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TWO-LOOP RENORMALIZATION OF THE QCD IN AN ARBITRARY GAUGE

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Перенормировка квантовой хромодинамики в двухпетлевом приближении в произвольной калибровке

Функция Гелл-Манна-Лоу и аномальные размерности пропагаторов квантовой хромодинамики вычислены в двухпетлевом приближении для произвольного значения калибровочного параметра.

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Two-Loop Renormalization of the QCD in an Arbitrary Gauge

The Gell-Mann-Low function and anomalous dimensions of vector, quark and ghost propagators are calculated at the two-loop level in an arbitrary gauge.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

The earliest use of the renormalization group (RG) can be found in papers[1]. The renormalization group equations give the high energy behaviour of the Green functions. Gross, Wilczek and Politzer [2] have discovered that the non-Abelian gauge theories like QCD possess asymptotic freedom, what means that the high momentum regime is controlled by a weak coupling region.

The effective strength of interaction actually decreases at short distances, what results in the Bjorken scaling up to the logarithmic deviations. This has strengthened the growing conviction that QCD describes the strong interactions. An important development of RG equations resulted from the use of the dimensional regularization and renormalization[3,4]. The two-loop renormalization of the non-Abelian gauge theories in particular gauges is studied in papers[5], whereas for arbitrary gauge parameter, α , in pure Yang-Mills in ref.[6].

In this work we present our calculations of $\beta(g)$ and anomalous dimensions of vector, quark and ghost propagators at the two-loop level for an arbitrary gauge parameter, α , in QCD.

2. The RG Equations for QCD

The Lagrangian of QCD in terms of "bare" fields, A^a_μ, ψ, φ, "bare" gauge coupling, g_B, and gauge parameter, α_B, is:

$$\mathcal{L} = -\frac{1}{4}\{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_B f^{abc} A^b_\mu A^c_\nu\}^2 - \frac{1}{2\alpha_B}(\partial_\mu A^a_\mu)^2 + \qquad (2.1)$$

$$+\sum_k i\bar{\psi}_k \gamma_\mu(\partial_\mu - i g_B T^a A^a_\mu)\psi_k + \overset{*}{\varphi}{}^a \partial_\mu(\partial_\mu \delta^{ac} + g_B f^{abc} A^b_\mu)\varphi^c,$$

where k is the flavor index, and T^a are the generators of the quark representation of the colour gauge group, G. They obey the commutation relations

$$[T^a, T^b] = i f^{abc} T^c$$

in which the coefficients f^{abc} are the structure constants of G. A^a_μ, ψ_k, φ in equation (2.1) are the vector, quark and Faddeev--Popov ghost fields, respectively. The statement of renormalizability implies that "bare" fields and couplings can be expressed in terms of the renormalized ones so that the Green-functions and cross-sections become finite. In the dimensional renormalization scheme these formulas are as follows:

$$A_\mu = Z_2^{1/2} A^R_\mu, \quad \varphi = \tilde{Z}_2^{1/2} \varphi_R, \quad \psi_k = \tilde{\tilde{Z}}_2^{1/2} \psi^R_k, \quad \alpha_B = Z_2 \alpha_R, \tag{2.2}$$

$$g_B = \mu^\varepsilon Z_1 Z_2^{-3/2} g_R = \mu^\varepsilon \tilde{Z}_1 Z_2^{-1/2} \tilde{Z}_2^{-1} g_R = \mu^\varepsilon \tilde{\tilde{Z}}_1 Z_2^{-1/2} \tilde{\tilde{Z}}_2^{-1} g_R ,$$

where $\varepsilon = (4-n)/2$, n is the space-time dimension, μ is the extra mass used in the dimensional regularization scheme, and Z_1, $\tilde{Z}_1$, $\tilde{\tilde{Z}}_1$ are the remormalization constants of vertices

$$f^{abc} \partial_\mu (A^R_\nu)^a (A^R_\mu)^b (A^R_\nu)^c , \quad f^{abc} \overset{*}{\varphi}{}^a_R (A^R_\mu)^b \varphi^c_R , \quad \bar{\psi}^R A^R_\mu \psi^R ,$$

respectively. The renormalization constants Z are expanded over the inverse powers of ε :

$$Z = 1 + \sum_{\nu=1}^{\infty} \frac{b_\nu(g_R^2, \alpha_R)}{\varepsilon^\nu} , \tag{2.3}$$

where b_ν are calculated within perturbation theory. The gauge invariance leads to the Slavnov-Taylor identities /7/:

$$Z_1 Z_2^{-1} = \tilde{Z}_1 \tilde{Z}_2^{-1} = \tilde{\tilde{Z}}_1 \tilde{\tilde{Z}}_2^{-1} . \tag{2.4}$$

Hence, it is sufficient to calculate only four constants; $Z_1, \tilde{Z}_1, Z_2, \tilde{Z}_2$ others are found by (2.4).

