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THE SOLITONS IN GENERAL RELATIVITY AND IN THE BORN-INFELD SCALAR FIELD MODEL IN TWO DIMENSIONAL SPACE-TIME

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Барбашов Б.М., Нестеренко В.В., Червяков А.М. **E2 - 11669**

Солитоны в теории тяготения и в модели скалярного поля Борна-Инфельда в двумерном пространстве-времени

Показано, что в двумерном пространстве-времени теория тяготения Эйнштейна с постоянной кривизной и скалярное поле Борна-Инфельда, рассматриваемое в так называемом геометрическом подходе, описываются одним и тем же нелинейным уравнением $U_{tt} - U_{xx} = Re^{U}$. Это уравнение имеет солитонные решения. Среди устойчивых решений есть периодические солитоны. Показано, что солитоны можно интерпретировать как новые частицы с отличной от нуля массой покоя, причем эта интерпретация имеет смысл и на классическом и на квантовом уровнях. Периодический солитон порождает серию резонансов с эквидистантным спектром масс. Полученные результаты имеют место и в теории релятивистской струны в трехмерном пространстве Минковского.

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Barbashov B.M., Nesterenko V.V., Chervjakov A.M. **E2 - 11669**

The Solitons in the General Relativity and in the Born-Infeld Scalar Field Model in the Two Dimensional Space-Time

It is shown that in the two dimensional space-time the general relativity with a constant curvature and the Born-Infeld scalar field considered in the so-called geometrical approach are described by the same nonlinear Liouville equation $U_{tt} - U_{xx} = Re^{U}$. This equation has soliton solutions. There are periodical solitons among the stable ones. It is shown that the solitons may be interpreted as new particles with the nonzero rest mass. This interpretation makes sense on the both classical and quantum stages. The periodical soliton product series of the resonances with equidistant mass spectrum. The obtained results take place in the theory of relativistic string in the three dimensional Minkowski space as well.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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I. Introduction

In the last years it has been discovered that a number of the nonlinear field models admits the soliton solutions /1/. In the elementary particle physics these solutions can be interpreted as particles different from quanta of the initial field /2/. So one nonlinear field describes the particles of several kinds. However, this can be demonstrated in the complete form only in the nonlinear models which are very far from the physical reality. A classic example here is the field satisfying the Sine-Gordon equation in two dimensional space-time /3/. The investigation of the soliton solutions even in such abstract models is of certain interest from a methodological viewpoint, at least.

In this paper it is shown that in two dimensional space-time the general relativity with a constant scalar curvature, the Born-Infeld massless scalar field and the relativistic string are described by the same nonlinear equation. This equation admits the soliton solutions which may be stable or unstable. There are periodical solutions among the stable ones. The addition to the canonical energy-momentum tensor of the term dependent on the soliton velocity allows us to define the total energy, momentum and the rest mass of the soliton. In quantum theory these soliton solutions can be interpreted as massive, either stable or unstable, particles with respect to the stability of the corresponding classical solution. The quanta of the initial field remain massless after the separation of the soliton. In the quantum case, the periodical soliton generates a series of resonances which have the equidistant mass spectrum beginning from the first excited state. This periodical soliton appears well-suited to the theory of the closed relativistic string. The obtained results show that the nonlinear models containing only a massless field can generate a rich spectrum of the massive particles and resonances, and this spectrum cannot be obtained in perturbation theory in principle.

2. The Gravitation Theory with a Constant Scalar Curvature in Two Dimensional Space-Time

It is widely believed that the Einstein gravitation theory in space-time with dimension $n<4$ is not interesting/4/. For example, if $n=2$ the Einstein tensor is identically equal to zero

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \equiv 0 . \qquad (1)$$

This is the consequence that in two dimensional case the action functional of the gravitational field can be transformed, by the Gauss-Bonnet theorem/5/, to the contour integral

$$S_g = -\frac{1}{2\varkappa}\iint_{\Omega}\sqrt{-g}\,R\,d^2x = -\frac{1}{4\varkappa}\oint_C k_g\,dS + const ,$$

where k_g is the geodesic curvature of the boundary C of the region Ω. As a consequence, the variation of S_g does not result in the equations for $g_{\mu\nu}$. But if we shall consider the scalar curvature R in (1) as a fixed function of co-ordinates

then identity (1) transforms to the equations for $g_{\mu\nu}$ 1). Just in this fashion the equations (1) will be considered below.

