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Квантование по Бору-Зоммерфельду n-мерных нейтральных и заряженных пульсонов

Из условия квантования Бора-Зоммерфельда численным интегрированием на ЭВМ найден спектр масс 1) нейтральных и 2) имеющих элементарный заряд Q=1 n-мерных пульсонов (локализованных осциллирующих протяженных решений) в модели уравнения Клейна-Гордона с логарифмической нелинейностью. Численные эксперименты указывают на устойчивость по Ляпунову рассматриваемых пульсонов при любом n.

Работа выполнена в Лаборатории вычислительной техники и автоматизации ОИЯИ.

Препринт Объединенного института ядерных исследований. Дубна 1978

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The Bohr-Sommerfeld Quantization of n-Dimensional Neutral and Charged Pulsons

The spectrum of masses of 1) neutral and 2) having elementary charge Q=1, n-dimensional pulsons (i.e., localized oscillating extended solutions) is found by numerical integration using a computer in the framework of the Klein-Gordon equation with the logarithmic nonlinearity. Computer experiments point out that the pulsons under consideration are apparently stable at any n.

The investigation has been performed at the Laboratory of Computing Techniques and Automation, JINR.

Preprint of the Joint Institute for Nuclear Research. Dubna 1978

The development of the methods for a quantization of the Lorentz-invariant nonlinear (LIN) field equations is the problem of a present interest of the high energy theoretical physics. Indeed, the quantum chromodynamics, which is intensively developed to describe the physics of hadrons, deals with nonlinear Yang-Mills fields. Then one of the possible ways to imagine a "naked" "structureless" particle having nonzero mass, for example, leptons (an electron, a muon) is to suppose that it is an extended field "bunch" of a finite size, kept as a whole by self-action forces.

Such models are described at the classical level by the LIN equations. The quantization of the LIN models is carried out, as a rule, by semiclassical methods (see, for instance, refs. /1,2,3/). The first step to fulfil this programme is the investigation of the localized solutions (LS) of the LIN equations.

The most studied class of the LS of the nonlinear evolutionary partial differential equations are soliton solutions. We use the term "soliton" to denote LS of the type $R(\vec{x})\exp(-i\omega t)$ having a finite energy, charge, etc., where $R(\vec{x})$ may be, in general, a scalar, a spinor or a vector. It is clear, that LS of nonlinear and, in particular, of the LIN equations, of more general kind can exist, for example, the periodic ones, $u(\vec{x}, t+T) = u(\vec{x}, t)$ or still more general ones, which have $||u(\vec{x}, t+T)|| = ||u(\vec{x})||$ and lead to periodic in time physical quantities (classical densities of energy, charge), $H(x, t+T) = H(x, t)$, $q(\vec{x}, t+T) = q(\vec{x}, t)$. We call such periodic LS "pulsons"/6/ for brevity.

The "bions" (bound states of two solitons) of the famous sine-Gordon equation (SG) are the examples of pulsons in the case of one space dimension ($n=1$). In the case $n=3$ the spherically symmetric (ss) scalar pulsons have been found and their stability has been investigated[6] (these pulsons are weakly radiating and hence only approximately periodic). The importance of studying of classical LS of the LIN equations, which periodically depend on time, as a possible starting-point for development of the nonlinear quantum field theory (QFT) is demonstrated by the following analogy pointed out by the authors of paper[2b] "...the Bohr orbits of hydrogen are *not* time-independent solutions to classical equation of motion but rather are motions which are periodic in time".

The results of a semiclassical quantization (in the case $n=1$) of the *completely integrable* SG-equation by the functional integration technique (FIT) show that a semiclassical results may coincide[1,2] with exact quantum results[7] even at large coupling constants. The Bohr-Sommerfeld quantization (BSQ) of the SG-bions leads to the mass spectrum coinciding with one obtained by FIT[1,2]. Therefore, one might hope, that the BSQ will give the reasonable mass spectrum even in those cases when one cannot carry out the quantization by the FIT. The BSQ becomes particularly valuable in the realistic case $n=3$ because up to now not a single completely integrable LIN at $n=3$ is found, having the extended LS with the finite energy, charge, etc.

