terms of the form

$$
k\,q_0^{-\gamma}t^\delta\int_0^\infty\int_0^\infty e^{iq_0\sqrt{s}}\left(\cos\sqrt{x}\right)^{(\alpha)}\Big|_{x=t\tau} f(\tau,s)\,\psi(k(s-\tau),\tau)\,ds\,d\tau \tag{5.20}
$$

Here $\gamma\ge 0$, $\delta\ge 0$ and function $\psi(x,\tau)$ has the form

$$
C_1\,x^p\left(\frac{\partial}{\partial x}\right)^{\pm q}\left(\frac{\partial}{\partial\tau}\right)^{q_1}\varphi_n(x,\tau), \tag{5.21}
$$

where p, q and q_1 are nonnegative integers (for $q<0$ $\left(\frac{\partial}{\partial x}\right)^q\varphi_n(x,\tau)$ is the primitive of order $-q$ of $\varphi_n(x,\tau)$ on x from $\mathcal{S}\otimes\mathcal{S}_+$. It exists, see the remark after (5.5). The numbers C_1 do not depend on τ and s. The function $f(\tau,s)$ has the form:

$$
C_2\,\tau^{\pm p_1/2}s^{\pm p_2/2}\left(\frac{\partial}{\partial\tau}\right)^{q_2}\left(\frac{\partial}{\partial s}\right)^{q_3}f_n(\tau,s), \tag{5.22}
$$

where p_1, p_2, q_2, q_3 are integer nonnegative numbers; the numbers C_2 do not depend on τ or s. The number n has been choosen large so that all the expressions (5.22) should be continuous in R^2 (with supports in the quadrant $\tau\ge 0$, $s\ge 0$).

The expression (5.20) (owing to (5.21) and (5.21) is estimated for some $N\ge 0$ by the integral

$$
kC_3\int_0^\infty\int_0^\infty\frac{(1+\tau^2+s^2)^N d\tau\,ds}{[1+k^2(s-\tau)^2+\tau^2]^{N+2}} \le C_3\int_{-\infty}^\infty\int_{-\infty}^\infty\frac{\left[1+\tau^2+\left(\frac{t}{k}+\tau\right)^2\right]^N d\tau\,ds}{(1+t^2+\tau^2)^{N+2}} \tag{5.23}
$$

and therefore is uniformly bounded when $k\to\infty$, $0\le t\le 1$, $\nu\ge 5$

Let now $t\ge 1$. Applying the operator $\left(2q_0\frac{\partial}{\partial\tau}\right)^\alpha$ to the expression (5.18) for g_k after integration by parts we obtain that the expressions (5.19) can be represented as a finite sum of terms of the form

$$K t^{-\delta}\left(\frac{K}{q_0}\right)^2\left(1+\frac{4q_0^2}{K^2}\right)^{\ell}\left[1+\frac{(t-q_0^2)^2}{2q_0}\right]^{m/2}\times \tag{5.24}$$

$$\times\int_0^\infty\int_0^\infty e^{\pm i q_0\sqrt{s}\,\pm i\sqrt{t\tau}}\, f(\tau,s)\,\Psi(K(s-\tau),\tau)\,ds\,d\tau\,.$$

where $\delta\ge 0$ and functions Ψ and f have the form (5.21) and (5.22), respectively.

Using the formulae

$$t\,e^{\pm i\sqrt{t\tau}} = -\left(4\tau\frac{\partial^2}{\partial\tau^2}+2\frac{\partial}{\partial\tau}\right)e^{\pm i\sqrt{t\tau}}$$

$$q_0^2\, e^{\pm iq_0\sqrt{s}} = -\left(4s\frac{\partial^2}{\partial s^2}+2\frac{\partial}{\partial s}\right)e^{\pm iq_0\sqrt{s}}$$

we represent each of expressions (5.24) as a finite sum of terms of the form

$$K t^{-\delta} q_0^{-\gamma}\int_0^\infty\int_0^\infty e^{\pm iq_0\sqrt{s}\,\pm i\sqrt{t\tau}}\, f(\tau,s)\,\Psi(K(s-\tau),\tau)\,ds\,d\tau\,, \tag{5.25}$$

where $\gamma\ge 0$, $\delta\ge 0$ and Ψ and f have the form (5.21) and (5.22), respectively. Due to estimates of the type (5.23) the expression (5.25) is uniformly bounded when $K\to\infty$, $t\ge 1$, $|2q_0|\ge 5$.

