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N.B.Skachkov, I.L.Solovtsov

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Скачков Н.Б., Соловцев И.Л.

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Формфакторы мезонов и ковариантная трехмерная формулировка составной модели

Развит аппарат, позволяющий в рамках релятивистской кварковой модели найти явные выражения для формфакторов мезонов через ковариантные волновые функции системы двух кварков. Эти волновые функции подчиняются двухчастичному квазипотенциальному уравнению, в котором ковариантным образом выделено относительное движение кварков. Точный вид волновых функций найден благодаря переходу в релятивистское координатное представление путем применения вместо обычного разложения Фурье гармонического анализа на группе Лоренца и решения полученного таким образом релятивистского разностного уравнения. Полученные выражения для формфакторов преобразованы к трехмерному ковариантному виду, являющемуся непосредственным геометрическим релятивистским обобщением аналогичных выражений в нерелятивистской квантовой механике и обеспечивающим в кулоновском поле убывание формфактора мезона при $-t\to\infty$ по закону $F_\pi(t)\sim t^{-1}$.

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Skachkov N.B., Solovtsov I.L.

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Meson Form Factors and Covariant Three-Dimensional Formulation of Composite Model

An approach is developed which is applied in the relativistic quark model to obtain explicit expressions for meson form factors in terms of covariant wave functions of the two-quark system. These wave functions obey the two-particle quasipotential equation in which the relative motion of quarks is singled out in a covariant way. The exact form of the wave functions is found using the transition to the relativistic configurational representation with the help of the harmonic analysis on the Lorentz group instead of the usual Fourier expansion and then solving the relativistic difference equation thus obtained. The expressions found for form factors are transformed into the three-dimensional covariant form which is a direct geometrical relativistic generalization of analogous expressions of the nonrelativistic quantum mechanics and provides the decrease of the meson form factor by the low $F_\pi(t)\sim t^{-1}$ as $-t\to\infty$, in the Coulomb field.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

The idea of the composite quark nature of hadrons in conjunction with the assumption of scale invariance leads to a simple universal law of the form factor behaviour in the asymptotic region of large momentum transfers /1/.

However, there is still a problem concerning the relativistic covariant description of form factors (and other characteristics caused by the composite nature) throughout the whole energy and momentum transfer region. For this purpose it is necessary to have a more detailed knowledge of quark dynamics and in particular to know the covariant wave functions of relative motion of quarks. Our consideration here will be restricted to the spinless mesons as objects composed of two "spinless" quarks.

In the nonrelativistic theory the form factor $F(-\vec{q}^{2})$ ($\vec{q}=\vec{p}-\vec{k}$ - momentum transfer) is defined as the Fourier transform of the modulus squared of the wave function of quark relative motion $\rho(r)=z\,|\Psi^{\ell=0}_{Bk}(r)|^2$:

$$F(-q^2)=\int d\vec{r}\, e^{i\vec{q}\vec{r}}\rho(r)=4\pi\cdot z\cdot\int_0^\infty r^2dr\,\frac{\sin rq}{rq}\,|\Psi^{\ell=0}_{Bk}(r)|^2, \tag{1.1}$$

or in the momentum representation

$$F(-\vec{q}^{2})=\frac{z}{(2\pi)^3}\int d\vec{p}\;\Psi_{Bk}(\vec{p})\Psi_{Bk}(\vec{p}-\vec{q})\,. \tag{1.2}$$

Within the four-dimensional formalism of quantum field theory, the covariant wave functions (WF) are derived using the two-particle Bethe-Salpeter equation. However, in this approach, the WF of relative motion maintains an additional dependence on relative time which has no analog in the nonrelativistic quantum mechanics and complicates the direct covariant generalization of the nonrelativistic quark model.

Our task is to construct a covariant three-dimensional formalism in the relativistic theory as close as possible to the nonrelativistic one. In the momentum representation, an essential progress has been achieved in the three-dimensional covariant description of the form factors of composite systems through the covariant formulation of the two-particle quasipotential equation of Logunov-Tavkhelidze[2] in papers[3,4]. However, in the momentum representation that is used in[3,4] the equation for WF is an integral equation that makes it difficult to obtain solutions in a closed form required for investigations.

As it was shown in our previous paper[5] the explicit form of covariant WF can be obtained on the basis of the method of transition to the relativistic configurational representation[6] proposed earlier in the framework of the Kadyshevsky quasipotential equation[7].

The aim of this work is to derive the explicit form of the relativistic form factors and to keep the analogy with the nonrelativistic formalism, in particular, with the expressions (1.1), (1.2).

2. Covariant Quasipotential Equation in the Relativistic Configurational Representation (RCR)

The main difference between the quasipotential equation and the Bethe-Salpeter equation, where all quantities are defined off the mass shell, but in each vertex the energy-momentum conserves, consists in the following: in the quasipotential approach (QPA) all the momenta of particles belong to the mass shell

$$p_0^2 - \vec{p}^{\,2} = m^2 \quad . \tag{2.1}$$

But, like in the "old fashioned" perturbation theory all quantities (WF, the Kernel of the equation) are defined over the "energy" shell. Equation (2.1) defines the three-dimensional surface of a hyperboloid whose upper sheet serves as a model of the Lobachevsky space. Therefore in the quasipotential equation, it is convenient to expand over the complete system of functions which realize the unitary representations of the group of motions of that space, i.e., the Lorentz group.

