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О геометрическом подходе в теории релятивистской струны

В данной работе в геометрическом подходе к теории струны в четырехмерном пространстве Минковского используется релятивистски инвариантная калибровка $(\ddot{x} \pm \dot{x})^2 = -q_{\pm}^2$, которая применялась ранее в случае трехмерного пространства-времени. В отличие от результатов предыдущих работ систему уравнений на коэффициенты квадратичных форм поверхности удается свести вновь к одному нелинейному уравнению Лиувилля, но теперь уже на комплексно-значную функцию u. Далее показано, что и в случае произвольной размерности пространства-времени есть такие движения струны, которые описываются решениями уравнения Лиувилля с действительной функцией u. И как следствие этого, солитонные решения этого уравнения, которые исследовались ранее, имеют место и в случае любой размерности пространства-времени, в котором движется струна.

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On the Geometrical Approach to the Relativistic String Theory

The geometrical approach to the theory of the relativistic string is extended to the Minkowsky space of arbitrary dimension. It is shown that in four-dimensional space-time in the relativistic invariant gauge $(\ddot{x} \pm \dot{x})^2 = -q_{\pm}^2$ the problem is reduced to the nonlinear Liouville equation for the complex values function.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. In papers /1-4/ the so-called geometrical approach to the theory of the relativistic string was proposed. In this method as the co-ordinates of the string world sheet the coefficients are considered of its first and second fundamental forms (g_{ij} and $b_{a|ij}$, respectively) instead of the Lorentz vector $x^{\mu}(\sigma,\tau)$ which describes this sheet in a parametric form /5,6/. The advantage of this approach is the possibility to combine the equation of motion and the supplementary conditions of the string theory in the three-dimensional space-time into one non-linear equation

$$\ddot{u} - u'' = R e^{u} , \qquad (16)$$

where $e^{-u(\sigma,\tau)} = g_{11} = \dot{x}^2$, $\dot{u} = \partial u/\partial\tau$, $u' = \partial u/\partial\sigma$ *. It appears that this equation admits the soliton solutions which lead to a new mass spectrum in comparison with the theory of the dual string /4/. If the string is moving in four-dimensional space-time, then in the gauge $t = \tau$ the problem can be reduced to the Gauss surface theory in three-dimensional space again /1/. However, in this case there are two equations for the coefficients of the fundamental forms of the string world sheet. For the free infinite string these equations have the form

$$\frac{\partial}{\partial\sigma}(\mathrm{ctg}^2\theta\lambda') = \frac{\partial}{\partial\tau}(\mathrm{ctg}^2\theta\dot{\lambda}),$$

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* The infinite relativistic string is considered only and the boundary conditions are not discussed /2,3/.

$$\ddot{\theta} - \theta'' + \frac{\cos\theta}{\sin^3\theta}(\dot{\lambda}^2 - \lambda'^2) = 0. \quad (2)$$

In this paper in a geometrical approach to the string theory in the four-dimensional Minkowsky space the relativistic invariant gauge $(\dot{x} \pm \dot{x}') = -q_{\pm}^2$ proposed in papers /2-4/ for the string moving in three-dimensional space-time is used. In contrast to the results of paper /1/ the system of equations for the coefficients of the fundamental forms of the string model world sheet can be reduced now to one equation (1) again but with a complex valued function u. It is shown that in the case of space-time with arbitrary dimension there are such string motions which are described by one non-linear equation (1) with a real function u. And as a consequence the soliton solutions investigated in paper /4/ take place in a geometrical approach to the string theory in any dimensional space-time.

