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I.V.Amirkhanov, G.V.Grusha, R.M.Mir-Kasimov

INVESTIGATION OF ASYMPTOTIC BEHAVIOUR OF THE QUASIPOTENTIAL AMPLITUDE IN COMPLEX ORBITAL MOMENTUM PLANE

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Амирханов И.В., Груша Г.В., Мир-Касимов Р.М.

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Исследование асимптотического поведения квазипотенциальной амплитуды в плоскости комплексного орбитального момента

Исследовано асимптотическое поведение квазипотенциальной амплитуды рассеяния двух релятивистских частиц при больших значениях комплексного орбитального момента. В иллюстративных целях изучены аналитические и асимптотические свойства амплитуды для точно решаемого случая рассеяния на δ-потенциале.

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Investigation of Asymptotic Behaviour of the Quasipotential Amplitude in Complex Orbital Momentum Plane

The asymptotic behaviour of the quasipotential scattering amplitude of two relativistic particles at large values of complex orbital momentum is investigated. The analytic and asymptotic properties of the amplitude are exhibited on example of the exactly soluble case of the scattering on δ-potential.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

In this paper the asymptotic behaviour of the quasipotential amplitude [1-5] of two-particle scattering at large complex orbital momenta is investigated.

We proceed from quasipotential type equation [6] in relativistic configurational space. The differential form of this equation for the particles of equal masses has the form $(\hbar = c = m = 1)$:

$$\hat{h}_0 \psi_\ell(r,\chi) = \gamma_\ell(r,\chi) V(r,\chi) \psi_\ell(r,\chi), \qquad (1.1)$$

where

$$\hat{h}_0 = \frac{d^2}{dr^2} - U_\ell^{(1)}(r,\chi) - U_\ell^{(2)}(r,\chi)\frac{d}{dr}.$$

The functions $U_\ell^{(1,2)}(r,\chi)$ and $\gamma_\ell(r,\chi)$ depend on free solutions of equation (1.1) (relativistic analogs of Riccati-Bessel functions $s_\ell(r,\chi)$, $c_\ell(r,\chi)$ [5]).

The relativistic configurational r-representation is connected with momentum representation by Fourier transformation [4,5] with kernel:

$$\xi(\vec{q},\vec{r}) = (\mathrm{ch}\chi_q - \vec{n}\vec{n}_q \,\mathrm{sh}\chi_q)^{-1-ir}, \qquad (1.2)$$

where $\vec{r} = r\vec{n}$, $\vec{q} = \vec{n}_q \,\mathrm{sh}\chi_q$ $(\vec{n}^2 = 1)$.

As is shown in [7], in the case of equation (1.1) the partial amplitude of the scattering on local, spherically-symmetric, nonsingular quasipotentials $V(r,\chi)$, providing the fulfillment of inequalities:

$$\int_0^\infty r|V(r,\chi)|e^{\nu r}\, dr < \infty\,, \quad 0<\eta<\mu\,, \qquad (1.3)$$

$$\int_0^\infty \left(\frac{r}{r+\delta}\right)\left|\frac{\partial}{\partial\chi}V(r,\chi)\right|e^{\nu r}\, dr < \infty\,, \quad 0<\delta<1\,. \qquad (1.4)$$

is meromorphic in the direct product of the orbital momentum region $\mathrm{Re}\,\ell+1>\delta$ and the strip $|\mathrm{Im}\chi| < \frac{\mu}{2} < \pi(|\chi| \ge \epsilon > 0)$, belonging to the region of quasipotential analyticity in rapidity χ.

Considering the asymptotics of the scattering amplitude (Sec. 2) we confine ourselves by the region of real rapidities, corresponding to the elastic scattering. In Sec. 3 the exact soluble model of the scattering on the potential $V(r)=-g\delta(r-a)$ is studied.

2. ASYMPTOTIC BEHAVIOUR OF THE SCATTERING AMPLITUDE AT LARGE COMPLEX ORBITAL MOMENTA

In this section the method of asymptotic estimate, proposed in ref. [8] for Schrödinger equation, is used.

First let us consider the case of real momenta ℓ. We proceed from the Fredholm equation for the wave function $\psi_\ell^{(+)}(r,\chi)$ (ref. [4])

*Here and further the dependence of the quasipotential on the energy is not shown explicitly.

$$\psi_\ell^{(+)}(r,\chi)=s_\ell(r,\chi)+\int_0^\infty g_{q\ell}^{(+)}(r,r')V(r')\psi_{q\ell}^{(+)}(r',\chi)dr', \qquad (2.1)$$

where

$$g_{q\ell}^{(+)}(r,r') = -\frac{\nu_\ell(r')}{\mathrm{sh}\chi}[\,\theta(r-r')s_\ell(r',\chi)e_\ell^{(1)}(r,\chi) + \theta(r'-r)s_\ell(r,\chi)e_\ell^{(1)}(r',\chi)]. \qquad (2.2)$$

In Appendix A it is shown that at large real ℓ the kernel of this equation may be uniformly bounded by separable kernel, vanishing at $\ell\to\infty$:

$$|g_{q\ell}^{(+)}(r,r')| < \sqrt{rr'}\sqrt{\frac{\pi}{2\ell+1}}\,. \qquad (2.3)$$