Earlier the RCR has been applied to describe the form factors in paper[8]. The transform of the form factor in RCR is defined as follows[8] $\left(t = (p-k)^2 = 2M^2(1 - \operatorname{ch}\chi)\right)$

$$F(t) = \int d\vec{r}\, \xi(\vec{\Delta}_{p,k};\vec{r}) F(r) = 4\pi \int_0^\infty \frac{\sin rM\chi}{rM \operatorname{sh}\chi} F(r) r^2 dr, \tag{2.2}$$

where analogs of the nonrelativistic plane waves $e^{i\vec{q}\vec{r}}$, which realize the group of motions of the three-dimensional Euclidean momentum space, are the following functions

$$\xi(\vec{\Delta},\vec{r}) = \left[\frac{\Delta_\mu n^\mu}{M}\right]^{-1-irM}; \quad \vec{r} = r\vec{n}; \quad \vec{n}^2 = 1; \quad n^\mu = (1,\vec{n}). \tag{2.3}$$

The functions (2.3), realizing the unitary irreducible representations of the Lorentz group, have been found in[9] The spatial part of 4-vector $(\Delta_{p,k})^\mu \equiv (L_k^{-1} p)^\mu \quad \mu = 0,1,2,3$:

$$\vec{\Delta}_{p,k} \equiv \vec{p}(-)\vec{k} \equiv \overrightarrow{(L_k^{-1} p)} = \vec{p} - \frac{\vec{k}}{M}\left(p_0 - \frac{\vec{p}\vec{k}}{k^0 + M}\right) \tag{2.4}$$

$$\Delta^0_{p,k} \equiv \sqrt{M^2 + \vec{\Delta}^2_{p,k}} \equiv (L_k^{-1} p)^0 = \frac{p_0 k_0 - \vec{p}\vec{k}}{M} = \frac{p_\mu k^\mu}{M} \tag{2.5}$$

can be treated as a vector of a difference of two vectors $\vec{p}$ and $\vec{k}$ in the Lobachevsky space: $\vec{\Delta}_{p,k} = \vec{p}(-)\vec{k}$. In the nonrelativistic limit $\vec{\Delta}_{p,k} = \vec{p}(-)\vec{k} \to \vec{q} = \vec{p} - \vec{k}$ and $\xi(\vec{\Delta},\vec{r}) \to e^{i\vec{q}\vec{r}}$, therefore, the expansion (2.2) transforms into (1.1). In ref.[6], the expansion (2.2) was proposed to consider as a relativistic three-dimensional generalization of the Fourier transformation used for the transition to the configurational representation in nonrelativistic quantum mechanics. The group parameter r in (2.2) and (2.3) which has been proposed [6] to consider as a relativistic analog of the relative coordinate r in (1.1), i.e., the "relativistic coordinate", numerates the eigenvalues of the invariant Casimir operator of the Lorentz group $\hat{C}_L = \frac{1}{4} M_{\mu\nu} M^{\mu\nu}$ (where $M_{\mu\nu} = p_\mu \frac{\partial}{\partial p^\nu} - p_\nu \frac{\partial}{\partial p^\mu}$ are the group generators)

$$\hat{C}_L\, \xi(\vec{\Delta},\vec{r}) = \left(\frac{1}{M^2}+r^2\right)\xi(\vec{\Delta},\vec{r}) \quad ; \; 0 \le r < \infty \,. \tag{2.6}$$

The nonrelativistic coordinate r has the same group-theoretical meaning: its square is an eigenvalue of the Casimir operator $\hat{C}_9 = \left(i\frac{\partial}{\partial \vec{q}}\right)^2$ of the group of motions of the Euclidean momentum space

$$\hat{C}_9\, e^{i\vec{q}\vec{r}} = r^2 e^{i\vec{q}\vec{r}} \tag{2.7}$$

In the nonrelativistic limit $\hat{C}_L \to \hat{C}_9$.

An important property of the relativistic analog of the relative coordinate in (2.2) is the relativistic invariance of its modulus (as a parameter which numerates the eigenvalues of the invariant Casimir operator $\hat{C}_L$ of the Lorentz group). Therefore, the distribution $F(r)$ in (2.2) is an invariant function.

In ref./8/ it has been shown that the usual definition of the invariant r.m.s. $\langle r_0^2\rangle \equiv 6\frac{\partial F(t)}{\partial t}\big|_{t=0}/F(0)$ has the group-theoretical meaning of an eigenvalue of the Casimir operator of the Lorentz group:

$$\langle r_0^2\rangle_{inv.} \equiv \frac{6\frac{\partial F(t)}{\partial t}\big|_{t=0}}{F(0)} = \frac{\left\{\hat{C}_L\, F(t)\right\}\big|_{t=0}}{F(0)} \,. \tag{2.8}$$

By using (2.2) and (2.6), eq. (2.8) results in the expression for $\langle r_0^2\rangle$ in terms of the invariant distribution $F(r)$ /8/

$$\langle r_0^2\rangle_{inv} \equiv \frac{6\frac{\partial F(t)}{\partial t}\big|_{t=0}}{F(0)} = \frac{1}{M^2} + \frac{\int r^2 d\vec{r}\, F(r)}{\int d\vec{r}\, F(r)} \tag{2.9}$$

which is valid in any coordinate system.

This equality was used in /8/ for analysing the vector dominance model and its modification at short distances.

Our aim is to establish the connection of the invariant distribution $F(r)$ (2.2) with covariant wave functions of relative motion of quarks inside hadron. Note, however, that unlike (2.2), in the quark model the expansion is performed on the mass hyperboloid of a quark m rather than on that of particle with mass M (composed of quarks).

As a result, the connection of the "relativistic coordinate" (for relative motion of quarks) with r.m.s. of the composite particle will be somewhat different from (2.9).