II. Our consideration will be based on the following embedding theorem of the differential geometry /7,8/.

The symmetrical tensor g_{ij}, p symmetrical tensors $b_{\alpha|ij}$ and $p(p-1)/2$ vectors $\nu_{\alpha\beta|i}$ $(=-\nu_{\beta\alpha|i})$, $i,j = 1,2$, $\alpha,\beta = 3,4,\ldots, p+2$ determine two dimensional surface V_2 with fundamental tensor g_{ij} embedded into the real flat space S_{p+2} (the Riemann curvature tensor of this space vanishes identically), then and only then the following equations are satisfied:

$$R_{ijk\ell} = \sum_{\alpha} e_{\alpha}(b_{\alpha|ik}\, b_{\alpha|j\ell} - b_{\alpha|i\ell}\, b_{\alpha|jk}), \quad (3)$$

$$b_{\alpha|ij;k} - b_{\alpha|ik;j} = \sum_{\beta} e_{\beta}(\nu_{\beta\alpha|k}\, b_{\beta|ij} - \nu_{\beta\alpha|j}\, b_{\beta|ik}), \quad (4)$$

$$\nu_{\beta\alpha|j;k} - \nu_{\beta\alpha|k;j} + \sum_{\gamma} e_{\gamma}(\nu_{\gamma\beta|j}\, \nu_{\gamma\alpha|k} - \nu_{\gamma\beta|k}\, \nu_{\gamma\alpha|j}) + $$
$$+ g^{\ell m}(b_{\beta|\ell j}\, b_{\alpha|mk} - b_{\beta|\ell k}\, b_{\alpha|mj}) = 0, \quad (5)$$

where $R_{ijk\ell}$ is the Riemann-Christoffel curvature tensor of the two-dimensional surface V_2 which has only one essential component

$$R_{1212} = \frac{1}{2}\left(2\frac{\partial^2 g_{12}}{\partial u^2 \partial u^1} - \frac{\partial^2 g_{11}}{\partial u^2 \partial u^2} - \frac{\partial^2 g_{22}}{\partial u^1 \partial u^1}\right) +$$
$$+ g^{\ell m}(\Gamma_{m,21}\Gamma_{\ell,12} - \Gamma_{m,22}\Gamma_{\ell,11}), \quad (6)$$

$\Gamma_{i,jk}$ is the Christoffel symbol

$$\Gamma_{i,jk} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right),$$

e_{α} is a constant equal ± 1.

In the string theory the variables u^i are denoted by σ and τ: $u^1 = \tau$, $u^2 = \sigma$, $\partial x/\partial\tau = \dot{x}$, , $\partial x/\partial\sigma = x'$ and the surface V_2 is the world sheet of the string in Minkowsky space. The semicolon means here covariant differentiation with respect to the metric tensor

$$g_{ij} = \sum_{\mu=1}^{p+2} c_{\mu} \frac{\partial x^{\mu}}{\partial u^i}\frac{\partial x^{\mu}}{\partial u^j}, \quad g_{ik}\, g^{kj} = \delta_i^j .$$

The constants c_{μ}, $\mu = 1,2,\ldots, p+2$ take into account the metric signature of space S_{p+2} and they are equal to ± 1. In the theory of the relativistic string S_{p+2} is the Minkowsky space E^1_{p+2}, so we put $c_1 = -c_{\nu} = 1$, $\nu = 2,3,\ldots,p+2$. At any point of the surface V_2 embedded into the flat space S_{p+2} the system can be constructed of the orthonormal vectors η^{μ}_{α} which are orthogonal to the tangent vectors of the world surface of the string $\dot{x}^{\mu}$ and x'^{μ}:

$$\sum_{\mu=1}^{p+2} c_{\mu}\, \eta^{\mu}_{\alpha}\, \eta^{\mu}_{\beta} = \begin{cases} e_{\alpha}, & \alpha = \beta, \\ 0, & \alpha \neq \beta, \end{cases}$$

$$\sum_{\mu=1}^{p+2} c_\mu \eta^\mu_\alpha \dot{x}^\mu = \sum_{\mu=1}^{p=2} c_\mu \eta^\mu_\alpha x'^\mu = 0 ,$$

$\alpha, \beta = 3, 4, \ldots, p+2.$

In the chosen metric of space S_{p+2} the constants e_α are equal to +1 for the time like vectors η^μ_α, and $e_\alpha = -1$ for the space like η^μ_α. From the physical view-point in the string theory /5, 6/ one puts $g_{11} = \dot{x}^2 > 0$, $g_{22} = x'^2 < 0$ and all vectors orthogonal to $\dot{x}$ and x' are space-like and as a consequence, $e_\alpha = -1$, $\alpha = 3,4,\ldots,p+2$.