In the co-ordinate system on the surface, where

$$g_{00} = -g_{11} = \exp u(x,t),\quad g_{01} = 0,$$

we have the following equation for $u(x,t)$

$$u_{tt} - u_{xx} = Re^{u}. \qquad (2)$$

In the differential geometry (2) is the Gauss equation/5/ which connects the Gauss curvature of the surface $K=R/2$ with the coefficients of the first fundamental form of this surface

$$ds^2 = e^{u}(dt^2 - dx^2).$$

Before to investigate eq. (2) we show that this equation arises also in the geometrical approach to the Born-Infeld scalar field model in two dimensional space-time and in the theory of the relativistic string.

3. The Born-Infeld Scalar Field in Two Dimensional Space-Time

The action in this model is/7/

$$S = -\gamma\iint dt\,dx\,\sqrt{1+\gamma^{-1}(\varphi_x^2 - \varphi_t^2)} , \qquad (3)$$

where $\varphi_x = \partial\varphi(x,t)/\partial x$, $\varphi_t = \partial\varphi(x,t)/\partial t$, γ is a constant with the dimension of inverse length. The field $\varphi(x,t)$ obeys the nonlinear equation

$$(\gamma - \varphi_t^2)\varphi_{xx} + 2\varphi_x\varphi_t\varphi_{xt} - (\gamma + \varphi_x^2)\varphi_{tt} = 0 , \qquad (4)$$

---

1) As is well known, the two dimensional Riemann many-fold with a constant scalar curvature R can be considered as a hyperboloid in the three dimensional flat space /6/. If $R>0$, then the spacelike geodesics of this manyfold form the closed curves, if $R<0$, then the timelike geodesics are closed.

which admits the wave solution of an arbitrary form propagating with the speed of light

$$\varphi(x,t) = \Phi(x \pm t).$$

The function Φ is not fixed by the equation (4). By these properties eq. (4) differs from the well searched ones: the Sine-Gordon equation, the Korteweg-de-Vries equation and the nonlinear Schrödinger equation. All these equations have the soliton solutions of an exactly fixed form and propagating with an arbitrary velocity v /1/.

Now we present the so-called geometrical approach to the Born-Infeld scalar field model. In this approach the model under consideration is described by the nonlinear Liouville eq. (2) which has the soliton solutions of definite form.

In papers /8/ the Born-Infeld scalar field was searched in the parametrical representation via introducing the Lorentz vector $x_\mu(\sigma,\tau)$ with components

$$x^\mu(\sigma,\tau) = \left(t(\sigma,\tau),\, x(\sigma,\tau),\, y(\sigma,\tau) = \gamma^{-1/2}\varphi(x(\sigma,\tau), t(\sigma,\tau))\right).$$

In the new variables the action (3) coincides with that for the relativistic string /9/

$$S = -\gamma \iint d\tau\, d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} = -\gamma \iint d\tau\, d\sigma \sqrt{-g}\,, \tag{5}$$

where $\dot{x}_\mu = \partial x_\mu(\sigma,\tau)/\partial\tau$, $x'_\mu = \partial x_\mu(\sigma,\tau)/\partial\sigma$, $g = \det|g_{ij}|$, $g_{ij} = \partial x_\mu/\partial u^i \cdot \partial x_\mu/\partial u^j$ is the metric tensor on the string world surface $x_\mu(\sigma,\tau)$; $i,j = 0,1$, $u^0 = \tau$, $u^1 = \sigma$, $\mu = 0,1,2$.