Let us turn now to the quasipotential equation. In ref./2/ the single-time quasipotential wave function (WF) was defined which follows from the Bethe-Salpeter equation. The WF of the two-particle system with mass M, momentum $\vec{K}$ and moment J (here $J=0$) in the Bethe-Salpeter approach is defined as

$$\Psi_{BK}(x_1,x_2) = \langle 0|T\{\varphi_1(x_1)\,\varphi_2(x_2)\}|M,J,\vec{K}\rangle. \tag{2.10}$$

After the covariant equating of particle time [3,4] $x^0 = x_1^0 - x_2^0 = 0$ it reads:

$$\Psi_{BK}(p_1,p_2) = (2\pi)^4\, \delta^{(4)}(\mathcal{P}-K)\,\tilde{\Psi}_{BK}(q)\; ; \; \mathcal{P} = p_1+p_2, \tag{2.11}$$

where

$$\tilde{\Psi}_{BK}(q) = \int d^{(4)}x\, e^{i\frac{(p_1-p_2)}{2}x}\,\delta(\lambda_\mu x^\mu)\langle 0|T\{\varphi_1(x)\,\varphi_2(-x)\}|M,J,\vec{K}\rangle. \tag{2.12}$$

As a vector λ_μ, it is convenient to choose the 4-velocity of the system*) [5,6]

$$\lambda^\mu_{\mathcal{P}} = \mathcal{P}^\mu/\sqrt{\mathcal{P}^2} \quad ; \quad \lambda^\mu_{\mathcal{P}} = \lambda^\mu_K$$

so that in the c.m.s. $\vec{\mathcal{P}} = \vec{p}_1+\vec{p}_2 = 0$ and $\lambda x = x_1^0 - x_2^0 = 0$. Because of the presence of δ-function under the integral sign, the integration is performed over the three-dimensional hypersurface $\lambda x = 0$. As a result, the WF (2.12)

$$\tilde{\Psi}_{BK}(q) = \int d\vec{x}'\, \exp\left[\overrightarrow{L_K^{-1}\left(\frac{p_1-p_2}{2}\right)}\cdot \vec{x}'\right]\langle 0|\varphi_1(0,\vec{x}')\,\varphi_2(0,-\vec{x}')|M,J,\vec{0}\rangle$$

depends only on the three-dimensional vector $\overset{\circ}{\vec{p}} = (\overrightarrow{L_K^{-1} q})$, which coincides with the spatial component of covariantly defined vector of the momentum of the first particle in the c.m.s. of the two-particle system, introduced in ref./11/. In notation (2.5) this vector has the form /5/ (see also /12/):

*) In refs. /10/ the quasipotential formalism was constructed with the use of 4-vector λ^μ on the cone, i.e., $\lambda_0^2 - \vec{\lambda}^2 = 0$.

$$\vec{p}^{\,\mathcal{P}} \equiv \left(L_{\mathcal{P}}^{-1}\left(\frac{p_1-p_2}{2}\right)\right) = \vec{p}_1(-)m\vec{\lambda}_{\mathcal{P}} \equiv \vec{\Delta}_{p_1,m\lambda_{\mathcal{P}}} \tag{2.13}$$

and

$$\tilde{\Psi}_{BK}(q) \equiv \tilde{\Psi}_{B0}(\vec{\Delta}_{p_1,m\lambda_{\mathcal{P}}})\,;\quad \vec{\Delta}_{p_1,m\lambda_{\mathcal{P}}} = -\vec{\Delta}_{p_2,m\lambda_{\mathcal{P}}}. \tag{2.14}$$

The analogous covariant equating of particles time is also used for deriving the two-time Green function

$$(2\pi)^4\delta^{(4)}(\mathcal{P}-Q)\,\tilde{G}(\Delta_{p,m\lambda};\Delta_{q,m\lambda};\mathcal{P}^2) = \int d^{(4)}x_1\, d^{(4)}x_2\, d^{(4)}y_1\, d^{(4)}y_2 \cdot \tag{2.15}$$

$$e^{ip_1x_1+ip_2x_2-iq_1y_1-iq_2y_2}\,\delta^{(4)}(\lambda x_1-\lambda x_2)\,\delta^{(4)}(\lambda y_1-\lambda y_2)\,G(x_1,x_2;y_1,y_2).$$

The definition of the Green function of the Bethe-Salpeter equation

$$G(x_1,x_2;y_1,y_2) = \langle 0|T\{\varphi_1(x_1)\varphi_2(x_2)\varphi_1(y_1)\varphi_2(y_2)\}|0\rangle, \tag{2.16}$$

with the completeness condition of the system of state vectors, the integral representation of θ -function, and the definition of the single-time WF (2.10) produce the following form of (2.15) near the bound state pole

$$\tilde{G}(\Delta_{p,m\lambda};\Delta_{q,m\lambda};\mathcal{P}^2) = \tilde{G}^{ret} - \tilde{G}^{adv} = \tag{2.17}$$

$$= i(2\pi)^3\left\{\frac{\tilde{\varphi}_{BM}(\vec{\Delta}_{p,m\lambda})\,\tilde{\varphi}^+_{BM}(\vec{\Delta}_{q,m\lambda})}{\sqrt{\mathcal{P}^2} - M + i\varepsilon} - \frac{\tilde{\varphi}_{BM}(\vec{\Delta}_{p,m\lambda})\,\tilde{\varphi}^+_{BM}(\vec{\Delta}_{q,m\lambda})}{\sqrt{\mathcal{P}^2} + M + i\varepsilon}\right\},$$

where WF $\tilde{\varphi}_{BM}(\Delta_{p,m\lambda})$ [5] differ from (2.12)-(2.14) by the normalization

$$\tilde{\varphi}_{BM}(\vec{\Delta}_{p,m\lambda}) = 2^3\,\Delta^0_{p,m\lambda}/\sqrt{m}\;\tilde{\Psi}_{B0}(\vec{\Delta}_{p,m\lambda}). \tag{2.18}$$