Hence, the inequality follows

$$\int_0^\infty \sqrt{r}|V(r)||\psi_\ell^{(+)}(r,\chi)|dr < \frac{\int_0^\infty |s_\ell(r,\chi)|\cdot|V(r)|\sqrt{r}\,dr}{1-\sqrt{\frac{\pi}{2\ell+1}}\int_0^\infty r|V(r)|\,dr}\,, \qquad (2.4)$$

which is valid for large ℓ:

$$2\ell+1>\pi^2\left[\int_0^\infty r|V(r)|\,dr\right]^2\,. \qquad (2.5)$$

Using the definition of the scattering amplitude

$$A_\ell^{(+)}(\chi) = -\frac{1}{\mathrm{sh}\chi}\int_0^\infty \nu_\ell(r)\, s_\ell(r,\chi)V(r)\,\psi_\ell^{(+)}(r,\chi)dr\,, \qquad (2.6)$$

we obtain from (2.5) (comp. [8]):

$$|A_\ell^{(+)}(\chi)| < \frac{1}{|\mathrm{sh}\chi|}\cdot\frac{\int_0^\infty |s_\ell(r,\chi)|^2\cdot|V(r)|\cdot dr}{1-\sqrt{\frac{\pi}{2\ell+1}}\int_0^\infty r'|V(r')|dr'}\,. \qquad (2.7)$$

The estimate (2.7) shows that at $\ell \gg 1$ the upper bound for the absolute value of amplitude coincides with the Born approximation.

Then, if the requirement is fulfilled

$$\int_0^\infty r^2 |V(r)|\, dr < \infty , \qquad (2.8)$$

the scattering amplitude $|A_\ell^{(+)}(\chi)|$ decreases more rapidly than $1/\ell$ (see ineq. (A.12)).

Now let us consider quasipotentials satisfying the condition:

$$|V(r)| \le g \frac{e^{-\mu r}}{r} . \qquad (2.9)$$

Since

$$\frac{e^{-\mu r}}{r} = \frac{\operatorname{ch} r(\pi-\mu)}{r \operatorname{sh} \pi r} - e^{-\pi r} \frac{\operatorname{ch} \mu r}{r \operatorname{sh} \pi r} , \qquad (2.10)$$

then for all r inequality is fulfilled:

$$\frac{e^{-\mu r}}{r} \le \frac{\operatorname{ch} r(\pi-\mu)}{r \operatorname{sh} \pi r} . \qquad (2.11)$$

Here in the right-hand side we write the relativistic Fourier-image of mass M spinless particle propagator [5], $\cos\mu = 1 - M^2/2$, $0 < M < 2$, $0 < \mu < \pi$.

Expanding the propagator $[M^2 - (p_0 - q_0)^2 + (p-q)^2]^{-1}$ in partial waves at $p_0 = q_0 = E_q$, we obtain:

$$\frac{g}{\operatorname{sh}\chi} \int_0^\infty |s_\ell(r,\chi)|^2 \frac{\operatorname{ch} r(\pi-\mu)}{r \operatorname{sh} \pi r} dr = \frac{g}{2 \operatorname{sh}\chi} Q_\ell \left(1 + \frac{M^2}{2 \operatorname{sh}^2 \chi}\right), \qquad (2.12)$$

where $Q_\ell(z)$ is the Legendre function of second kind.

Using inequality (2.11) and asymptotic expansion of the function $Q_\ell(z)$ [13], we obtain the upper bound for the scattering amplitude on quasipotentials (2.9):

$$|A_\ell^{(+)}(\chi)| < O(\lambda^{1/2} e^{-a\lambda}) , \qquad (2.13)$$

where

$$z = \operatorname{ch} a = 1 + \frac{M^2}{2 \operatorname{sh}^2 \chi}, \quad \lambda = \ell + 1/2 .$$

Finally, from asymptotic behaviour of the functions $s_\ell(R,\chi)$ at $R \ll \ell$ (C.4) it follows, that for finite potentials of radius R $|A_\ell^{(+)}(\chi)|$ decreases more rapidly than any exponent.

Now we put $\ell = i\lambda - 1/2$ where λ is positive real number. In this case the Green function (2.2) increases infinitly at $\lambda \to \infty$. Indeed, considering the expressions for the functions $e_\ell^{(1,2)}(r,\chi)$ at $r \operatorname{sh}\chi \gg 1$ (B.9), we can easily be convinced that like the Hankel functions [8,12], they cannot be simultaneously bounded at real r and χ.

We shall try to continue equation (2.1) in complex z-plane ($z = r + iy$) deforming the path in integrals to majorize the kernel by separable function, vanishing at $\lambda \to \infty$ (comp. [8]).

In App. B for the points z lying on some curve $\Gamma(\lambda)$, connecting $z=0$ and $z = \infty + \frac{i\pi\lambda}{2y}$, the inequalities for Legendre functions are obtained:

$$|Q^{i\lambda}_{-1/2 - iz}(\operatorname{ch}\chi)| < c\, I_\lambda(z \operatorname{sh}\chi)\, e^{-\frac{3\pi\lambda}{2}} \left| \frac{\Gamma(1/2 - iz)}{\Gamma(1/2 - i\lambda - iz)} \right| , \qquad (2.14)$$

$$|P^{-i\lambda}_{-\frac{1}{2}+iz}(\mathrm{ch}\chi)| < cI_\lambda(z\,\mathrm{sh}\chi)\,e^{\frac{\pi\lambda}{2}}\left|\frac{\Gamma(\frac{1}{2}-iz)}{\Gamma(\frac{1}{2}+i\lambda-iz)}\right|, \qquad (2.14)$$

where $I_\lambda(z\,\mathrm{sh}\chi) = |\lambda^2 + z^2\mathrm{sh}^2\chi|^{-1/4}$, C is positive number.