Therefore the boundedness of the expressions (5.19) is proved. But then the following estimate

$$\rho(|\nu|)\,\nu^2 \sup_{\substack{\frac{\nu^2}{4}+\nu\xi\ge 0\\ 0\le\alpha\le m}} (1+\xi^2)^{m/2}\left|\frac{\partial^\alpha}{\partial\xi^\alpha}\, g_K\left(\frac{\nu}{2},\frac{\nu^2}{4}+\nu\xi\right)\right| \le$$

$$\le \frac{C_4}{K}\left(\frac{\nu}{K}\right)^{2+\tau}\frac{\rho(|\nu|)}{\rho(K)}\left(1+\frac{\nu^2}{K^2}\right)^{-\ell},\quad K\to\infty,\ |\nu|\ge 5$$

holds from which the estimate for the seminorm (5.12)

$$Q_m[g_K]\le\frac{C_4}{K}\int_{|\nu|\ge 5}\left(\frac{\nu}{K}\right)^{2+\tau}\frac{\rho(|\nu|)}{\rho(K)}\left(1+\frac{\nu^2}{K^2}\right)^{-\ell}d\nu\le \tag{5.26}$$

$$\le 2C_4\int_{5/K}^\infty\frac{\rho(Kt)\,(Kt)^{2+\tau}}{\rho(K)\,K^{2+\tau}}\left(1+t^2\right)^{-\ell}dt$$

follows.

We choose $\tau>-3-\alpha$ where α is an order of the automodel function ρ. By the Lemma 2.5 the function $\theta(t-5)\,t^{2+\tau}\rho(t)$ has a quasiasymptotic with respect to the automodel function $t^{2+\tau}\rho(t)$ of the order $\alpha+2+\tau>-1$. From this it follows that there exists $\ell=\ell(\tau,\alpha)$ such that the integral (5.26). converges and has a finite limit when $K\to\infty$. Therefore the seminorm $Q_m[g_K]$ is bounded when $K\to\infty$.

Analogously one can prove the boundedness of the seminorm (5.11) $P_m[g_K]$ when $K\to\infty$, and as a consequence, the boundedness of the sequence (5.2).

The boundedness of (5.2) results in the boundedness of the sequence

$$\frac{\Psi^a(K\lambda,t)}{\rho(K)},\quad K\to\infty \text{ in } \mathcal{S}'_T\,. \tag{5.2'}$$

From this and from the condition (4.22) and from relations (4.20) and (4.21) it follows that the sequence (5.2') converges on the test functions from $\mathcal{S}_T$ of the form

$$\int_{-t}^{t}\varphi(s-\lambda)\,ds\,,\quad \varphi\in\mathcal{S}\,.$$

But then from the equality

$$\left(\frac{\Psi^a(K\lambda,t)}{\rho(K)},\int_{-1}^{1}\varphi\left(\frac{s-\lambda b}{b}\right)ds\right)=$$

$$=\frac{\rho(K/b)}{b\,\rho(K)}\left(\frac{\Psi^a(\frac{K}{b}\lambda,t)}{\rho(K/b)},\int_{-1}^{1}\varphi\left(\frac{s-\lambda}{b}\right)ds\right),\ b>0$$

and from properties of automodel functions $\rho(K)$ (see §2.1) it follows that the sequence (5.2') converges on the test functions from $\mathcal{S}_T$ of the form

$$\int_{-t}^{t}\varphi(\lambda+xs)\,ds\,,\quad \varphi\in\mathcal{S}\,,\quad x>0\,. \tag{5.27}$$

By well-known theorems of functional analysis to prove the weak convergence of the sequence (5.2') in $\mathcal{S}'_T$ it is sufficient to establish that the closed linear hull of the functions of the form (5.27) in topology of $\mathcal{S}_T$ (let us denote it by H) coincides with $\mathcal{S}_{a,T}$ (the subspace of $\mathcal{S}_T$ which consists of odd functions on t).