From (2.2) and (2.3) one obtains

$$g_B^2 = \mu^{2\varepsilon} g_R^2 \left(1 + \sum_{\nu=1}^{\infty} \frac{a_\nu(g_R^2)}{\varepsilon^\nu}\right) . \tag{2.5}$$

The coefficients a_ν are independent of α_R . Changing our unit of mass μ , we have to change our renormalized parameters g_R, α_R in order to obtain the same g_B^2 , α_B and the renormalized S-matrix. This is just the renormalization invariance of the theories in the scheme of t'Hooft /4/. Differentiating equation (2.5) with respect to $\ln \mu^2$ under the condition $g_B^2 = const$, we obtain in the limit $\varepsilon \to 0$:

$$\left.\frac{d g_R^2(\mu)}{d \ln \mu^2}\right|_{g_B^2 = const} = \beta(g_R^2(\mu))$$

$$\beta(g^2) \equiv g^4 \frac{d a_1(g^2)}{d g^2} \tag{2.6}$$

$$\frac{d a_{\nu+1}}{d g^2} = \frac{d a_1}{d g^2} \left(a_\nu + g^2 \frac{d a_\nu}{d g^2} \right), \quad \nu = 1, 2, \ldots$$

It follows from the last equation that the ν-loop level coefficients of expansion (2.5) are determined by the one-loop coefficients.

Now, we define the RG equation for the renormalized vector propagator $D^R_{\mu\nu}$. By the definition

$$D^R_{\mu\nu}(k,\mu^2,g^2_R,\alpha_R) = \lim_{\varepsilon\to 0} \frac{\int A^R_\mu A^R_\nu \exp\mathcal{L}(A_\mu,\psi,\varphi,g_B,\alpha_B)\prod_x dA^R_\mu d\psi_R d\varphi_R}{\int \exp\mathcal{L}(A_\mu,\psi,\varphi,g_B,\alpha_B)\prod_x dA^R_\mu d\psi_R d\varphi_R}. \tag{2.7}$$

After passing to the "bare" fields

$$A^R_\mu \to Z_2^{-1/2} A_\mu\ ,\ \varphi_R \to \tilde{Z}_2^{-1/2}\varphi\ ,\ \psi_R \to \tilde{\tilde{Z}}_2^{-1/2}\psi$$

the integral (2.7) can be written as follows:

$$\lim_{\varepsilon\to 0} Z_2^{-1} \frac{\int A_\mu A_\nu \exp\mathcal{L}(A,\psi,\varphi,g_B,\alpha_B)\prod_x dA\, d\varphi\, d\psi}{\int \exp\mathcal{L}(A,\psi,\varphi,g_B,\alpha_B)\prod_x dA\, d\varphi\, d\psi}.$$

Hence, it follows that Z_2 is the renormalization constant of the vector propagator:

$$G^R_{\mu\nu}(k,\mu^2,g^2_R,\alpha_R) = \lim_{\varepsilon\to 0} Z_2^{-1}(g^2_R,\alpha_R,\varepsilon)\, G_{\mu\nu}(k,g^2_B,\alpha_B). \tag{2.8}$$

Differentiating this equation with respect to $\ln\mu^2$, one obtains

$$\left[\frac{\partial}{\partial\ln\mu^2} + \beta(g^2_R)\frac{\partial}{\partial g^2_R} + \delta(g^2_R,\alpha_R)\alpha_R\frac{\partial}{\partial\alpha_R} - \gamma(g^2_R,\alpha_R)\right] G^R_{\mu\nu}(k,\mu^2,g^2_R,\alpha_R) = 0, \tag{2.9}$$

where

$$\gamma(g^2_R,\alpha_R) = \delta(g^2_R,\alpha_R) = \frac{d\ln\alpha_R}{d\ln\mu^2} = \frac{d\ln Z_2^{-1}}{d\ln\mu^2}. \tag{2.10}$$

The last equation is a consequence of the equation $\alpha_B = Z_2\alpha_R$ and the requirement $\alpha_B = const$ under variation of μ.