Hence, we come to inequalities for relativistic analogs of the Riccati-Bessel functions ($z \in \Gamma(\lambda)$, $\ell = i\lambda - \frac{1}{2}$.:

$$|s_\ell(z,\chi)| < \frac{C}{2} I_\lambda(z\,\mathrm{sh}\chi)\left|\frac{\Gamma(1-iz)}{\Gamma(\frac{1}{2}-iz)}\right|\cdot\left|\frac{P_\lambda(z)\,\mathrm{sh}\,\pi z e^{\pi\lambda}}{\mathrm{ch}\,\pi(z+\lambda)}\right|,$$

$$|s_\ell(z,\chi)\nu_\ell(z)| < \frac{C}{2} I_\lambda(z\,\mathrm{sh}\chi)\left|z\frac{\Gamma(\frac{1}{2}-iz)}{\Gamma(1-iz)}\right|,$$

$$|e_\ell^{(1)}(z,\chi)| < CI_\lambda(z\,\mathrm{sh}\chi)\left|\frac{P_\lambda(z)\,\Gamma(1-iz)}{\Gamma(\frac{1}{2}-iz)}\right|, \qquad (2.15)$$

$$|e_\ell^{(1)}(z,\chi)\nu(z)| < CI_\lambda(z\,\mathrm{sh}\chi)\left|\frac{\Gamma(\frac{1}{2}-iz)\,z}{\Gamma(1-iz)}\right| \times$$

$$\times\left|\frac{\mathrm{ch}\,\pi(z+\lambda)e^{-\pi\lambda}}{\mathrm{sh}\,\pi z}\right|,$$

where

$$C = \sqrt{\frac{\mathrm{sh}\chi}{2\pi}}, \quad P_\lambda(z) = \left|\frac{\Gamma^2(\frac{1}{2}-iz)}{\Gamma(\frac{1}{2}-iz-i\lambda)\Gamma(\frac{1}{2}-iz+i\lambda)}\right|,$$

and obtain the upper bound for the Green function along the path $\Gamma(\lambda)$:

$$|g^{(+)}_{q\ell}(z,z')| < \frac{C^2}{\mathrm{sh}\chi} I_\lambda(z\,\mathrm{sh}\chi)\cdot I_\lambda(z'\,\mathrm{sh}\chi)\left|\frac{z'e^{\pi z'}}{2\,\mathrm{sh}\,\pi z'}\right| \times \qquad (2.16)$$

$$\times\left|\frac{\Gamma(1-iz)\Gamma(\frac{1}{2}-iz')}{\Gamma(1-iz')\Gamma(\frac{1}{2}-iz)}\right|\cdot X(z,\lambda),$$

where

$$X(z,\lambda) = \left|\frac{P_\lambda(z)e^{\pi(z+\lambda)}}{2\,\mathrm{ch}\,\pi(z+\lambda)}\right|, \qquad (2.17)$$

$$X(z,\lambda) = \left|\frac{P_\lambda(z)e^{\pi(z+\lambda)}}{2\,\mathrm{ch}\,\pi(z+\lambda)}\right|.$$

Choosing a quasipotential, which permits the integration in equation (2.1) and the definition of the scattering amplitude (2.6) along the path $\Gamma(\lambda)$, we take into account the chain of Green function poles at the points $z' = i, 2i, \ldots, ni$, where n is integer part of $\lambda/\mathrm{sh}\chi$. We put

$$V(z) = \hat{V}(z)\,(1-e^{-2\pi z}), \qquad (2.18)$$

where $\hat{V}(z)$ is a superposition of the Yukawa type potentials:

$$\hat{V}(z) = \frac{g}{z}\int_{\mu_0}^{L} \sigma(\mu)e^{-\mu z}\,d\mu, \quad 0 < \mu_0 < \pi. \qquad (2.19)$$

Using the estimate (2.16), we obtain inequality:

$$\left(1 - \frac{C^2}{\mathrm{sh}\chi}J(\lambda)\right)\cdot\left\{\int_{\Gamma(\lambda)}|\psi^{(+)}_\ell(z,\chi)|\cdot|\hat{V}(z)|\cdot|I_\lambda(z\,\mathrm{sh}\chi)| \times\right. \qquad (2.20)$$

$$\left.\times\left|\frac{\Gamma(1-iz)}{\Gamma(\frac{1}{2}-iz)}\right|\cdot X(z,\lambda)\,d|z|\right\} < CJ(\lambda),$$

where

$$J(\lambda) = \int_{\Gamma(\lambda)} \frac{|\hat{V}(z)| \cdot |z| \cdot d|z|}{|\lambda^2 + z^2 \operatorname{sh}^2\chi|^{1/2}} X(z,\lambda). \qquad (2.21)$$

In nonrelativistic limit ($|z| \gg \frac{h}{mc}$) the quantity $X(z,\lambda) \to 1$, $V(z) \equiv \hat{V}(z)$ and inequality (2.20) coincide with those, obtained in ref. [8].

However, in the relativistic region ($|z| \sim \frac{h}{mc}$) $X(z,\lambda)$ increases at $\lambda \gg 1$ (for example, $X(0,\lambda) = e^{\pi\lambda}/\pi$).