We note that the differentiation with respect to x of the expression (5.27) is continuous in $\mathcal{S}_T$ and therefore it gives the functions again from H. Differentiating (5.27) on χ $2h$-times and putting $\chi=0$ we conclude that H contains the functions $\varphi^{(2n)}_{(\lambda)}\, t^{2n+1}$, $\varphi\in\mathcal{S}$, $n=0,1\dots$ But $\mathcal{S}_T$ consists of functions which are defined on $\lambda\geq 0$ and $|t|\leq 1$. Therefore H contains functions $\varphi(\lambda)\,t^{2n+1}$, $\varphi\in\mathcal{S}$, $n=0,1\dots$ whose closed linear hull in $\mathcal{S}_T$ coincides with $\mathcal{S}_{a,T}$. Now we conclude $H=\mathcal{S}_{a,T}$ and from the convergence of the sequence (5.2') in $\mathcal{S}'_T$ from remark 2 § 4.2 the convergence of the sequence (4.24) in $\mathcal{S}'_{T_0}$ follows. The main theorem in completely proved.

6. About forbidden asymptotics in the Björken domain

From the previous sections it follows that any automodel asymptotics in the Björken domain does not contradict the general principles of local quantum field theory. In other words for an arbitrary automodel function ρ one can construct the distribution $F(\varphi)$ from $\mathcal{S}'(\mathbb{R}^4)$ with the properties 1)-4) § 1 which has an asymptotic in the Björken limit with respect to ρ (see definition 1, § 4.2), For construction of such a distribution one can take as its weight function

$$\Psi(\lambda,\vec{u})=\frac{d^N}{d\lambda^N}\left[\theta(\lambda-1)\,\lambda^N\rho(\lambda)\right]\times\Psi_0(\vec{u}),\quad \alpha+N>-1,$$

where α is an order of ρ and Ψ_0 is an arbitrary rotation invariant distribution from $\mathcal{S}'(\mathbb{R}^3)$ with the support in the unique ball. (Due to the Lemmas 2.5 and 2.3 the distribution $\Psi(\lambda,\vec{u})$ has a quasiasymptotic at ∞ with respect to ρ, i.e., satisfies the condition (ii) of the main Theorem).

In this connection there arises a question whether any arbitrary asymptotic in the Björken domain is allowed or there exist asymptotics which contradicts the general principles of local quantum field theory and, therefore they are forbidden. The following theorem shows that forbidden asymptotics do exist.

<u>Theorem</u>. Let the distribution $F(\varphi)$ from $\mathcal{S}'(\mathbb{R}^4)$ satisfy the conditions 1)-4) § 1 and it has an asymptotics in the Björken domain with respect to a positive continuous function ρ such that $\theta(t)\,t^N\rho(t)$ has a quasiasymptotic at ∞ with respect to some automodel functions ρ_1 of an order $N+\alpha>-1$ ($N\geq 0$ is an integer). Then $t^N\rho(t)/\rho_1(t)\to C\neq 0$, $t\to\infty$. So ρ is an automodel function of order α.

<u>Remark 1</u>. Under the conditions of the theorem $F(\varphi)$ has an asymptotic in the Björken domain with respect to any positive continuous function ρ_2 such that $\rho(t)/\rho_2(t)\to C\neq 0$, $t\to\infty$. But in this case, according to the Lemma 2.6, the function $\theta(t)\,t^N\rho_2(t)$ also has a quasiasymptotic at ∞ with respect to ρ_1 and therefore it is automodel.