The principle of least action, as applied to the functional (5), leads to the problem of determination of the extremal surface in the three dimensional pseudoeuclidean space (t, x, y). On the searched surface the isothermal co-ordinate system always can be chosen in the form

$$\dot{\vec{x}}^2 = g_{00} = -g_{11} = -\vec{x}'^2, \quad g_{01} = (\dot{\vec{x}}\vec{x}') = 0. \tag{6}$$

In this case the equations of motion $\delta\sqrt{-g}/\delta x_\mu = 0$ are reduced to the D'Alambert equation for $x_\mu(\sigma,\tau)$

$$\ddot{x}_\mu - x''_\mu = 0. \tag{7}$$

Instead of searching for the vector $x_\mu(\sigma,\tau)$ which describes the co-ordinates of the string world surface through the joint solution of equations of motion (7) and of the nonlinear conditions (6) one can look for the first and second fundamental forms of the string world surface /10-12/. According to the basic theorem in the differential geometry /5/ the coefficients of these forms determine the position of the surface in the space up to translations and rotations. The first fundamental form is the squared interval between the two neighbouring points on the surface and has the form

$$ds^2 = dx_\mu dx^\mu = g_{00}(d\tau^2 - d\sigma^2) = \dot{\vec{x}}^2(d\tau^2 - d\sigma^2). \tag{8}$$

The second fundamental form is the square of the perpendicular from the given point of the surface to the tangent plane at the neighbouring point

$$d\ell^2 = L\, d\tau^2 + 2M\, d\tau\, d\sigma + N\, d\sigma^2. \tag{9}$$

The functions L, M and N define the projections of the vectors $\ddot{\vec{x}}$, $\dot{\vec{x}}'$ and $\vec{x}''$, respectively, onto the unit space-like vector $\vec{m}$ orthogonal to the $\dot{\vec{x}}$ and $\vec{x}'$

$$\ddot{\vec{x}} = \Gamma^0_{00}\dot{\vec{x}} + \Gamma^1_{00}\vec{x}' + L\vec{m},$$
$$\dot{\vec{x}}' = \Gamma^0_{01}\dot{\vec{x}} + \Gamma^1_{01}\vec{x}' + M\vec{m}, \qquad (10)$$
$$\vec{x}'' = \Gamma^0_{11}\dot{\vec{x}} + \Gamma^1_{11}\vec{x}' + N\vec{m},$$

where Γ^i_{jk} are the Christoffel symbols of the second kind on the string world surface

$$\Gamma^i_{jk} = \Gamma^i_{kj} = \frac{1}{g} g^{il}\left(\frac{\partial g_{lj}}{\partial u^k} + \frac{\partial g_{lk}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^l}\right),$$
$$i,j,k,l = 0,1, \quad u^0 = \tau, \; u^1 = \sigma.$$

From the equations (7) and (10) it follows that

$$L = N. \qquad (11)$$

The arbitrariness in the choice of the co-ordinate system on the searched surface, remaining after imposing the conditions (6) can be used to fix the coefficients of the second fundamental form

$$(\ddot{x} \pm \dot{x}')^2 = \Psi''^2_{\pm}(\sigma \pm \tau) = -q^2_{\pm}, \qquad (12)$$

where $\Psi_{\pm}$ are arbitrary functions in the general solution of eq.(7) $x_\mu(\sigma,\tau) = \left(\Psi_{+\mu}(\sigma+\tau) + \Psi_{-\mu}(\sigma-\tau)\right)/2$.

Taking into account the invariance of eqs.(6-7) under the conformal transformations $\tilde{\sigma} \pm \tilde{\tau} = f_{\pm}(\sigma \pm \tau)$ it can be shown easily that the condition (12) may always be satisfied by the corresponding choice of the functions $f_{\pm}$ /12/. The quantities $q_{\pm}$ are arbitrary given beforehand functions of the variables $\sigma \pm \tau$, respectively. For simplicity we shall take these functions as constants.

Substituting the expansions (10) into (12) and taking into account that $\vec{m}^2 = -1$, we get

$$(L \pm M)^2 = q^2_{\pm}. \qquad (13)$$

So, only the coefficient $g_{00} = \dot{\vec{x}}^2$ remains unfixed in the first and in the second fundamental forms. It must be defined from the Gauss and Peterson-Codazzi equations connecting the coefficients of both fundamental forms of the surface.

The Peterson-Codazzi equations /5/ in the case under consideration have the form

$$\frac{\partial L}{\partial \sigma} - \frac{\partial M}{\partial \tau} = 0, \quad \frac{\partial M}{\partial \sigma} - \frac{\partial L}{\partial \tau} = 0,$$

hence it follows that

$$L = [\lambda_+(\sigma+\tau) + \lambda_-(\sigma-\tau)]/2, \qquad (14)$$
$$M = [\lambda_+(\sigma+\tau) - \lambda_-(\sigma-\tau)]/2,$$

where $\lambda_{\pm}$ are arbitrary functions. It is obvious that (14) does not contradict (13) if we put $q_{\pm} = \lambda_{\pm}$.