Let quasipotential provides the inequality for $J(\lambda)$

$$\frac{C^2}{\operatorname{sh}\chi} J(\lambda) < 1, \qquad (2.22)$$

this means, in fact, that the quasipotential is spherically unsymmetric in the region $|z| \lesssim h/mc$. Then from (2.20) we find the upper bound for the scattering amplitude:

$$|A^{(+)}_{i\lambda - \frac{1}{2}}(\chi)| < \frac{C'' J(\lambda)}{1 - \frac{C^2}{\operatorname{sh}\chi} J(\lambda)} < \text{const}, \qquad (2.23)$$

coinciding with the nonrelativistic one [8].

Finally, let us consider the behaviour of the partial amplitude in the region $0 < |\arg\lambda| < \pi/2$.

Nonrelativistic discussion of this case is based usually on the theorem about the $\hat{S}_\ell$ -matrix boundness:

$$|\hat{S}_\ell(q)| < e^{-\pi \operatorname{Im}\lambda} \quad \text{for } \operatorname{Re}\lambda > 0, \operatorname{Im}\lambda < 0, \qquad (2.24)$$

proved by Regge [10] for any potentials satisfying the condition:

$$\int_0^\infty r|V(r)|\,dr < \infty.$$

We proceed from the differential equation (1.1) at $\operatorname{Re}\lambda > 0$. Introducing its regular at $r=0$ solutions $\psi_\ell^{(0)}(r,\chi)$ and $\psi_\ell^{(0)*}(r,\chi)$, we can come to the relation:

$$K(\lambda,\chi) = \chi \int_0^\infty \frac{|\psi_\ell^{(0)}(r,\chi)|^2}{|W_q(s_\ell, c_\ell)|^2} [\operatorname{Im} R_\ell(r,\chi) + V(r) \operatorname{Im}\gamma_\ell(r,\chi)]\,dr = \qquad (2.25)$$

$$= -\frac{1}{\operatorname{sh}\chi} |\tau_\ell(\chi)|^2 \operatorname{sh} 2(\operatorname{Im}\delta_\ell(\chi) - \pi \operatorname{Im}\lambda/2).$$

Here $R_\ell(r,\chi)$ is a relativistic analog of centrifugal barrier [4], and $\tau_\ell(\chi)$ is the modulus of the Jost function [7].

The relation (2.25) shows that the scattering phase $\delta_\ell(\chi)$ is bounded by inequality

$$\operatorname{Im}\delta_\ell(\chi) > \frac{\pi}{2} \operatorname{Im}\lambda \qquad (2.26)$$

in the regions of λ-plane satisfying condition

$$K(\lambda,\chi) < 0, \quad (\operatorname{Re}\lambda > 0). \qquad (2.27)$$

Since the function $R_\ell + V \cdot \gamma_\ell$ is Hermitian, the real semiaxis $\operatorname{Re}\lambda > 0$ devides the regions in which $\operatorname{Im}\delta_\ell(\chi)$ has upper and lower bounds. In the general case, the complex structure of $K(\lambda,\chi)$ as a function of λ complicates the investigation of the dependence of sign $K(\lambda,\chi)$ on sign $\operatorname{Im}\lambda$.

As is known, for the special case of scattering on the nonrelativistic Yukawa potentials in the frame of the potential [10] and quasipotential [1] theory, one can prove that the partial amplitude decreases at $|\lambda| \to \infty$ along any ray including imaginary axes. This is the case when the analytic continuation from physical values of ℓ is unique and the total amplitude has the Regge-behaviour.

Concluding, we have shown in this section, that for the relativistic Yukawa-type potentials (2.19):

1) $|\hat{S}_\ell(x) - 1| < Ce^{-a\lambda}$ (λ is large real number);

2) $\hat{S}_\ell(\chi)$ is analytic in ℓ and bounded by inequality (2.28) in the region $K(\lambda,\chi) < 0$, $\mathrm{Re}\,\nu > 0$. To finish the proof of the Regge-behaviour of the total scattering amplitude on the relativistic Yukawa-type potentials (2.19), one must prove for this case the inequality (2.23) and show that $\mathrm{sign}\,K(\lambda,\chi) = \mathrm{sign}\,\mathrm{Im}\lambda$ in semiplane $\mathrm{Re}\lambda \geq 0$.

3. SCATTERING ON δ-POTENTIAL

Let us consider in this section the exactly soluble problem of scattering on δ-potential

$$V(r) = -g\delta(r-a), \quad g > 0. \tag{3.1}$$

Solving the equation for $\psi_\ell^{(+)}(r,\chi)$ (2.1) with potential (3.1), we obtain the expression for the scattering amplitude:

$$A_\ell^{(+)}(\chi) = \frac{g}{\mathrm{sh}\chi} \cdot \frac{\nu_\ell(a)\, s_\ell^2(a,\chi)}{1 - \frac{g}{\mathrm{sh}\chi}\nu_\ell(a)\, s_\ell(a,\chi) e_\ell^{(1)}(a,\chi)} \tag{3.2}$$

and for the scattering matrix

$$\hat{S}_\ell(\chi) = \frac{f_\ell^{(-)}(\chi)}{f_\ell^{(+)}(\chi)} = \frac{1 - \frac{g}{\mathrm{sh}\chi}\nu_\ell(a)\, s_\ell(a,\chi) e_\ell^{(2)}(a,\chi)}{1 - \frac{g}{\mathrm{sh}\chi}\nu_\ell(a)\, s_\ell(a,\chi) e_\ell^{(1)}(a,\chi)}. \tag{3.3}$$

In the limit $a \gg \frac{h}{mc}$, $\mathrm{sh}\chi = \frac{q}{mc} \ll 1$ the expressions (3.2) and (3.3) coincide with the nonrelativistic ones [14].