The world sheet of the string is the minimal surface, so in the isothermic co-ordinate system /9/

$$g_{11} = \dot{x}^2 = -x'^2 = -g_{22}, \quad g_{12} = \dot{x}x' = 0 \tag{7}$$

the vector $x^\mu(\sigma,\tau)$ which describes this surface obeys the d' Alambert equation

$$\ddot{x}^\mu = x''^\mu = 0, \quad \mu = 1,2,\ldots, p+2 . \tag{8}$$

If the surface is embedded into the three-dimensional flat space $(p=1)$, then there are no vectors $\nu_{\alpha\beta|i}$, and eqs. (3-5) are reduced to the first two ones, in this case eq. (3) being the Gauss equation and (4) being the Codazzi equation /10/.

In coordinate system (7) eq. (3), by virtue of (6), takes the form

$$\ddot{g}_{11} - g''_{11} + g^{11} [(g'_{11})^2 - (\dot{g}_{11})^2] =$$

$$= -2 \sum_{\alpha=3}^{p+2} (b_{\alpha|11} b_{\alpha|22} - b_{\alpha|12})^2 . \tag{9}$$

The tensors $b_{\alpha|ij}$ are defined by the derivation formulae

$$x^\mu_{;ij} = \sum_{\alpha=3}^{p+2} e_\alpha b_{\alpha|ij} \eta^\mu_\alpha , \tag{10}$$

where η^μ_α is the system of the orthogonal vectors introduced above. Taking into account (8) we get from the expansion (10)

$$b_{\alpha|11} = b_{\alpha|22} , \quad \alpha = 3,4,\ldots,p+2 .$$

In addition to eq. (7), one can impose on the variables $x^\mu(\sigma,\tau)$ the following conditions /2-4/

$$(\dot{x} \pm \dot{x}')^2 = -q^2_\pm , \tag{11}$$

where $q_\pm$ are any constants. Substituting (10) into (11) we get

$$\sum_{\alpha=3}^{p+2} (b_{\alpha|11} \pm b_{\alpha|12})^2 = q^2_\pm .$$

If the string moves in four-dimensional Minkowsky space $(p=2, \alpha=3,4)$, then the following variables

$$b_{3|11} \pm b_{3|12} = q_\pm \cos\alpha_\pm , \quad b_{4|11} \pm b_{4|12} = q_\pm \sin\alpha_\pm$$

can be introduced.

Eqs. (3-5) take now the form

$$\ddot{g}_{11} - g''_{11} + g^{11} [(g'_{11})^2 - (\dot{g}_{11})^2] = -2q_+ q_- \cos(\alpha_+ - \alpha_-), \tag{12}$$

$$\dot{\alpha}_+ - \alpha'_+ = \nu_1 - \nu_2 , \quad \dot{\alpha}_- + \alpha'_- = \nu_1 + \nu_2 , \tag{13}$$

$$\nu'_1 - \dot{\nu}_2 + g^{11} q_+ q_- \sin(\alpha_+ - \alpha_-) = 0 , \tag{14}$$

where $\nu_1 = \nu_{43|1}$, $\nu_2 = \nu_{43|2}$. In terms of variables $\theta = \alpha_+ - \alpha_-$ and $g_{11} = e^{-u}$ eqs. (12) and (14) part from the system (12-14)

$$\ddot{u} - u'' = 2q_+ q_- e^u \cos\theta, \tag{15}$$

$$\ddot{\theta} - \theta'' = 2q_+ q_- e^u \sin\theta. \tag{16}$$

These equations can be reduced to one by means of complex valued function $w = u + i\theta$

$$\ddot{w} - w'' = Re^{w} , \qquad (17)$$

where $R = 2q_{+}q_{-}$. So the theory of relativistic string in four-dimensional space-time in gauge (11) is reduced to the Liouville equation /11/ again. But in contrast to three-dimensional Minkowsky space this equation is for the complex valued function w.