We also define the next function

$$\tilde{\varphi}^r_{BM}(\vec{\Delta}_{p,m\lambda}) = (2\Delta^0_{p,m\lambda})^{3/2}\,\tilde{\Psi}_{B0}(\Delta_{p,m\lambda}) \tag{2.19}$$

The spectral representation (2.17) contains poles at points both with positive and negative mass. Therefore, if we want to have a formalism close in form to the nonrelativistic one, we should follow[13] and use the construction procedure of the quasipotential equation only with the help of the retarded Green function $\tilde{G}^{ret}(\Delta_{p,m\lambda};\Delta_{q,m\lambda};\mathcal{P}^2)$. As a result of the usual procedure[2-4], we obtain two possible types of equations[5]

$$\left[\mathcal{P}^2 - 4\Delta^{0\,2}_{p,m\lambda}\right]\tilde{\varphi}_{BM}(\vec{\Delta}_{p,m\lambda}) = \frac{1}{(2\pi)^3}\int V(\vec{\Delta}_{p,m\lambda};\vec{\Delta}_{k,m\lambda};\mathcal{P}^2)\,\tilde{\varphi}_{BM}(\vec{\Delta}_{k,m\lambda})\,\frac{d^{(3)}\vec{\Delta}_{k,m\lambda}}{m^{-1}\Delta^0_{k,m\lambda}} \tag{2.20}$$

$$2\Delta^0_{p,m\lambda}\left[\sqrt{\mathcal{P}^2} - 2\Delta^0_{p,m\lambda}\right]\tilde{\varphi}^r_{BM}(\vec{\Delta}_{p,m\lambda}) = \frac{1}{(2\pi)^3}\int V^{ret}(\vec{\Delta}_{p,m\lambda};\vec{\Delta}_{k,m\lambda};\mathcal{P}^2)\,\tilde{\varphi}^r_{BM}(\vec{\Delta}_{k,m\lambda})\,\frac{d^{(3)}\vec{\Delta}_{k,m\lambda}}{m^{-1}\Delta^0_{k,m\lambda}} \tag{2.21}$$

Equation (2.20) corresponds to the formulation of the quasipotential approach in terms of the Green function (2.17) while eq. (2.21)-in terms of the retarded Green function G^{ret}. (Note, that eq. (2.21) coincides with the equation obtained in the Kadyshevsky approach[7] on the basis of the covariant Hamilton formulation of QFT). The procedure of constructing quasipotentials V and V^r from matrix elements of the relativistic scattering amplitude is presented in[2-4, 7, 13].

As is shown in[5], under the Lorentz transformation the WF (2.12)-(2.14) is transformed by the law

$$U(L)\tilde{\Psi}_{B0}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}) = \tilde{\Psi}_{B0}(\vec{\Delta}_{Lp,m\lambda_{L\mathcal{P}}}) = \tag{2.22}$$

$$= \tilde{\Psi}_{B0}(R\{V^{-1}(L^{-1},\mathcal{P})\}\,\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}).$$

After the transformation (2. 2) with the substitution of vector $\vec{\Delta}_{p_1,m\lambda_{\mathcal{P}}}$ (2.13) into the function (2.3), equations (2.20) and (2.21) take the form[5]:

$$\left[\mathcal{P}^2 - H_0^2\right]\tilde{\varphi}_{BM}(\vec{r}) = V(\vec{r})\,\tilde{\varphi}_{BM}(\vec{r}), \tag{2.23}$$

$$\hat{H}_0\left[\sqrt{\mathcal{P}^2}-\hat{H}_0\right]\tilde{\varphi}^{\,r}_{BM}(\vec{r}) = V^{ret}(\vec{r})\,\tilde{\varphi}^{\,r}_{BM}(\vec{r}), \tag{2.24}$$

The free Hamiltonian $\hat{H}_0$ is a finite-difference operator /5,6/. It should be noted that since the "relativistic coordinate" r is now conjugated to the covariant momentum vector of the particle in the c.m.s. of the two-particle system $\Delta_{p_1, m\lambda\mathcal{P}}$ *) the operator $\hat{H}_0$ in (2.23) and (2.24) is the Lorentz invariant /5/.

When quasipotentials V and V^r in eqs. (2.20) and (2.21) do not depend on $\mathcal{P}^2$ the normalization conditions for the WF are of the form

$$\int \frac{d^{(3)}\vec{\Delta}_{p_1,m\lambda\mathcal{P}}}{m^{-1}\,\Delta^0_{p_1,m\lambda\mathcal{P}}}\cdot\tilde{\varphi}^{+}_{BM}(\vec{\Delta}_{p_1,m\lambda\mathcal{P}})\,\tilde{\varphi}_{BM}(\vec{\Delta}_{p_1,m\lambda\mathcal{P}}) = 1 \tag{2.25}$$

$$\int \frac{d^{(3)}\vec{\Delta}_{p_1,m\lambda\mathcal{P}}}{m^{-1}\,\Delta^0_{p_1,m\lambda\mathcal{P}}}\cdot\tilde{\varphi}^{+r}_{BM}(\vec{\Delta}_{p_1,m\lambda\mathcal{P}})\cdot\frac{2\Delta^0_{p_1,m\lambda\mathcal{P}}}{M}\,\tilde{\varphi}_{BM}(\vec{\Delta}_{p_1,m\lambda\mathcal{P}}) = 1 \tag{2.26}$$

or in the RCR:

$$\int d\vec{r}\,\left|\tilde{\varphi}_{BM}(\vec{r})\right|^2 = 1 \tag{2.27}$$

$$\int d\vec{r}\;\tilde{\varphi}^{r+}_{BM}(\vec{r})\cdot\frac{\hat{H}_0}{M}\cdot\tilde{\varphi}^{r}_{BM}(\vec{r}) = 1 \tag{2.28}$$