From analytic properties of relativistic analogs of the Riccati-Bessel functions it follows, that the scattering amplitude $A_\ell^{(+)}(\chi)$ (3.2) and the scattering matrix $\hat{S}_\ell(\chi)$ (3.3) are meromorphic in the direct product of the regions $\mathrm{Re}\,\ell + 1 > 0$ and $|\mathrm{Im}\chi| < \pi$, $|\chi| \geq \epsilon > 0$. Considering the transformation for the argument $\chi \to \chi e^{\pm i\pi}$ of the functions $s_\ell(a,\chi)$, $e_\ell^{(1,2)}(a,\chi)$, we can be convinced that in expression (3.3)

$$f_\ell^{(\pm)}(\chi e^{\pm i\pi}) \neq f_\ell^{(\mp)}(\chi)$$

and, correspondingly,

$$[f_\ell^{(+)}(i\eta)]^* \neq f_{\ell^*}^{(+)}(i\eta), \tag{3.4}$$

if ℓ is unequal to integer number.

The equation of motion of the pole $\ell(\chi)$ has the form:

$$1 - \frac{g}{\mathrm{sh}\chi}\nu_\ell(a)\, a_\ell(a,\chi) e_\ell^{(1)}(a,\chi) = 0, \tag{3.5}$$

or, equivalently,

$$1 + iag e^{-i\pi\ell} P^{-\lambda}_{-1/2 - ia}(\mathrm{ch}\chi) \cdot Q^{\lambda}_{-1/2-ia}(\mathrm{ch}\chi) = 0 , \qquad (3.6)$$

where $P^{-\lambda}_{-1/2-ia}(\mathrm{ch}\chi)$, $Q^{\lambda}_{-1/2-ia}(\mathrm{ch}\chi)$ are the Legendre functions, $\lambda = \ell + 1/2$.

Now let us investigate equation (3.5) in limit $2a\,\mathrm{sh}\frac{\chi}{2} \ll 1$ /15/. At $\mathrm{Re}\,\lambda > 0$ we obtain:

$$\lambda = \frac{ga}{2}\{1 - \frac{2\,\mathrm{sh}^2\frac{\chi}{2}}{1-\lambda^2}(a^2 - 1/2) + \ldots \qquad (3.7)$$

$$- e^{-i\pi\lambda}\frac{\pi\lambda}{\Gamma^2(1+\lambda)\sin\pi\lambda}(\mathrm{th}\frac{\chi}{2})^{2\lambda}\frac{a^{(\lambda+1/2)}}{a^{(-\lambda+1/2)}}\},$$

where

$$a^{(\lambda)} = e^{i\pi\lambda/2}\,\Gamma(-ia+\lambda)/\Gamma(-ia) .$$

From (3.7) we can easily find the threshold behaviour of the scattering amplitude poles (3.2):

$$\lambda(0) = \frac{ga}{2} > 0. \qquad (3.8)$$

Solution (3.8) coincides with the nonrelativistic one /14/. The roots of nonlinear equation (3.7) are complex in the scattering region $\chi^2 > 0$. In the region of bound states $\chi^2 < 0$ $(\chi = i\eta,\ 0 < \eta < \pi)$ the consideration of correctness to (3.8) of an order of $O(\sin^2\eta/2)$ leads to complex Regge trajectories $\ell(i\eta)$.

Another phenomenon, having no nonrelativistic analog, can be seen in the halfplane $\mathrm{Re}\lambda < 0$.

Indeed, at $\mathrm{Re}\,\ell + 1 \le 0$ the function $f^{(-)}_{\ell}(\chi)$ has a series of simple poles at the points

for which:

$$\lambda + 1/2 + ia = -n,\quad n = 0,1,2,\ldots \qquad (3.9)$$

The position of these "kinematical" poles does not depend on χ

$$\mathrm{Re}\,\ell_n = -1 - n,\quad \mathrm{Im}\,\ell_n = -a. \qquad (3.10)$$

At the points

$$\mathrm{Re}\,\ell_n = -1 - n,\quad \mathrm{Im}\,\ell_n = a$$

$\hat{S}_{\ell}(\chi)$ has a series of zeroes.

Now let us find the asymptotics of the scattering amplitude $A^{(+)}_{\ell}(\chi)$ (3.2) at $|\lambda| \to \infty$ $(|\arg\lambda| < \pi/2)$, complex χ.

Using the expansion of the functions $s_{\ell}(a,\chi)$, $e^{(1,2)}_{\ell}(a,\chi)$ at $a \ll |\lambda|$, $\mathrm{Re}\lambda \to \infty$ (C.3), (C.4), we obtain:

$$|A^{(+)}_{\ell}(\chi)| \le \frac{\pi|\mathrm{sh}\chi|}{2|\lambda|}|\mathrm{th}\frac{\chi}{2}|^{2\,\mathrm{Re}\lambda}\{1 - \frac{ag}{2|\lambda|}\}^{-1}. \qquad (3.11)$$

Thus, the scattering amplitude exponentially vanishes at $|\lambda| \to \infty$, $|\arg\lambda| < \pi/2$ if χ lies in the rapidities strip

$$|\mathrm{th}\frac{\chi}{2}| < 1 , \qquad (3.12)$$

or, equivalently, the energy region $(E = \mathrm{ch}\chi)$ is bounded by the condition

$$\mathrm{Re}\,E > 0 . \qquad (3.13)$$

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APPENDIX A

Let us prove the uniform boundness of $g^{(+)}_{q\ell}(r, r')$ at real χ and large real ℓ.