<u>Remark 2.</u> The theorem forbids such asymptotics which have quasiasymptotics but are not automodel. The class of such functions is rather wide. It comprises some oscillating functions, for example

$$\rho(t)=1+\alpha\sin t^\beta,\quad |\alpha|<1,\quad \beta>0\ .$$

<u>Proof of the theorem.</u> It is sufficient to prove the theorem for the case $N=0$. Due to the main Theorem, (i)$\leftrightarrow$(ii), our theorem will be proved if it will be established that the weight function $\Psi(\lambda,\vec{u})$ of $F(\varphi)$ has a quasiasymptotic at ∞ with respect to ρ_1.

First of all like in the proof of the implication (i)$\leftrightarrow$(ii), we shall prove the (weak) boundedness of the sequence

$$\frac{\Psi(k\lambda,\vec{u})}{\rho_1(k)},\quad k\to\infty\quad \text{in}\quad \mathcal{S}'_+\otimes\mathcal{S}'(\mathbb{R}^3). \tag{6.1}$$

The difference is that in the expression (5.9) for g_k the function $\rho(k)$ should be replaced by $\rho_1(k)$ and so the inequality (5.26) $(\tau=0)$ assumes the form

$$Q_m[g_k]\leq C\int_{5/k}^{\infty}\frac{\rho(kt)}{\rho_1(k)}\,t^2\,(1+t^2)^{-\ell}\,dt\ . \tag{6.2}$$

The boundedness of the right-hand side of (6.2) for sufficiently large $\ell\geq 0$ follows from the quasiasymptotical properties of the function $\theta(t)\,\rho(t)$ with respect to $\rho_1(t)$.

From (6.1) the boundedness of the sequence (compare with (5.2'))

$$\frac{\Psi^a(k\lambda,t)}{\rho_1(k)},\quad k\to\infty\quad \text{in}\quad \mathcal{S}'_T \tag{6.1'}$$

follows.

Let us now reveal on which function from $\mathcal{S}_{a,T}$ the sequence (6.1') converges. By the assumptions of the theorem for any $\varphi \in \mathcal{S}$ and χ (with the properties a), b), c) (see definition 1 § 4.2) there exists a limit

$$\left(\frac{\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot)}{\nu\,\rho(\nu)},\ \varphi(\cdot) \right) \to (C,\varphi),\quad \nu \to \infty,\quad C \neq 0. \tag{6.3}$$

Applying the Lemma 2.6 to (6.3) one concludes that there exists the limit

$$\frac{1}{k\,\rho(k)} \int \left(\mathcal{F}(\nu,\cdot)\,\chi(\nu,\cdot), \varphi(\cdot) \right) g\left(\frac{\nu}{k} - x\right) \frac{d\nu}{\nu} \to (C_1,\varphi),\quad k\to\infty,\quad C_1 \neq 0 \tag{6.4}$$

for all real x and $g \in \mathcal{D}(-1,1)$. Using now the formula (4.20) we rewrite the left-hand side of (6.4) in the form

$$\frac{\pi}{2k\rho_1(k)} \int \left(\psi^a(\nu\lambda,t),\ \varphi_\nu(\lambda,t) \right) g\left(\frac{\nu}{k} - x\right) d\nu = \tag{6.5}$$

$$= \frac{\pi}{2\rho_1(k)} \left(\psi^a(k\lambda,t),\ \int \varphi_{k\nu}\left(\frac{\lambda}{\nu},t\right) g(\nu - x)\,\frac{d\nu}{\nu} \right).$$

From this and also from (4.23) and from boundedness of (6.1') we conclude that the sequence (6.1') converges on functions of the form

$$\int \frac{g(\nu-x)}{\nu} \int_{-t}^{t} \varphi\left(s - \frac{\lambda}{\nu}\right) ds\, d\nu,\quad \varphi \in \mathcal{S},\ g \in \mathcal{D}(-1,1),\ x>0 \tag{6.6}$$