In what follows we shall illustrate our method for finding the pion form factor by solving eq. (2.23) with the potential **)

$$V(r) = -e^2/r \tag{2.29}$$

The corresponding covariant WF coincides with those found in 2nd ref. of /6/ for the c.m.s. and for S-state these have the form

$$\tilde{\varphi}^{l=0}_{BM}(r) = Const\cdot e^{-xrm}\;{}_2F_1\left(1-irm, 1-\frac{e^2}{\sin 2x}; 2; 1-\exp(-2ix)\right) \tag{2.30}$$

$$\sqrt{\mathcal{P}^2} = 2m\cos x$$

*) The modulus $|\vec{\Delta}_{p_1,m\lambda\mathcal{P}}|$ and vector [5] $\Delta^0_{p_1,m\lambda\mathcal{P}}$ are invariant quantities and on the energy shell

$2|\vec{\Delta}_{p_1,m\lambda\mathcal{P}}| = \sqrt{M^2-4m^2}$; $2\Delta_{p_1,m\lambda\mathcal{P}} = M$; $\mathcal{P}^2 \equiv (p_1+p_2)^2 = M^2$.

**) At present it is accepted that inside the hadron quarks interact through the exchange by the massless gluon.

In the RCR to the propagator $(p-k)^{-2}$ there corresponds the quasipotential (2.29) at distances $r > m^{-1}$.

Though the quasipotential V^{ret} in (2.24) is constructed from diagrams of field theory different from those for $V^{/14/}$ in (2.23) (V^{ret} is a sum of irreducible diagrams arising in the Hamilton covariant formulation of quantum field theory by Kadyshevsky /15/), we consider as an example eq. (2.24) with the same potential (2.29).

We look for the solution to eq. (2.24) with (2.29) in the form of the Laplace integral

$$\tilde{\varphi}^{r,l=0}_{BM}(r) = \int_\alpha^\beta dp\, e^{-pr} f(p). \tag{2.31}$$

Substitution (2.31) into (2.24) gives a simple differential equation for the function $f(p)$ and certain relations which allow one to choose complex points α and β in the integral (2.31). As a result, for the function $\tilde{\varphi}^{r,l=0}_{BM}(r)$ we get the integral representation

$$\tilde{\varphi}^{r,l=0}_{BM}(r) = C_1\cdot e^{-rmx}\cdot\int_C dy\, e^{-rm(y-x)}\left[\sin\frac{y+x}{2}\right]^{\frac{\alpha}{\sin 2x}-1}\cdot \left[\sin\frac{y-x}{2}\right]^{-\frac{\alpha}{\sin 2x}-1}\cdot\left[\sin\frac{y+\frac{\pi}{2}}{2}\right]^{\frac{\alpha}{2\cos x}-1}\cdot\left[\sin\frac{y-\frac{\pi}{2}}{2}\right]^{-\frac{\alpha}{2\cos x}-1}. \tag{2.32}$$

The correct asymptotic behaviour is provided by the choice of contour C as drawn in Fig. 1 and by the following quantisation condition

$$\frac{e^2}{\sin 2x} = N \quad (N = 1,2,3\ldots) \quad ; \quad x = \arccos\frac{\sqrt{\mathcal{P}^2}}{2m^2} \tag{2.33}$$

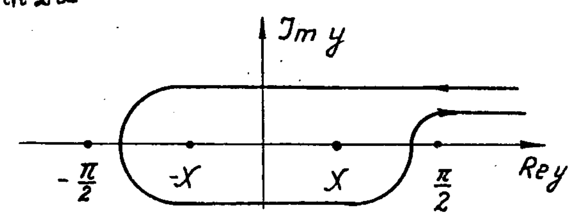

Fig. 1

For the ground state $(N = 1)$ eq. (2.32) gives

$$\tilde{\varphi}^{r,l=0}_{BM}(r) = Const\cdot r\cdot e^{-rmx}. \tag{2.34}$$

3. Form Factor of the Relativistic Two-Particle System

The matrix element of the local current operator between bound states is obtained with the help of the five-point Green-like function[18]

$$R(x_1,x_2;y_1,y_2) = \langle 0|T\{\varphi_1(x_1)\varphi_2(x_2)\,\mathcal{J}(0)\,\varphi_1(y_1)\varphi_2(y_2)\}|0\rangle .$$

As follows from ref.[4], the Fourier transform of the covariant two-time Green function

$$\tilde{R}(\Delta_{p_1,m\lambda_{\mathcal{P}}};\Delta_{q_1,m\lambda_Q};\mathcal{P},Q) = \int d^{(4)}x_1\cdot d^{(4)}x_2\cdot d^{(4)}y_1\cdot d^{(4)}y_2\cdot \tag{3.1}$$

$$\cdot\,\delta^{(4)}\big[\lambda_{\mathcal{P}}(x_1-x_2)\big]\cdot\delta^{(4)}\big[\lambda_Q(y_1-y_2)\big]\cdot R(x_1,x_2;y_1,y_2)$$

near poles of bound states A and B (with masses M_A and M_B resp.) can be represented in the form

$$R(\Delta_{p,m\lambda_{\mathcal{P}}};\Delta_{q,m\lambda_Q};\mathcal{P};Q) = \frac{\tilde{\varphi}_{AM_A}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\langle A\vec{\mathcal{P}}|\mathcal{J}(0)|B\vec{Q}\rangle\tilde{\varphi}_{BM_B}(\vec{\Delta}_{q,m\lambda_Q})}{(\sqrt{\mathcal{P}^2}-M_A)\cdot(\sqrt{Q^2}-M_B)} \tag{3.2}$$

with the help of the completeness condition for the system of state vectors, integral representation of θ -function (corresponding to T -product) and definition of the covariant two-particle WF (2.10), (2.18).