We consider the integral representations for the Legendre functions [12] ($\nu = -\frac{1}{2} - ir$, $\lambda = \ell + \frac{1}{2}$):

$$P_\nu^{-\lambda}(\mathrm{ch}\chi) = \left(\frac{\mathrm{sh}\chi}{2}\right) \frac{1}{\sqrt{\pi}\,\Gamma(\lambda+\frac{1}{2})} \int_{-\pi/2}^{\pi/2} \cos t^{2\lambda} (\mathrm{ch}\chi + \sin t \cdot \mathrm{sh}\chi)^{\nu-\lambda} dt$$

(comp. with the Poisson integral for the Bessel functions [12])

$$Q_\nu^{\lambda}(\mathrm{ch}\chi)\Gamma(\tfrac{1}{2}-\lambda) = e^{i\lambda\pi}\sqrt{\pi}\left(\frac{\mathrm{sh}\chi}{2}\right)^{-\lambda} \int_0^\infty \mathrm{sh}\,t^{2\lambda} (\mathrm{ch}\chi + \mathrm{sh}\chi \cdot \mathrm{ch}\,t)^{-\nu+\lambda+1}.$$

It is easy to verify that for real ℓ and χ:

$$|P_\nu^{-\lambda}(\mathrm{ch}\chi)| \le |P_{-1/2}^{-\lambda}(\mathrm{ch}\chi)|,$$
$$|Q_\nu^{\lambda}(\mathrm{ch}\chi)| \le |Q_{-1/2}^{\lambda}(\mathrm{ch}\chi)|. \qquad (A.3)$$

Further, for maxima of modulus of the gamma-functions ratios the inequalities

$$\max\left|\frac{\Gamma(1-ir)}{\Gamma(\ell+1-ir)}\right| \le \max\left|\frac{\Gamma(1-ir')}{\Gamma(\ell+1-ir')}\right|, \text{ at } r > r', \qquad (A.4)$$

$$\max\left|\frac{\Gamma(\ell+1-ir)}{\Gamma(1-ir)}\right| \le \max\left|\frac{\Gamma(\ell+1-ir')}{\Gamma(1-ir')}\right|, \text{ at } r < r' \qquad (A.5)$$

are valid ($\ell > 0$).

One can prove this fact using the expression:

$$\frac{\Gamma(u)}{\Gamma(u+v)} = e^{\gamma v} \prod_{n=0}^{\infty} \left(1 + \frac{v}{u+n}\right) e^{-\frac{v}{n+1}} :$$

Now, using the definitions of relativistic analogs of the spherical Bessel functions and (A.3)-(A.5), we find at $r < r'$:

$$\max|s_\ell(r,\chi)| \le \max|s_\ell(r',\chi)|, \qquad (A.6)$$

$$\max|e_\ell^{(1)}(r,\chi)| \ge \max|e_\ell^{(1)}(r',\chi)|, \qquad (A.7)$$

$$\max|s_\ell(r,\chi)e_\ell^{(1)}(r',\chi)| \le \max|s_\ell(r,\chi)e_\ell^{(1)}(r,\chi)|, \qquad (A.8)$$

(comp. with [9]).

At $r = r'$ we use the simple method of estimate, proposed in ref. [8].

From the expansion [16]:

$$\frac{e^{ir\chi_t}}{\mathrm{sh}\chi_t} = \frac{1}{r\,\mathrm{sh}^2\chi} \sum_{\ell=0}^{\infty} (2\ell+1) s_\ell^*(r,\chi) e_\ell^{(1)}(r,\chi) P_\ell(\cos\theta), \qquad (A.9)$$

where

$$\mathrm{ch}\chi_t = \mathrm{ch}^2\chi - \cos\theta \cdot \mathrm{sh}^2\chi,$$

we find:

$$\frac{1}{r^2\mathrm{sh}^2\chi}|s_\ell^*(r,\chi)e_\ell^{(1)}(r,\chi)| < \frac{1}{r}\int_{-1}^{+1} \frac{P_\ell(\cos\theta)\,d\cos\theta}{|\mathrm{sh}\chi_t(\theta)|}. \qquad (A.10)$$

Since

$$sh\chi_t(\theta) = \sqrt{2q^2(1-\cos\theta)(1+q^2\sin\frac{\theta}{2})},$$

we obtain hence at $\ell \gg 1$ (see [8]):

$$\frac{1}{r^2 sh^2\chi}|s^*_\ell(r,\chi)\, e^{(1)}_\ell(r,\chi)| < \frac{1}{r\, sh\chi}\sqrt{\frac{\pi}{2\ell+1}}. \qquad (A.11)$$

Let us denote, that inequality

$$\frac{1}{r\, sh\chi}|s_\ell(r,\chi)| < \frac{1}{\sqrt{2\ell+1}} \qquad (A.12)$$

is also valid.