Now the Gauss equation remains only. It can be obtained more easily in the following way. Using equations (6), (11) and (13) we find the Gauss curvature of the string world surface

$$K = \frac{LN - M^2}{g_{00}g_{11} - g^2_{01}} = \frac{L^2 - M^2}{-g^2_{00}} = \frac{q_+ q_-}{(\dot{\vec{x}}^2)^2}$$

and substitute it into eq.(2)

$$u_{\tau\tau} - u_{\sigma\sigma} = 2(q_+ q_-)e^{u}, \qquad (15)$$

where $e^{-u} = g_{00} = \dot{\vec{x}}^2$.

From comparison of (15) and (2) a direct connection follows between the internal geometries of the minimal surfaces and of the surfaces with a constant Gauss curvature in three dimensional pseudoeuclidean space E^1_3 : the products of the corresponding coefficients of the first fundamental forms of these surfaces are equal to unity. This enables us to simulate the minimal surfaces in the E_3^1 with the aid of the hyperboloids in E_3^1 -space[6].

Equation (15) has to be complemented by the boundary conditions if the relativistic string is finite. For example, for a closed string $0 \leq \sigma \leq \pi$ we have

$$u(0,\tau) = u(\pi,\tau).$$

The boundary conditions for the free finite string and for the string with masses at the ends were considered in papers[11,12].

4. Investigation of the Soliton Solutions

Now we shall study the fundamental equation (2). The general solution of this equation, obtained by Liouville,[13], is

$$u(x,t) = \ln\left(\frac{8}{|R|}\frac{|f'(x+t)g'(x-t)|}{[f(x+t)-g(x-t)]^2}\right), \qquad (16)$$

where f and g are arbitrary functions and prime denotes the differentiation with respect to the function argument. However, the general solution of eq. (2) is not of great interest, physically. From this point of view the particular solutions of eq. (2) - the solitons[2)] are more attractive. They have the form

$$u(x,t) = F(x - vt), \qquad (17)$$

where $|v| < 1$. To see these solutions in formula (16) is not easy. It is more simple to insert (17) into (2) that gives the following ordinary differential equation for F

2) Eq. (2) in variables $x \pm t$ was studied in paper[14] by the inverse scattering method. However, soliton solutions of this equation were not discussed from the viewpoint of the particle physics there.

$$(1-v^2)F'' = -Re^F.$$

If $R > 0$, then we obtain

$$g_{00} = e^{u_1} = \frac{m^2}{32R}\operatorname{sech}^2\left(\frac{m}{8}\frac{x - vt - x_0}{\sqrt{1-v^2}}\right), \qquad (18)$$

where m is arbitrary constant. As will be shown below, m is the soliton mass (18). If $R < 0$, then we get the solitons of the two kinds

$$e^{u_2} = \frac{m^2}{32|R|}\operatorname{csch}^2\left(\frac{m}{8}\frac{x - vt - x_0}{\sqrt{1-v^2}}\right), \qquad (19)$$

$$e^{u_3} = \frac{m^2}{32|R|}\operatorname{cosec}^2\left(\frac{m}{8}\frac{x - vt - x_0}{\sqrt{1-v^2}}\right). \qquad (20)$$

Having the obvious solutions (18-20), one can easily choose the functions f and g in the general solution (16) so that (16) will result in (18-20). For example, for the soliton (18) taking into account the formula

$$\operatorname{sech}^2\tfrac{1}{2}(y-z) = \frac{4e^y e^z}{(e^y + e^z)^2},$$

we have to put

$$f(x+t) = \frac{8}{m}\exp\left\{\frac{m}{8}\sqrt{\frac{1-v}{1+v}}(x+t-x_0)\right\},$$

$$g(x-t) = -\frac{8}{m}\exp\left\{-\frac{m}{8}\sqrt{\frac{1+v}{1-v}}(x-t-x_0)\right\}.$$

The solutions (18) and (19) are solitary waves, moving with a velocity less than the speed of light. The solution (20) is the periodical soliton, describing the "comb" of the waves. It should be noted that (19) and (20) have the nonintegrable singularities $e^{u_i} \sim z^{-2}$, when $z \to 0$, $z = (x - vt - x_0)/\sqrt{1-v^2}$, $i = 1, 2$; and thereby formally they do not satisfy the requirements usually imposed on the soliton solutions /1,2/. However, to simplify the terminology, we shall call these solutions as solitons, and it will be shown that their singularities do not lead to the principal difficulties in attempting to interpret these solutions as extended particles.