Representing $G^R_{\mu\nu}(k)$ in the form

$$G^R_{\mu\nu}(k) = (k^2 g^{\mu\nu} - k^\mu k^\nu)d^R + \alpha_R k_\mu k_\nu$$

we can rewrite equation (2.9) in the form

$$\left[\frac{\partial}{\partial\ln k^2} - \beta(g^2_R)\frac{\partial}{\partial g^2_R} - \delta(g^2_R,\alpha_R)\alpha_R\frac{\partial}{\partial\alpha_R} + \gamma(g^2_R,\alpha_R)\right] d^R\left(\frac{k^2}{\mu^2},g^2_R,\alpha_R\right) = 0. \tag{2.11}$$

Analogous equations can be written for the dimensionless parts of other propagators G, with γ_G instead of γ:

$$\gamma_G(g^2_R,\alpha_R) \equiv \frac{d\ln Z_G^{-1}}{d\ln\mu^2}. \tag{2.12}$$

Substituting (2.3) into (2.12), taking into account the finiteness of γ_G and comparing the coefficients at $\varepsilon^{-\nu}$, one obtains

$$\gamma_G(g^2_R,\alpha_R) = g^2_R\frac{\partial b_1}{\partial g^2_R}, \tag{2.13}$$

$$g^2_R\frac{\partial b_{\nu+1}}{\partial g^2_R} = \beta(g^2_R)\frac{\partial b_\nu}{\partial g^2_R} + g^2_R\frac{\partial b_1}{\partial g^2_R}b_\nu + \delta\alpha_R\frac{\partial b_\nu}{\partial\alpha_R}. \tag{2.14}$$

The functions γ, δ in (2.10) are determined analogously to (2.13) with the only defference that one has to replace b_1 by the corresponding expansion coefficient of Z_2.

In conclusion let us write the solution of equation (2.11)/8/:

$$d_R(x,g^2_R,\alpha_R) = d_R(1,g^2(x),\alpha(x))\exp\left\{-\int_1^x \gamma(g^2(t),\alpha(t))\frac{dt}{t}\right\}, \tag{2.15}$$

where

$$\frac{dg^2(x)}{d\ln x} = \beta(g^2(x))\ ,\ g^2(1) = g^2_R$$

$$\frac{d\ln\alpha(x)}{d\ln x} = \delta(g^2(x),\alpha(x)),\ \alpha(1) = \alpha_R. \tag{2.16}$$

Here $g^2(x)$, $\alpha(x)$ are called the "effective" charge and gauge parameters, respectively.

Our aim is to calculate the Gell-Mann-Low function $\beta(g)$ and anomalous dimensions γ for the vector, quark and ghost propagators in the QCD. In what follows we represent the results.

3. Results

$$Z_2 = 1 - \frac{A}{\varepsilon}\left(\left(\frac{\alpha}{2} - \frac{13}{6}\right)C_2 + \frac{4tN}{3}\right) - \frac{A^2}{\varepsilon}\left(\left(\frac{\alpha^2}{8} + \frac{11\alpha}{16} - \frac{59}{16}\right)C_2^2 + 2NtT^2 + \frac{5}{2}NtC_2\right) + \frac{A^2}{\varepsilon^2}\left(\left(\frac{\alpha^2}{4} - \frac{17\alpha}{24} - \frac{13}{8}\right)C_2^2 + NtC_2\left(1 + \frac{2\alpha}{3}\right)\right) \tag{2.17}$$

$$\tilde{Z}_2 = 1 + \frac{A}{\varepsilon}\left(\frac{3}{4} - \frac{\alpha}{4}\right)C_2 + \frac{A^2}{\varepsilon}\left(\left(\frac{\alpha}{32} + \frac{95}{96}\right)C_2^2 - \frac{5NtC_2}{12}\right) + \frac{A^2}{\varepsilon^2}\left(\left(\frac{3\alpha^2}{32} - \frac{35}{32}\right)C_2 + \frac{\alpha^2}{2}T^2\right)$$

$$\tilde{\tilde{Z}}_2 = 1 - \frac{A}{\varepsilon}\alpha T^2 - \frac{A^2}{\varepsilon}T^2\left(\left(\frac{25}{8} + \alpha + \frac{\alpha^2}{8}\right)C_2 - Nt - \frac{3}{4}T^2\right) + \frac{A^2}{\varepsilon^2}T^2\left(\left(\frac{\alpha^2}{4} + \frac{3\alpha}{4}\right)C_2 + \frac{\alpha^2}{2}T^2\right)$$