This follows from the expansion

$$1 = \frac{1}{r^2 sh^2\chi}\sum_{\ell=0}(2\ell+1)\, s^*_\ell(r,\chi)\, s_\ell(r,\chi).$$

Taking into account the property (A.8), we obtain from (A.11) the desired estimate:

$$|g^{(+)}_{q\ell}(r,r')| < \sqrt{rr'}\sqrt{\frac{\pi}{2\ell+1}}. \qquad (A.13)$$

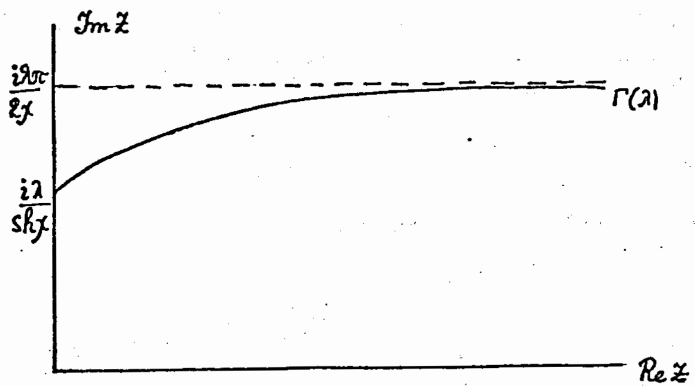

Fig. 1. The curve $\Gamma(\lambda)$.

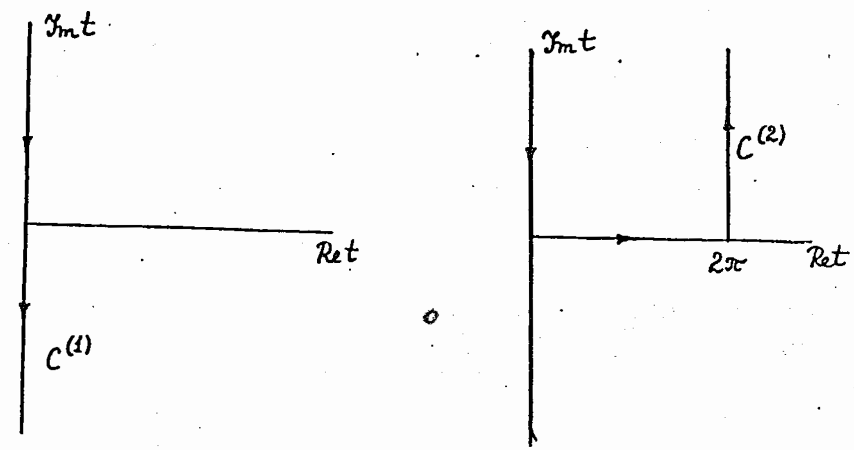

Fig. 2a,b. The contours in integrals (B.1), (B.2).

APPENDIX B

Using the definitions from [12] one can represent the Legendre functions in the form of contour integrals (see Figs. 2a,b):

$$Q^{i\lambda}_{-1/2-iz}(ch\chi) = e^{-\frac{3\pi\lambda}{2}}\frac{\Gamma(1/2-iz)}{\Gamma(1/2-iz-i\lambda)}\cdot\frac{i}{2}\int_{-i\infty+\eta}^{i\infty-\eta} R(t,z)\,dt, \qquad (B.1)$$

$$P^{-i\lambda}_{-1/2-iz}(ch\chi) = e^{\pi\lambda}\frac{\Gamma(1/2-iz)}{\Gamma(1/2-iz+i\lambda)}\cdot\frac{1}{2\pi}\int_{i\infty+\eta}^{2\pi+i\infty-\eta} R(t,z)\,dt, \qquad (B.2)$$

where

$$R(t,z) = e^{-\lambda(t-\pi/2)}(\mathrm{ch}\chi + \mathrm{sh}\chi\cdot\cos t)^{iz-\frac{1}{2}} ,$$

$\eta < \arg z < \pi - \eta$, λ and χ are real.

Integrals (B.1), (B.2) are the analogs of the Zommerfeld representation for the Bessel functions /12/.

Representing $R(t,z)$ in the form:

$$R(t,z) = e^{\lambda f(t,z)}(\mathrm{ch}\chi + \mathrm{sh}\chi\cdot\cos t)^{-\frac{1}{2}} , \qquad (B.3)$$

Let us find the asymptotic estimate for integrals (B.1), (B.2).

First we discuss the case $\mathrm{Im}\, z < \frac{\lambda}{\mathrm{sh}\chi}$. Let us put

$$z = iy,\quad y\,\mathrm{sh}\chi = \lambda\sin\phi,\quad \epsilon \le \phi \le \frac{\pi}{2} - \epsilon \quad (\eta = 0). \qquad (B.4)$$

The mesh points t_0 are defined by the condition:

$$\cos t_0 = \frac{-\mathrm{ch}\chi\cdot\mathrm{sh}\chi \pm i\cos\phi\sin\phi}{\mathrm{sh}^2\chi + \sin^2\phi}^{*)} . \qquad (B.5)$$

Since $\mathrm{Re}\cos t < 0$, we consider two mesh points with $\mathrm{Im}\, t_0 \gtrless 0$, $\pi > \mathrm{Re}\, t_0 > \pi/2$. Note, that $f(t,z) = \ln(\mathrm{ch}\chi + \mathrm{sh}\chi\cdot\cos t)$ has the branch points defined by condition $\cos t_b = -\frac{\mathrm{ch}\chi}{\mathrm{sh}\chi}$.

Thus, $\mathrm{Re}\, t_b = \pi$, and at $\phi \ne 0$ $\mathrm{Re}\, t_0 \ne \mathrm{Re}\, t_0'$.