Apart from one-soliton solutions the equation (2) has the n-soliton ones as it follows from the results of paper /14/. Such a solution describes one soliton, moving with arbitrary velocity and $(n-1)$ solitons, moving with the speed of light. Only the soliton (18) follows from the n-soliton solution at $n=1$, while the other solitons (19-20) cannot be obtained in such a way. For simplicity, we shall restrict our consideration to the one-soliton solutions only.

The solitons (18-20), viewed as solutions of the Einstein equations (1), do not lead to the Euclidean metric at infinity. And what is more, for the solitons (18,19) the metric tensor vanishes in this case. So, the infinity is an essential singularity of these solutions from the viewpoint of the gravitation theory /15/. This is the reason that makes it impossible to use the energy-mementum pseudotensor of the gravitational field for finding the soliton energy. It is not easy to solve this problem in the theory of the relativistic string, as the Hamiltonian constructed in accordance with the canonical rules vanishes in this theory /9/. This difficulty can be overcome if we consider eq. (2) as the Euler equation for the Lagrangian

$$\mathcal{L} = \frac{1}{2}(u_t^2 - u_x^2) + Re^u. \tag{21}$$

The potential energy in this Lagrangian $V(u) = -Re^u$ is a monotonous function of variable u. Usually, the solitons are considered in the models with spontaneous symmetry breaking where $V(u)$ has at least two minima /2/. This difference between the model under consideration and usually studied ones leads to some peculiarities. For instance, it is impossible to introduce the topological charge for the solitons of eq. (2).

Now let us show that using the Lagrangian (21) we can define the total energy, momentum, and mass of the solutions (18-20) in a correct relativistic correlation. We take the energy-momentum tensor for the solitons in the following form

$$\Theta^{\mu\nu} = T^{\mu\nu} + \frac{m^2}{32}\eta^{\mu\nu} - \frac{m^2}{16} v^\mu v^\nu, \tag{22}$$

where $T^{\mu\nu}$ is the canonical energy-momentum tensor of the field $u(x,t)$, corresponding to the Lagrangian density (21)

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu u)}\partial^\nu u - \eta^{\mu\nu}\mathcal{L},$$

$\mu, \nu = 0, 1$, $\eta^{00} = -\eta^{11} = 1$, v^μ is the velocity vector of the soliton $v^0 = \frac{1}{\sqrt{1-v^2}}$, $v^1 = \frac{v}{\sqrt{1-v^2}}$. The term supplemented to $T^{\mu\nu}$ in formula (22) does not depend on the co-ordinates x, t and leads to the convergent integrals at infinity for the energy and momentum of the solitons

$$P^i = \int \Theta^{i0}[u_c(x,t)]dx. \tag{23}$$

Substituting the solutions $u_i(x,t)$ (18-20) into (23) we obtain

$$E_1 = -\frac{m}{\sqrt{1-v^2}}\ , \quad P_1 = -\frac{mv}{\sqrt{1-v^2}}\ ,$$
$$E_i = \frac{mI_i}{\sqrt{1-v^2}}\ , \quad P_i = \frac{mI_i v}{\sqrt{1-v^2}}\ , \quad i=2,3\ , \tag{24}$$

where $I_2 = \int_0^\infty dx\, csch^2 x\,,\ I_3 = \int_0^{\pi/2} dx\,(ctg^2 x - 1)$. So in the first case one can consider the constant m as a mass of soliton. For the solitons (19) and (20) the masses are equal to mI_i , $i=2,3$. The integrals I_i , divergent at zero, can be regularized by introducing a cut-off, for example. However, it will be sufficient that between E_i and P_i in formulae (24) a correct relativistic correlation does exist. Just this fact enables us to interpret the solitons as extended particles with nonzero rest mass even in the classical theory.