$$\tilde{Z}_1 = 1 - \frac{A}{\varepsilon}\frac{\alpha C_2}{2} + \frac{A^2}{\varepsilon}\left(-\frac{\alpha^2}{16} - \frac{5\alpha}{16}\right)C_2^2 + \frac{A^2}{\varepsilon^2}\left(\frac{\alpha^2}{4} + \frac{3\alpha}{8}\right)C_2^2$$

$$\beta(g^2) = -\beta_1 g^4 - \beta_2 g^6, \quad \beta_1 = \frac{1}{(4\pi)^2}\left(\frac{11}{3}C_2 - \frac{4tN}{3}\right),$$

$$\beta_2 = \left(\frac{34}{3}C_2^2 - 4NtT^2 - \frac{20Nt}{3}C_2\right)\frac{1}{(4\pi)^4}$$

$$\delta = \gamma = (\delta_1 - \delta_2\alpha)g^2 + (\delta_3 - \delta_4\alpha - \delta_5\alpha^2)g^4,$$

$$\delta_1 = \left(\frac{13}{6}C_2 - \frac{4tN}{3}\right)\frac{1}{(4\pi)^2}, \quad \delta_2 = \frac{C_2}{2(4\pi)^2}$$

$$\delta_3 = \left(\frac{59}{8}C_2^2 - 4NtT^2 - 5NtC_2\right)\frac{1}{(4\pi)^4}, \quad \delta_4 = \frac{11}{8}\frac{C_2^2}{(4\pi)^4}, \quad \delta_5 = \frac{C_2^2}{4(4\pi)^4}$$

$$\gamma_\psi = -\gamma_0\alpha g^2 - (\gamma_1 + \gamma_2\alpha + \gamma_3\alpha^2)g^4,$$

$$\gamma_0 = \frac{T^2}{(4\pi)^2}, \quad \gamma_1 = \left(\frac{25}{4}C_2 - 2tN - \frac{3}{2}T^2\right)\frac{T^2}{(4\pi)^4}$$

$$\gamma_2 = \frac{2C_2T^2}{(4\pi)^4}, \quad \gamma_3 = \frac{C_2T^2}{4(4\pi)^4}$$

$$\gamma_\varphi = \left(\frac{3}{4} - \frac{\alpha}{4}\right)C_2A + \left(\left(\frac{\alpha}{16} + \frac{95}{48}\right)C_2^2 - \frac{5NtC_2}{6}\right)A^2$$

Here N is the number of flavours, $A = \frac{g^2}{(4\pi)^2}$; C_2, t, T^2 are the parameters of colour group and its representation:

$$C_2\delta^{ab} = f^{acd}f^{bcd}$$

$$\mathrm{Tr}\,T^aT^b = t\,\delta^{ab}$$

$$T^aT^a = T^2 I,$$

I being unit matrix.

Taking $SU(n)$ as a colour-gauge group and its fundamental representation as a quark representation, one has

$$C_2 = n, \quad t = \frac{1}{2}, \quad T^2 = \frac{n^2-1}{2n}.$$

It is interesting that the renormalization constant of the vertex $\bar{\varphi}\varphi A$, $\tilde{Z}_1$, does not depend on the quark sector. We do not know whether this fact is conserved at higher loop level or not.

In the following section we discuss the asymptotic of the "effective" parameters and propagators.

4. High Energy Asymptotics of QCD

For the investigation of the asymptotics it is convenient to introduce the notation:

$$g^2 \equiv u \ , \quad \ln x = t$$

Then equation (2.15) takes the form:

$$\frac{du}{dt} = \beta(u) \ , \quad u(0) = u_R \tag{4.1}$$

$$\frac{d \ln \alpha(t)}{dt} = \delta(u,\alpha) \ , \quad \alpha(0) = \alpha_R$$

$$d_R(t, u_R, \alpha_R) = d_R(0, u(t), \alpha(t)) \exp - \int_0^t \gamma(u(t), \alpha(t)) dt$$

or

$$t = \int_{u_R}^{u} \frac{du}{\beta(u)} \tag{4.2}$$

$$\frac{d \ln \alpha(u)}{du} = \frac{\delta(u,\alpha)}{\beta(u)}$$

$$d_R(t, u_R, \alpha_R) = d_R(0, u, \alpha(u)) \exp - \int_{u_R}^{u} \gamma(u, \alpha(u)) \frac{du}{\beta(u)} \ .$$

Now we write out the solutions of equations (4.2) at the one- and two-loop level under the condition that $\alpha_R \neq 0$.