Let us denote by H the closure (in $\mathcal{S}_T$) of the linear hull consisting of functions of the form (6.6). We have to prove that $H = \mathcal{S}_{a,T}$. Tending in (6.6) g to δ we conclude that H contains functions of the form

$$\frac{1}{x} \int_{-t}^{t} \varphi\left(s - \frac{\lambda}{x}\right) ds,\quad \varphi \in \mathcal{S},\ x > 0$$

and therefore functions of the form (5.27). Like in the proof of the main theorem in § 5, we derive from this that $H = \mathcal{S}_{a,T}$. The Theorem is proved.

7. Asymptotical properties of the two-point Wightman function

1. Many-dimensional generalization of the Hardy-Littlewood theorem. Let Γ be a convex closed proper solid cone in $\mathbb{R}^n$ with the vertex at 0. We denote by C the interior of the conjugate cone

$$\Gamma^* = [\, y : (y,\xi) \geq 0,\ \forall \xi \in \Gamma \,]$$

$C = \operatorname{int} \Gamma^* \neq \phi$; $T^C = \mathbb{R}^n + iC$ is the tube domain over the cone C; θ_Γ is the characteristic function of the cone Γ; $\mathrm{pr}\,\Gamma = \Gamma \cap S$ is the intersection of Γ with the unique sphere S;

$$\mathcal{K}_C(z) = \int_\Gamma e^{i(z,\xi)} d\xi = i^n \Gamma(n) \int_{\mathrm{pr}\,\Gamma} \frac{d\sigma}{(z,\sigma)^n}$$

is the Cauchy-Szegö kernel of the domain T^C.

Let μ be a non-negative measure in $\mathbb{R}^n$ with support in the cone Γ. We denote by μ_0 the primitive of μ with respect to the cone Γ:

$$\mu_0 = \mu * \theta_\Gamma = \int_{\Gamma \cap (\xi - \Gamma)} \mu(d\xi).$$

We say that the function $f(\xi)$ bounded in the neighbourhood of infinity of $\mathbb{R}^n$ has a weak limit $f_0(\sigma)$ from $\mathcal{L}^\infty(S)$ when $|\xi| \to \infty$ if for any $\varphi \in \mathcal{L}^1(S)$ the following relation

$$\lim_{\tau\to\infty} \int_S f(\tau\sigma)\,\varphi(\sigma)\,d\sigma = \int_S f_0(\sigma)\,\varphi(\sigma)\,d\sigma$$

holds.

Theorem. Let μ be a tempered nonnegative measure with support in the convex proper solid cone Γ and μ_0 be its primitive with respect to Γ. The following propositions are equivalent

(i) For some $\alpha \geq 0$ there exists the limit

$$\lim_{\rho\to+0} \rho^\alpha \int_\Gamma e^{-\rho(s,\xi)} \mu(d\xi) = h(s) \neq 0,\quad s \in \mathrm{pr}\,C. \tag{7.1}$$

(ii) The function $\mu_0(\xi)\,|\xi|^{-\alpha}$ has a weak limit $H(\sigma) \neq 0$ from $\mathcal{L}^\infty(S)$ when $|\xi| \to \infty$.

In this case the functions $h(\mathfrak{z})$ and $H(\sigma)$ are related by the integral equation

$$\Gamma(n+\alpha)\int_{\mathrm{pr}\,\Gamma}\frac{H(\sigma)\,d\sigma}{(\mathfrak{z},\sigma)^{n+\alpha}}=h(\mathfrak{z})\,\mathcal{K}_C(i\mathfrak{z}),\quad \mathfrak{z}\in \mathrm{pr}\,C\,. \tag{7.2}$$

Remark 1. In the case $n=1$, $\Gamma=[0,\infty)$ the integral equation (7.2) is reduced to the following relation

$$\Gamma(1+\alpha)\,H(1)=h(1)$$

and a weak limit of the function $\mu_0(\xi)\,\xi^{-\alpha}$ appears to be its usual limit when $\xi\to\infty$. In this case we have the classical Hardy-Littlewood Theorem.