Let us go to the investigation of the soliton stability. At first, we consider the static soliton solutions which are defined by formulae (18-20) at $v=0$. We represent the solution of eq.(2) in the following form

$$u(x,t) = U_i(x) + e^{i\omega t}\cdot \Psi(x)\,. \tag{25}$$

Substituting (25) into (2) leads to the equation for $\Psi(x)$ which has the form of the one-dimensional Schrödinger equation with the potential $V[U_i(x)] = -R\, e^{U_i(x)}$:

$$\left(-\frac{d^2}{dx^2} + V[U_i(x)]\right)\Psi(x) = \omega^2 \Psi(x)\,, \tag{26}$$

where

$$V[U_1(x)] = -\frac{m^2}{32} sech^2\left(\frac{m}{8}x\right), \tag{27}$$

$$V[U_2(x)] = \frac{m^2}{32} csch^2\left(\frac{m}{8}x\right), \tag{28}$$
$$V[U_3(x)] = \frac{m^2}{32} cosec^2\left(\frac{m}{8}x\right). \tag{29}$$

If in eq. (26) $\omega^2 > 0$, then solution $U_i(x)$ is stable in the classical theory and by virtue of the relativistic invariance $U_i(x,t)$ is stable, as well. When $\omega^2 < 0$, then the correction to $U_i(x)$ in eq.(25) increases exponentially with time and the soliton is unstable.

Eq. (26) with potentials (27-29) can be solved exactly [16]. In the first case ω^2 has one negative value $\omega_1^2 = -m^2/64$ and the continuous spectrum, beginning from the translation mode $\omega_c^2 = 0$. Because of $\omega_1^2 < 0$ the soliton solution $U_1(x,t)$ is not stable already in the classical theory.

For the potential (28) ω^2 has the continuous spectrum $\omega^2 > 0$ and the translation mode $\omega = 0$ which again adjoins the continuous spectrum. The solution $U_2(x,t)$ is stable.

The potential (29) can be reduced to the Pöschl-Teller potential [16]. The periodicity of this potential turns out to be unessential for the solutions of eq. (26) as the neighbouring potential wells are separated by the impenetrable barriers. Therefore, we can restrict ourselves to consideration of one of such wells. In this case there are only discrete spectrum $\omega_n^2 = m^2(n+1)^2/64$, $n=1,2,\ldots$ and zero frequency mode $\omega_0 = 0$. The solution $U_3(x,t)$ is stable.

So in the classical theory the soliton $U_1(x,t)$ can be considered as an unstable particle with mass m and lifetime $\sim 8m^{-1}$. The solitons $U_2(x,t)$ and $U_3(x,t)$ describe the stable particles.

5. Quantum Theory

There are many approaches to the construction of quantum theory for the field models, having the particle like classical solutions /2,17,18/. The ideas of these methods differ at the first sight but the basic equations determining the spectrum of the states, turn out to be practically the same. Without going into details, we shall follow the so-called canonical quantization of the particle like solutions which is more close to the usual field approach although a somewhat formal /18/.

The field $\mathcal{U}(x,t)$ will be represented in the form

$$\mathcal{U}(x,t)=\mathcal{U}_c(x-x_0)+\Psi(x,t), \qquad (30)$$

where $\mathcal{U}_c(x-x_0)$ are the soliton solutions (18-20) with $v=0$. The co-ordinates of the system will be the center-of-mass position of the soliton $x_0(t)=x_0+vt$ and the field $\Psi(x,t)$. The canonical conjugate momenta are $p(t)$ and $\mathcal{T}(t)$, respectively.

After substituting (30) into (22) the total Hamiltonian can be divided into the free term and the interaction Hamiltonian

$$H=H_0+H_I,$$

where $H_0=m+\frac{1}{2}\int dx\left(\mathcal{T}^2+\Psi'^2-\Psi^2 R e^{\mathcal{U}_c}\right)$ is the Hamiltonian of the particle with mass m and of the field $\Psi(x,t)$, imbedded into the external classical field $-Re^{\mathcal{U}_c}$. The Hamiltonian H_I, which describes the interaction of $\Psi(x,t)$ with the soliton, depends on x_0, p, Ψ and $\mathcal{T}$. The explicit form of H_I is complicated enough /18/ and we do not write it here.