Further, following the paper[4], we derive the momentum representation of the matrix elements of the current operator in terms of the covariant single-time quasipotential WF of two bound states A and B (for details, see App. I).

a) For the WF defined by eq. (2.23) these are

$$\langle A\vec{\mathcal{P}}|\mathcal{J}(0)|B\vec{Q}\rangle = \frac{Z_1+Z_2}{(2\pi)^3}\int\frac{d^{(3)}\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}}{m^{-1}\,\Delta^{\circ}_{p,m\lambda_{\mathcal{P}}}}\cdot\tilde{\varphi}^{+}_{AM}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\cdot$$

$$\cdot\,\frac{\sqrt{\mathcal{P}^2}\,\Delta^{\circ}_{q,m\lambda_Q}+\sqrt{Q^2}\cdot\Delta^{\circ}_{p,m\lambda_{\mathcal{P}}}}{\Delta^{\circ}_{p,m\lambda_{\mathcal{P}}}\cdot\Delta^{\circ}_{q,m\lambda_Q}}\cdot\tilde{\varphi}_{BM}(\vec{\Delta}_{q,m\lambda_Q}) \tag{3.3}$$

b) For the WF (2.24) (i.e., with the retarded Green function $\tilde{G}^{ret}$) one has

$$\langle A\vec{\mathcal{P}}|\mathcal{J}(0)|B\vec{Q}\rangle =$$

$$= \frac{Z_1+Z_2}{(2\pi)^3}\int\frac{d^{(3)}\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}}{m^{-1}\,\Delta^{\circ}_{p,m\lambda_{\mathcal{P}}}}\cdot\tilde{\varphi}^{r+}_{AM}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\,\tilde{\varphi}^{r}_{BM}(\vec{\Delta}_{q,m\lambda_Q}) \tag{3.4}$$

The matrix element (3.4) is more close in form to the nonrelativistic one than the matrix element (3.3). In expressions (3.3) and (3.4) the vector of Lobachevsky space $\vec{\Delta}_{q,m\lambda_Q}$ is related to the integration variable $\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}$ as follows

$$\Delta_{q,m\lambda_Q} = L_Q^{-1}\cdot L_{\mathcal{P}}\cdot\Delta_{p,m\lambda_{\mathcal{P}}}, \tag{3.5}$$

where L_Q and $L_{\mathcal{P}}$ are matrices of the pure Lorentz transformations. The product of two pure Lorentz transformations, in general, is not pure Lorentz transformation on the resulting vector $\vec{\Delta}_{Q,\mathcal{P}} = \vec{\mathcal{P}}(-)\vec{Q} = (L_Q^{-1}\mathcal{P})$ but contains an additional Wigner rotation

$$L_Q^{-1}\cdot L_{\mathcal{P}} = L^{-1}_{\Delta_{\mathcal{P},Q}}\cdot V(L_Q,\mathcal{P}).$$

For the spherical-symmetric WF of S-state of the two-particle system it can be easily shown that the matrix element (3.4) is an invariant function $F(t)$ which depend only on the invariant quantity the square of modulus of vector $\Delta^2_{\mathcal{P}Q}$ related to the momentum transfer of the system $t=(Q-\mathcal{P})^2$ by the formula ($M_A = M_B = M$):

$$t = (\mathcal{P}-Q)^2 = 2M^2 - 2M\sqrt{M^2+\vec{\Delta}^2_{\mathcal{P}Q}}\,. \tag{3.6}$$

As a result, the function $F(\vec{\Delta}^2_{\mathcal{P},Q})$, called the system form factor, can be written as a convolution of the quasipotential WF in the Lobachevsky space

$$F(\vec{\Delta}^2_{\mathcal{P},Q}) = \frac{Z_1+Z_2}{(2\pi)^3}\int \frac{d^{(3)}\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}}{m^{-1}\Delta^0_{p,m\lambda_{\mathcal{P}}}} \cdot \tilde{\varphi}^{r\,+}_{BM}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\,\tilde{\varphi}^{r}_{BM}\left(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}(-)\frac{m}{M}\vec{\Delta}_{\mathcal{P},Q}\right). \tag{3.7}$$

The form factor (3.7) is a direct geometrical generalization (in the sense of the change of the Euclidean to the Lobachevsky geometry) of the corresponding nonrelativistic expression (1.2), that is the convolution in the Euclidean space. By applying the "addition theorem" to the relativistic "plane waves" (2.3)[6,9]

$$\int d\omega_{\vec{n}}\, \xi\left(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}(-)\frac{m}{M}\vec{\Delta}_{\mathcal{P},Q};\vec{r}\right) = \int d\omega_{\vec{n}}\, \xi(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}};\vec{r})\,\xi^*\left(\frac{m}{M}\vec{\Delta}_{\mathcal{P},Q};\vec{r}\right) \tag{3.8}$$

the form factor $F(\vec{\Delta}^2_{\mathcal{P},Q})$ can be represented in the form of the relativistic Fourier transform of the modulus squared of the covariant quasipotential WF $\tilde{\varphi}^r_{BM}(\vec{r})$ (2.24)

$$F(t) = F(\vec{\Delta}^2_{\mathcal{P},Q}) = (Z_1+Z_2)\int d^{(3)}\vec{r}\,\xi^*\left(\frac{m}{M}\vec{\Delta}_{\mathcal{P},Q};\vec{r}\right)\left|\tilde{\varphi}^r_{BM}(\vec{r})\right|^2. \tag{3.9}$$

$$\xi\left(\frac{m}{M}\vec{\Delta}_{\mathcal{P},Q},\vec{r}\right) = \left(\frac{mM^{-1}\Delta_0 - mM^{-1}\vec{\Delta}\vec{n}}{m}\right)^{-1-irm} = \left(\frac{\Delta_0-\vec{\Delta}\vec{n}}{M}\right)^{-1-irm}$$