Remark 2. The solution of the integral equation (7.2) is unique in the class of distributions $\mathcal{E}'(S)$ with support in $\mathrm{pr}\,\Gamma$ (if it exists).

Remark 3. In a recent paper by Drozzinov and Zavialov it has been proved that the function $\mu_0(\tau\sigma)\,\tau^{-\alpha}$ has a usual limit when $\tau\to\infty$, the convergence being uniform on any compact set from pr int Γ and the limit function $H(\sigma)$ being continuous on pr int Γ.

Example. Let the cone C be regular, i.e., $\mathcal{K}_C(z)$ be a divisor of the identity element in the algebra $H(C)$ (see /26/) so that $\mathcal{K}_C^{\alpha}(z)\in H(C)$ for any real α.

Let us introduce the distributions $\theta_\Gamma^{\alpha}(\xi)$ by the following formula

$$\mathcal{K}_C^{\alpha}(z)=L\,[\,\theta_\Gamma^{\alpha}\,], \tag{7.3}$$

where L is the Laplace transform; $\theta_\Gamma^{\alpha}(\xi)$ are tempered homogeneous distributions of degree $n\alpha-n$ with support in the cone $C^*=\Gamma$ and $\theta_\Gamma^{\alpha}*\theta_\Gamma^{\beta}=\theta_\Gamma^{\alpha+\beta}$, $\theta_\Gamma^{0}=\delta$, $\theta_\Gamma^{1}=\theta_\Gamma$; $\theta_\Gamma^{\alpha}(\xi)$ are tempered functions when $\alpha\geq 1$ (see /26/).

Let in (7.1)

$$h(\mathfrak{z})=M\,\mathcal{K}_C^{\alpha/n}(i\mathfrak{z}),\quad \alpha\geq 0\,.$$

It is easy to see that the function

$$H(\sigma)=M\,\theta_\Gamma^{1+\frac{\alpha}{n}}(\sigma) \tag{7.4}$$

is a (unique) solution of the integral equation (7.2).

2. Asymptotical properties of the two-point Wightman function. Two-point Wightman function is as follows

$$W_2(x_1-x_2)=\langle\Psi_0|\,A(x_1)\,B(x_2)\,|\Psi_0\rangle=F[\mu](x_1-x_2)$$

where μ is a non-negative $L_+^{\uparrow}$ -invariant, tempered measure with the support in the closed future light cone $\bar{V}^+$.

The cone $\bar{V}^+$ is self-conjugate, $(\bar{V}^+)^*=\bar{V}^+$ regular and

$$\mathcal{K}_{V^+}(z)=8\pi\,(z^2)^{-2},\quad z^2=z_0^2-z_1^2-z_2^2-z_3^2\,.$$

The Wightman function $W_2(x)$ has an analytic continuation into the future tube $T^+=\mathbb{R}^4+iV^+$. This continuation is given by the Laplace transform of the measure μ: $W_2(z)=L[\mu](z)=F[e^{-yp}\mu](x)$ (we remind that the Fourier or Laplace transforms are defined here on the basis of the Lorentz scalar product $zp=z_0p_0-(\vec{z},\vec{p})$, see §1).

Applying the many-dimensional generalization of the Hardy-Littlewood Theorem, we obtain that the following propositions are equivalent:*)

(i) There exists the limit

$$\lim_{\rho\to+0}\rho^{\alpha}\,W_2(i\rho\mathfrak{z})=h(\mathfrak{z})\not\equiv 0,\quad \mathfrak{z}\in\mathrm{pr}\,V^+ \tag{7.5}$$

for some $\alpha\geq 0$.