Obviously the Liouville solution /11/ can be generalized to the complex equation (17). This equation, as in the case of real function u /4/, has the soliton solution

$$e^{w/2} = A\sqrt{\frac{2}{R}}\,\mathrm{sech}\left(A\frac{\sigma - v\tau}{\sqrt{1-v^2}} + \delta\right), \qquad (18)$$

where A and δ are arbitrary complex constants: $A = a_1 + ia_2$, $\delta = \delta_1 + i\delta_2$. Separating the real and imaginary parts in eq. (18) we get

$$e^{u} = \frac{4(a_1^2 + a_2^2)}{|R|\,(\mathrm{ch}\,2z_1 + \cos 2z_2)} ,$$

$$\mathrm{tg}\frac{\theta}{2} = \frac{a_2 + a_1\,\mathrm{th}\,z_1\,\mathrm{tg}\,z_2}{a_1 - a_2\,\mathrm{th}\,z_1\,\mathrm{tg}\,z_2} ,$$

where $z_i = a_i(\sigma - v\tau)/\sqrt{1-v^2} + \delta_i$, $i = 1,2$.

Equations (15) and (16) are the Euler equations for the Lagrangian density

$$\mathfrak{L} = \frac{1}{2}(\dot{u}^2 - u'^2) - \frac{1}{2}(\dot{\theta}^2 - \theta'^2) + Re^{u}\cos\theta . \qquad (19)$$

The corresponding Hamiltonian is

$$\mathcal{H} = \frac{1}{2}(\pi_u^2 + u'^2) - \frac{1}{2}(\pi_\theta^2 + \theta'^2) + Re^{u}\cos\theta ,$$

where $\pi_u = \partial\mathfrak{L}/\partial\dot{u} = \dot{u}$, $\pi_\theta = \partial\mathfrak{L}/\partial\dot{\theta} = -\dot{\theta}$. The free Hamiltonian of the field θ is included in $\mathcal{H}$ with sign minus.

3. Let us go to arbitrary dimension $(p>2)$ of the space-time into which the world sheet of the string is embedded. We shall consider the sets of variables $b_{\alpha|11}$ and $b_{\alpha|12}$, $\alpha = 3,4,\ldots,p+2$ as co-ordinates of two Euclidean vectors $b^1 = (b_{3|11}, b_{4|11}, \ldots, b_{p+2|11})$ and $b^2 = (b_{3|12}, b_{4|12}, \ldots, b_{p+2|12})$. Introducing the variable θ as the angle between these vectors we can reduce eq. (3) to eq. (15) again. Let us show that the particular solution of the system (3)-(5) is $\theta = 0$, $\nu_{\beta\alpha|j} = 0$ and $b_{\alpha|ij}$ are constants. Equation (4) is satisfies identically in this case and eq. (5) gives

$$\frac{b_\alpha^1}{b_\beta^1} = \frac{b_\alpha^2}{b_\beta^2} , \qquad \alpha,\beta = 3,4,\ldots,p+2 .$$

This is identity also as for $\theta = 0$ we have $b_\alpha^1 = \lambda b_\alpha^2$ where λ is constant. The essential equation now is eq. (15) which is reduced to eq. (1). So in the case of space-time with any dimension there are such string motions which are described by one real Liouville equation (1). All results obtained in paper /4/ concerning the soliton solutions of eq. (1) and the mass spectrum generated by these solutions are valid here as a particular case.

4. The theory of the infinite relativistic string in four-dimensional space-time in gauge (11) is reduced to the investigation of eqs. (15) and (16) or one complex eq. (17). In paper /1/, as it was noted above, the same problem was considered in gauge $t = \tau$ and eqs. (2) were obtained different from eqs. (15) and (16). Here the following question arises: Do the results of the geometrical approach to the relativistic string theory depend on the choice of gauge? In any theory with gauge invariance the physical results have to be gauge independent. To answer this question, we have to investigate the reduction of eqs. (2) to (15), (16). If this reduction is impossible, the mass spectrum of eqs. (2) and (15), (16) should be compared taking into account the soliton solutions in these systems.

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