We deform now the contours C_1 and C_2 so, that they pass through at least one mesh point and lie in the region $\mathrm{Re} f(t,iy) \le 0$. This region is defined by the condition

---

*) The nonrelativistic limit of (B.5) is:

$$\mathrm{Re}\, t_0 = \frac{\pi}{2},\quad \mathrm{Im}\, t_0 = \pm\mathrm{Arch}\frac{\lambda}{y} \qquad (\text{comp. } /8/).$$

$$\frac{\sin\phi}{\mathrm{sh}\chi}\ln|\mathrm{ch}\chi + \cos t\cdot\mathrm{sh}\chi| \ge \frac{\pi}{2} - \mathrm{Re}\, t . \qquad (B.6)$$

Calculating

$$|\mathrm{ch}\chi + \mathrm{sh}\chi\cdot\cos t_0| = \frac{\sin\phi}{\sqrt{\mathrm{sh}^2\chi + \sin^2\phi}},$$

$$\left|\frac{d^2 f(t)}{dt^2}\right| = \frac{\cos\phi}{\sin\phi}\sqrt{\mathrm{sh}^2\chi + \sin^2\phi},$$

we obtain for both integrals (B.1) and (B.2):

$$\left|\int R(t,iy)\,dt\right| < \frac{\mathrm{const}}{|\lambda^2 - y^2\mathrm{sh}^2\chi|^{1/4}} \qquad (B.7)$$

(comp. with the asymptotic estimate of the Bessel functions /8/).

For $\mathrm{Im}\, z > \frac{\lambda}{\mathrm{sh}\chi}$ it is enough to consider the region

$$\frac{\lambda}{\mathrm{sh}\chi} < \mathrm{Im}\, z < \frac{\pi\lambda}{2\chi} \quad (0 < \mathrm{Re}\, z < \infty). \qquad (B.8)$$

Indeed, at $\mathrm{Re}\, z \to \infty$ the functions

$$|e_\ell^{(1,2)}(z,\chi)| \underset{\ell = i\lambda - \frac{1}{2}}{\sim} |\exp i(z\chi + \frac{\pi}{4} - \frac{i\pi\lambda}{2})| \qquad (B.9)$$

are bounded on line $\mathrm{Im}\, z = \pi\lambda/2\chi$.

It is obvious that the condition

$$\mathrm{Re} f(t,z) = \lambda(\frac{\pi}{2} - \mathrm{Re}\, t) + \mathrm{Re}(iz\ln(\mathrm{ch}\chi + \mathrm{sh}\chi\cdot\cos t)) \le 0 \qquad (B.10)$$

puts the restriction to the position of points Z. In the nonrelativistic limit $\mathrm{Re}\,f(t,z)$ is the algebraic function of $\mathrm{ch\,Im}\,t$, that allows one to show the explicit form of equation for proper curve in Z-plane /8/.
In the case of Legendre functions, however, expression (B.10) is the transcendental function of $\mathrm{ch\,Im}\,t$. In this paper we assume, that at least one continuous curve, connecting $z = \frac{i\lambda}{\mathrm{sh}\,\chi}$ and $z = \infty + \frac{i\pi\lambda}{2\chi}$, exists (Fig. 1), for which the contours C_1 and C_2 may be deformed so, that they pass through the mesh points, and condition (B.10) is satisfied.

Then, taking into account the contribution of mesh points, we find the estimate for integral entering to (B.1), (B.2):

$$|\int R(t,z)\,dt| \lesssim \frac{\mathrm{const}}{|\lambda^2 + z^2\,\mathrm{sh}^2\chi|^{1/4}} . \qquad (B.11)$$

Here $\Gamma(\lambda)$ lies on the curve $z = i\lambda/\mathrm{sh}\,\chi$ (Fig. 1), in which the neighbourhood of point $z = \frac{i\lambda}{\mathrm{sh}\,\chi}$ is excluded.

APPENDIX C

Here we consider the general case of complex rapidities χ and complex momenta ℓ.
At $\mathrm{Re}\,\lambda \to \infty$ the asymptotic expansions

$$P_\nu^{\lambda}(\mathrm{ch}\,\chi) \sim \frac{1}{\Gamma(1+\lambda)}\left(\mathrm{th}\,\frac{\chi}{2}\right)^{\lambda} , \qquad (C.1)$$

$$Q_\nu^{\lambda}(\mathrm{ch}\,\chi) \sim \frac{1}{2\Gamma(1+\lambda)}\, e^{i\pi\lambda}\left(\mathrm{cth}\,\frac{\chi}{2}\right)^{\lambda}\Gamma(\lambda-\nu)\Gamma(1+\nu+\lambda) \qquad (C.2)$$

are valid /12/.
In expansions (C.1), (C.2) $|\mathrm{th}\frac{\chi}{2}| < 1$, that corresponds to energy halfplane $\mathrm{Re}\,E = \mathrm{Re\,ch}\,\chi > 0$. From (C.1), (C.2) it follows:

$$s_\ell(r,\chi)\, e_\ell^{(1,2)}(r,\chi)\, \nu_\ell(r) \sim \frac{r\,\mathrm{sh}\,\chi}{2\lambda} . \qquad (C.3)$$

At $r \ll |\ell|$, using asymptotic expansion of the gamma-functions entering into definition of $s_\ell(r,\chi)$ /4, 6/, we obtain:

$$s_\ell(r,\chi) \sim \lambda^{ir-\frac{1}{2}}\left(\mathrm{th}\frac{\chi}{2}\right)^{\lambda}\sqrt{\frac{\pi\,\mathrm{sh}\,\chi}{2}}\; e^{-\frac{i\pi}{2}(\lambda+\frac{1}{2})} . \qquad (C.4)$$

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