I. One-loop level

$$u(t) = u_R (1 + \beta_1 u_R t)^{-1} \sim (\beta_1 t)^{-1}$$

$$\alpha(u) = \alpha_0 \Big(1 + \frac{\alpha_0 - \alpha_R}{\alpha_R} \Big(\frac{u}{u_R}\Big)^{\delta_1/\beta_1}\Big)^{-1} , \quad \alpha_0 = \frac{\delta_1}{\delta_2} \tag{4.3}$$

$$\Gamma_{\psi\psi} \sim u^{-\alpha_0 \gamma_0/\beta_1} \Big(1 + \frac{\alpha_0 - \alpha_R}{\alpha_R} \Big(\frac{u}{u_R}\Big)^{\delta_1/\beta_1}\Big)^{\gamma_0/\delta_2}$$

$$\Gamma_{AA} = \frac{\alpha_R}{\alpha} = \frac{\alpha_R}{\alpha_0} \Big(1 + \frac{\alpha_0 - \alpha_R}{\alpha_R} \Big(\frac{u}{u_R}\Big)^{\delta_1/\beta_1}\Big)$$

For $n = 4$, $N = 4, 5, 6$ all the parameters in (4.3) are positive.

II. Two-loop level

$$u(t) \sim (\beta_1 t)^{-1} \Big(1 - u_0 \frac{\ln t}{t} + o\Big(\frac{\ln t}{t}\Big)\Big) \ , \quad u_0 = \beta_2/\beta_1^2$$

$$\alpha(u) = \alpha_0 (1 + cu) \Big(1 + \frac{\alpha_0 - \tilde{\alpha}_R}{\tilde{\alpha}_R} \Big(\frac{u}{u_R}\Big)^{\delta_1/\beta_1}\Big)^{-1} + o(u)$$

$$\Gamma_{\psi\psi} \sim u^{-\alpha_0 \gamma_0/\beta_1} \Big[\Big(1 + \frac{\alpha_0 - \tilde{\alpha}_R}{\tilde{\alpha}_R} \Big(\frac{u}{u_R}\Big)^{\delta_1/\beta_1}\Big)^{\gamma_0/\delta_2} (1 - du)\Big] + o(u) \tag{4.4}$$

$$\tilde{\alpha}_R = \frac{\alpha_R}{1 + c u_R} \ , \quad c = \frac{1}{\delta_1 - \beta_1} \Big(\delta_3 - \frac{\beta_2}{\beta_1} \delta_1 - \Big(\delta_4 - \frac{\beta_2}{\beta_1} \delta_2\Big) \alpha_0 - \delta_5 \alpha_0^2\Big)$$

$$d = \frac{1}{\beta_1} \Big[\gamma_1 + \alpha_0 \gamma_0 c + \Big(\gamma_2 - \frac{\beta_2}{\beta_1} \gamma_0\Big) \alpha_0 + \delta_3 \alpha_0^2\Big] .$$

The two-loop approximation gives the corrections of an order of u and also of an order of u^{δ_1/β_1} . The "effective" gauge parameter $\alpha(x)$ at the one- and two-loop level approaches to α_0 (if $\alpha_R \neq 0$. For $\alpha_R = 0$ it equals zero for all x).

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References

1. M.Gell-Mann, F.Low. Phys.Rev., 95, 1300, 1954.
   N.N.Bogolubov, D.V.Shirkov. DAN SSSR, 103, 203, 1955.
   N.N.Bogolubov, D.V.Shirkov. Introduction to the Theory of Quantized Fields, Nauka, Moscow, 1973.
2. D.J.Gross, F.Wilczek. Phys.Rev.Lett., 30, 1343 (1973).
   H.D.Politzer. Phys.Rev.Lett., 30, 1346, 1973.
3. G. t'Hooft, M.Veltman. Nucl.Phys., B44, 189, 1972.
4. G. t'Hooft. Nucl.Phys., B61, 455, 1973.
   J.C.Collins, A.J.Macfarlane. Phys.Rev., D10, 1201, 1974.
5. A.A.Belavin, A.A.Migdal. Pizma JETP, 19, 317, 1974.
   D.R.T.Jones. Nucl.Phys., B75, 531, 1974.
   W.E.Caswell. Phys.Rev.Lett., 33, 244, 1974.
6. A.A.Vladimirov, O.V.Tarasov. Nucl.Phys. (USSR) 25, 1104, 1977.
7. A.Slavnov. Teor.Math.Fiz., 10, 153, 1972.
   J.Taylor, Nucl.Phys., B33, 436, 1971.
8. S.Weinberg. Phys.Rev., D8, 3497, 1973.

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