(ii) The function $\mu_0(p)\,|p|^{-\alpha}$, where $\mu_0=\mu*\theta_\Gamma$ has a weak limit $H(\sigma)$ from $\mathcal{L}^{\infty}(S)$ when $|p|\to\infty$.

In addition

$$h(\mathfrak{z})=C\,(\mathfrak{z})^{-\alpha/2},\quad H(\sigma)=C\,\frac{8\,\Gamma(\alpha/2+5/2)}{\sqrt{\pi}\,\Gamma(\alpha+4)\,\Gamma(\alpha/2-1)}\,(\sigma^2)^{\alpha/2}. \tag{7.6}$$

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*) This result has been obtained by K.A.Bukin who used another method /38/.

Proof. From (7.5) and from $L_+^\uparrow$ -invariance of the measure μ it follows that the function

$$|y|^{-\alpha} h\left(\frac{y}{|y|}\right) = \lim_{\rho \to 0} \rho^\alpha \int_{\bar{V}^+} e^{-\rho y p} \mu(dp) = \lim_{\rho \to +0} \rho^\alpha \int_{\bar{V}^+} e^{-\rho(y,p)} \mu(dp)$$

is $L_+^\uparrow$ -invariant and homogeneous of degree α in V^+. Hence $h(\xi)$ can be represented as follows

$$h(\xi) = C(\xi)^{-\alpha/2} = C(8\pi)^{-\alpha/4} \mathcal{K}_{V^+}^{\alpha/4}(i\xi), \quad \xi \in pr\, V^+ .$$

From this and (7.4) we obtain

$$H(\sigma) = C(8\pi)^{-\alpha/4}\, \theta_{\bar{V}^+}^{1+\alpha/4}(\sigma), \quad \sigma \in pr\, \bar{V}^+ . \tag{7.7}$$

But the function $\theta_{\bar{V}^+}^{1+\alpha/4}(p)$ is $L_+^\uparrow$ -invariant and homogeneous of degree α with support in the cone $\bar{V}^+$. Therefore

$$\theta_{\bar{V}^+}^{1+\alpha/4}(\sigma) = C_\alpha (\sigma^2)^{\alpha/2}, \quad \sigma \in pr\, \bar{V}^+ . \tag{7.8}$$

We are left now to calculate the coefficient C_α. We have

$$\mathcal{K}_{V^+}^{1+\alpha/4}(i, \vec{0}) = (8\pi)^{1+\alpha/4} = \int_{\bar{V}^+} \theta_{\bar{V}^+}^{1+\alpha/4}\left(\frac{\xi}{|\xi|}\right) |\xi|^\alpha e^{-\xi_0} d\xi =$$

$$= C_\alpha \int_{pr\,\bar{V}^+} (\sigma^2)^{\alpha/2} \int_0^\infty \tau^{3+\alpha} e^{-\tau\sigma_0} d\tau\, d\sigma = C_\alpha \Gamma(4+\alpha) \int_{pr\,\bar{V}^+} (\sigma^2)^{\alpha/2} \sigma_0^{-4-\alpha} d\sigma =$$

$$= 4\pi C_\alpha \Gamma(4+\alpha) \int_0^{\pi/4} (\cos^2\theta - \sin^2\theta)^{\alpha/2} (\cos\theta)^{-4-\alpha} \sin^2\theta\, d\theta =$$

$$= 4\pi C_\alpha \Gamma(4+\alpha) \int_0^1 (1-u^2)^{\alpha/2} u^2 du = 2\pi C_\alpha \Gamma(4+\alpha) B\left(\tfrac{3}{2}, \tfrac{\alpha}{2}+1\right) =$$

$$= \pi^{3/2} C_\alpha \Gamma(4+\alpha) \frac{\Gamma(\alpha/2+1)}{\Gamma(\alpha/2+5/2)} .$$

From this formula, and (7.8) and (7.7) the second formula (7.6) follows.

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