For S-state, integrating over angles gives

$$F(t) = (Z_1+Z_2)\frac{y}{\operatorname{sh} y}\cdot 4\pi\int_0^\infty dr\,\frac{\sin rmy}{rmy}\left|\tilde{\varphi}^{r,\ell=0}_{BM}(r)\right|^2, \tag{3.10}$$

where $y = \operatorname{Arch}\left(1-\frac{t}{2M^2}\right)$ is the rapidity and $\tilde{\varphi}^{r,\ell=0}_{BM}(r)$, the radial WF. The form factor (3.10), being a generalization of the nonrelativistic form factor (1.1), differs from it in two points. First, (3.10) contains the integration over invariant "relativistic coordinate" r which is conjugate not to the momentum transfer, like in (1.1) but to the rapidity my [6], that has the meaning of a distance in the Lobachevsky space. Second, (3.10) contains an additional relativistic geometrical

factor $y/\operatorname{sh} y$ which vanishes in the nonrelativistic limit. (The meaning of this factor is discussed in refs.[8]).

Now let us consider a particular example, the form factor of a meson in the case of the Coulomb interaction between quarks. It is known that a nonrelativistic model based on the Coulomb potential predicts the dipole decrease of the pion form factor $F_\pi(t) = t^{-2}$ [17] that contradicts the prediction of the dimensional quark counting rules, i.e., the decrease of type $F_\pi(t) \sim t^{-1}$ [1]. In our approach, the covariant quasipotential WF (2.34) produces the following behaviour of $F_\pi(t)$ at large transfer momenta $-t \to \infty$

$$F_\pi(t) \equiv F(\vec{\Delta}^2_{\mathcal{P},Q}) \underset{-t\to\infty}{\approx} \frac{1}{y^3 \operatorname{sh} y} \approx \frac{1}{|t|\left(\ln\frac{|t|}{M^2}\right)^3}, \tag{3.11}$$

which differs from that predicted by the dimensional quark counting rules by the logarithmic factor $\left(\ln\frac{|t|}{m^2}\right)^3$ only.

Note also that the approach developed can help one to find the relativistic form factor of a system with the nonrelativistic internal motion of constituents. In this case the relativistic relative "coordinate" r coincides with the nonrelativistic one, and eq. (2.24) turns into the Schrödinger equation [7,6]. Now, by comparing (1.1) and (3.10) it can be easily concluded that

$$F_{rel.}(t) = F_{rel.}(\vec{\Delta}^2_{\mathcal{P},Q}) = y/\operatorname{sh} y \cdot F_{nonrel.}(my), \tag{3.12}$$
$$y = \operatorname{Arch}\left(1 - t/2M^2\right).$$

It should be noted that formula (3.12) has been obtained in the consistent relativistic theory without any approximations of the type of an expansion in powers of v^2/c^2. On the other hand, the results of paper[19] can be applied only if the interaction Lagrangian is restricted to terms of first order in v^2/c^2, as it was shown for classical models in ref.[20]. The consideration of higher orders results in essential difficulties in determining the transformation properties of WF[21]. Expanding y in (3.12) in powers of $1/c^2$ one can easily verify that formula (3.12) coincides, within terms $1/c^4$, with the corresponding formula of paper[19].

On the basis of results of paper[22] formula (3.12) can be written in the form

$$F_{rel}(\Delta^2_{\mathcal{P},Q}) \approx \frac{1}{y\, sh\, y}\tilde{V}(my) \sim \frac{1}{|t|\ln\left(\frac{|t|}{M^2}\right)}\tilde{V}\left(m\ln\frac{|t|}{M^2}\right),$$

with $\tilde{V}(my)$ - the Fourier-Bessel transform of $V(r)$. From the latter expression it is clear what class of potentials provides the asymptotic behaviour of $F_{rel}(t)$ consistent with predictions of the dimensional quark counting.

Conclusion

Let us summarize the most essential results of the paper.

1. The relativistic configurational representation (RCR) allows one to express the particle form factor in terms of the invariant distribution $F(r)$ (2.2) in any reference frame[8] and not only in the Breit one (as in the case of the Fourier-Bessel transformation (1.1)).

2. We establish the connection of the invariant distribution $F(r)$ with the covariant quasipotential WF of the system of two particles (quarks).

3. The quasipotential WF are solutions of the covariant two-particle quasipotential equations (2.23) and (2.24), and for some potentials they can be found explicitly (see, e.g., (2.30) and (2.34)).

4. The relativistic coordinate r in eqs. (2.23) and (2.24) is conjugated to the covariant momentum vector of particle in the c.m.s. of the system $\Delta_{p,m\lambda_{\mathcal{P}}}$ (2.13). The invariance of its modulus $2|\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}| = inv.$ leads to the invariance of the modulus of the relativistic relative coordinate r in (2.2) and (3.10). Therefore the RCR with the group parameter (2.6) playing the role of the invariant relativistic relative "coordinate", allows one to describe the system internal motion, responsible for the particle structure, in an invariant way.

5. The invariance of the modulus of the relativistic relative "coordinate"("incompressibility" under the Lorentz transformations) is the main difference from the Licht-Pagnamenta formalism[19] widely used in quark theory. This property allows

one to obtain the simple formula (3.12) for the invariant form factors of the systems with nonrelativistic internal motion.

Our further purpose is the application of the developed formalism for calculation of form factors for other types of quark-antiquark interaction and the inclusion of spin on the basis of an approach developed in refs.[7,23].