It follows from the Hamilton equations with H_0 that

$$\ddot{\Psi}-\Psi''=Re^{\mathcal{U}_c}\Psi.$$

Below we will use the usual method of quantization in external field /19/ by the expansions

$$\Psi(x,t)=\sum_k\frac{1}{\sqrt{2\omega_k}}\left[b_k\Psi_k(x)e^{-i\omega_k t}+b_k^+\Psi_k^+(x)e^{i\omega_k t}\right],$$

$$\mathcal{T}(x,t)=\sum_k(-i)\sqrt{\tfrac{1}{2}\omega_k}\left[b_k\Psi_k(x)e^{-i\omega_k t}-b_k^+\Psi_k^+(x)e^{i\omega_k t}\right],$$

where $\Psi_k(x)$ compose a complete set of the solutions of eq. (26) with excluded translation mode

$$\sum_k\Psi_k(x)\Psi_k^+(y)=\delta(x-y)-\frac{1}{m}\mathcal{U}_c'(x)\mathcal{U}_c'(y).$$

The symbol $\sum_k$ represents also here the integration over k, if necessary. The canonical commutation relations

$$[b_k,b_{k'}^+]=\delta_{kk'},\quad [b_k,b_{k'}]=[b_k^+,b_{k'}^+]=0$$

are postulated and the Hilbert space of the states $|P,\{k_i\}\rangle=b^+_{\{k_i\}}|P\rangle$ is constructed. Here P is a common momentum and $\{k_i\}$ is a set of meson momenta. Perturbation theory can be developed on this basis as usual.

After the transition to the normal product of the operators b_k the free Hamiltonian H_0 takes the form

$$H_0=m+\sum_k\omega_k b_k^+ b_k. \qquad (31)$$

Calculating the matrix elements of H_0 over the state vectors $|P,\{k_i\}\rangle$, we obtain the energy spectrum of the system in zero approximation of perturbation theory. It is obvious that this spectrum is defined completely by the eigenvalues ω in eq.(26).

The soliton (18), being unstable in classical theory, will be unstable in quantum case as well because the correction to the energy (31) from the continuous spectrum is pure imaginary in this case.

In quantum theory the solution (19) corresponds to the stable particle with mass m and field $\Psi(x,t)$ describes the massless mesons. In the model under consideration there is no conservation law of the soliton number (the topological charge), therefore the quantum transitions of the soliton particles into the meson states are not in principle forbidden.

The periodical soliton (20) gives the most rich spectrum

$$E_o = m, \quad E_n = m + \frac{m}{8}(n+1), \quad n = 1, 2, \ldots$$

It is interesting that the spectrum is equidistant if we do not consider the lowest state with energy E_o. The spacing of the energy levels is defined by the soliton mass $m/8$ and can take any value. The states of this spectrum are time-independent only if the interaction H_I is neglected. The Hamiltonian H_I leads to the transition between these states and in reality we have here a series of resonances.

The periodical soliton and the spectrum created by it are well-suited to the theory of the closed relativistic string /9/. In the usual approach, this model has the equidistant spectum of the stationary states which form a basis for the construction of the dual resonance models. The zero width of the energy levels is an essential defect of these models. In this connection, the mass spectrum obtained by taking into account the soliton solutions in the theory of the relativistic string is more realistic.

6. Conclusion

The basis of the geometrical approach to the relativistic string theory and to the Born-Infeld scalar field model is the change of the variables $x_\mu(\sigma,\tau)$ to one function $u(\sigma,\tau) = -\ln \dot{\vec{x}}^2$. Mathematically, the string co-ordinates $x_\mu(\sigma,\tau)$ and the function $u(\sigma,\tau)$ carry the same information about the dynamics of the system because we can reconstruct $x_\mu(\sigma,\tau)$ from the known function $u(\sigma,\tau)$ by integrating formulae (10). However, this transition cannot be considered as a canonical transformation, and quantization in terms of the variables $x_\mu(\sigma,\tau)$ and $u(\sigma,\tau)$ gives different results. In this connection, a question arises: what are variables which should be used to quantize the nonlinear models? Probably, the physical evaluation of the final results may be here the only decisive criterion.

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