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Appendix

Let us introduce, following ref.[4] the generalized vertex function Γ related to the five-point function R (3.1) as follows:

$$R(\mathcal{P},Q) = G(\mathcal{P})\cdot\Gamma(\mathcal{P},Q)\cdot G(Q), \qquad (A.1)$$

where multiplication implies the invariant integration with volume element of momentum space $d\Omega_\Delta = \frac{d^3\vec{\Delta}_{p,m\lambda}}{m^{-1}\Delta^0_{p,m\lambda}}$ realized on the hyperboloid (2.1).

Graphically, this representation is of the following form[4] (see Fig. 2)

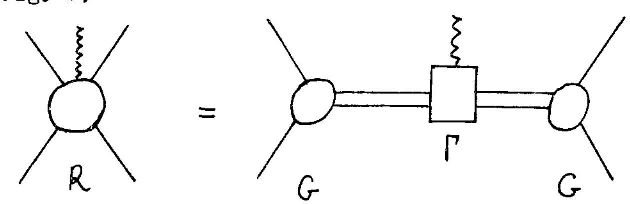

Fig. 2.

In the presence of a bound state with the two-time Green function has a pole representation according to (2.17). Comparing (3.2) with (A.1) and applying the pole representation (2.17) to G, one gets the current matrix element between states A and B in terms of the quasipotential WF equation (2.10) and generalized vertex function Γ in the following form

$$\langle A\vec{\mathcal{P}}|J(0)|B\vec{Q}\rangle = \frac{1}{(2\pi)^6}\int\frac{d^3\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}}{m^{-1}\Delta^0_{p,m\lambda_{\mathcal{P}}}}\cdot\frac{d^3\vec{\Delta}_{q,m\lambda_Q}}{m^{-1}\Delta^0_{q,m\lambda_Q}}\cdot \qquad (A.2)$$
$$\cdot\Psi^+_{A0}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\Gamma(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}};\vec{\Delta}_{q,m\lambda_Q};\mathcal{P},Q)\Psi_{B0}(\vec{\Delta}_{q,m\lambda_Q})$$

Within the quasipotential approach in terms of the retarded Green function relation (A.1) should be changed as follows:

$$R^{ret}(\mathcal{P},\mathcal{Q}) = G^{ret}(\mathcal{P})\cdot\Gamma^{ret}(\mathcal{P},\mathcal{Q})\cdot G^{ret}(\mathcal{Q}), \qquad (A.3)$$

where $R^{ret}(\mathcal{P},\mathcal{Q})$ is the retarded part of the five-point function (3.1), and the current matrix element is:

$$\langle A\vec{\mathcal{P}}|J(0)|B\vec{\mathcal{Q}}\rangle = \frac{1}{(2\pi)^6}\int\frac{d^3\vec{\Delta}_{p,m\lambda_{\mathcal{P}}}}{m^{-1}\Delta^0_{p,m\lambda_{\mathcal{P}}}}\cdot\frac{d^3\vec{\Delta}_{q,m\lambda_{\mathcal{Q}}}}{m^{-1}\Delta^0_{q,m\lambda_{\mathcal{Q}}}}\cdot \qquad (A.4)$$
$$\cdot\tilde{\varphi}^{tr}_{A0}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}})\,\Gamma^{ret}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}};\vec{\Delta}_{q,m\lambda_{\mathcal{Q}}};\mathcal{P},\mathcal{Q})\,\tilde{\varphi}^{r}_{B0}(\vec{\Delta}_{q,m\lambda_{\mathcal{Q}}})$$

The vertex function Γ will be calculated approximately as an expansion in interaction constant. In the impulse approximation (see Fig. 3)

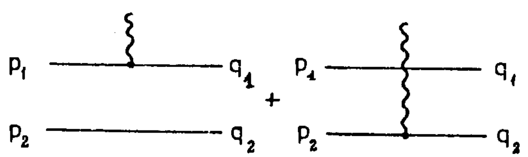

Fig. 3.

it is of the form

$$\Gamma_0(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}};\vec{\Delta}_{q,m\lambda_{\mathcal{Q}}};\mathcal{P},\mathcal{Q}) = \frac{2^6}{(2m)^2}\left(\sqrt{\mathcal{P}^2}\cdot\Delta^0_{q,m\lambda_{\mathcal{Q}}}+\sqrt{\mathcal{Q}^2}\cdot\Delta^0_{p,m\lambda_{\mathcal{P}}}\right)\cdot \qquad (A.5)$$
$$\cdot\langle p_1,p_2|J(0)|q_1\,q_2\rangle$$

$$\Gamma_0^{ret}(\vec{\Delta}_{p,m\lambda_{\mathcal{P}}};\vec{\Delta}_{q,m\lambda_{\mathcal{Q}}};\mathcal{P},\mathcal{Q}) = \frac{2^6}{m^2}\left(\Delta^0_{p,m\lambda_{\mathcal{P}}}\cdot\Delta^0_{q,m\lambda_{\mathcal{Q}}}\right)\cdot \qquad (A.6)$$
$$\cdot\langle p_1p_2|J(0)|q_1\,q_2\rangle \; .$$

For the interaction Lagrangian

$$\mathcal{L}(x) = z_1\varphi_1^+\varphi_1 A + z_2\varphi_2^+\varphi_2 A$$

with scalar field A we get

$$\langle p_1\,p_2|J(0)|q_1\,q_2\rangle = \frac{z_1}{(2\pi)^3}2p_{20}\,\delta^{(3)}(\vec{p}_2-\vec{q}_2) + (1\leftrightarrow 2) = \qquad (A.7)$$
$$= \frac{z_1}{(2\pi)^3}\,2m\,\delta^3(\vec{p}_2(-)\vec{q}_2) + (1\leftrightarrow 2) \; .$$

Substituting (A.5) and (A.6) into (A.2) and (A.4) and allowing for (A.7) and its invariance produce expressions (3.3) and (3.4).

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