Apparently, the first example of the LIN equation admitting at arbitrary n (in particular, at $n=3$) an exact analytical soliton solution has been pointed out in paper[8]:

$$u_{\tau\tau} - \nabla_\xi^2 u + m^2 u - \ell^{-2}\ln(|u|^2 a^{n-1})u = 0. \qquad (1)$$

In the present paper the BSQ of the LS of this equation is carried out.

The properties of the classical LS of Eq. (1) obtained in refs.[8,9] and in this paper, together with the proposed "improved" modification of the model (1) are presented in Sec. 1. The BSQ of these LS gives the mass spectra of neutral "particles" (Sec. 2) and "particles" having an elementary charge $Q=1$ (Sec. 3).

1. *SOLITONS AND PULSONS OF THE KLEIN-GORDON EQUATION WITH THE LOGARITHMIC NONLINEARITY*

By introducing dimensionless variables $t, \vec{x}, \phi$

$$t = \tau\ell^{-1},\ \vec{x} = \vec{\xi}\ell^{-1},\ u = \ell^{\frac{1-n}{2}} G\phi,\ G^2 = \left(\frac{\ell}{a}\right)^{n-1}\exp[n+(\ell m)^2] \qquad (2)$$

equation (1) is transformed to the invariant form

$$\phi_{tt} - \nabla_x^2\phi - n\phi - \ln(|\phi|^2)\phi = 0, \qquad (3)$$

The initial equation (1) is obtained from the variational principle with the Lagrangian density

$$\mathcal{L} = |u_\tau|^2 - |\nabla_\xi u|^2 - (m^2+\ell^{-2})|u|^2 + \ell^{-2}|u|^2\ln(|u|^2 a^{n-1}) =$$
$$= G^2\ell^{-(n+1)}[|\phi_t|^2 - |\nabla_x\phi|^2 - (1-n)|\phi|^2 + |\phi|^2\ln|\phi|^2] \qquad (4)$$

in the case of the charged (complex) field and

$$\mathcal{L} = \frac{1}{2}[u_\tau^2 - (\nabla_\xi u)^2 - (m^2+\ell^{-2})u^2 + \ell^{-2}u^2\ln(u^2 a^{n-1})] =$$
$$= \frac{1}{2}G^2\ell^{-(n+1)}[\phi_t^2 - (\nabla\phi)^2 - (1-n)\phi^2 + \phi^2\ln\phi^2], \qquad (5)$$

in the case of the real (neutral) field.

The invariants of equation (1) are written in the form: the complex field energy

$$E_c = \int d^n\xi [\,|u_\tau|^2 + |\nabla_\xi u|^2 + (m^2 + \ell^{-2})|u|^2 -$$

$$- \ell^{-2}|u|^2 \ln(|u|^2 a^{n-1})\,] =$$

$$= G^2 \ell^{-1} \int d^n x [\,|\phi_t|^2 + |\nabla_x \phi|^2 + (1-n)|\phi|^2 - |\phi|^2 \ln|\phi|^2],$$

$$(1-n)|\phi|^2 - |\phi|^2 \ln|\phi|^2 = U(|\phi|), \qquad (6)$$

the real field energy

$$E_r = \frac{1}{2} G^2 \ell^{-1} \int d^n x [\,\phi_t^2 + (\nabla_x \phi)^2 + (1-n)\phi^2 - \phi^2 \ln \phi^2], \quad (7)$$

and the charge

$$Q = i\int d^n\xi [\,u_\tau u^* - u_\tau^* u] = iG^2 \int d^n x [\,\phi_t \phi^* - \phi_t^* \phi\,]. \qquad (8)$$

The soliton solutions of (3) are [8]

$$\phi(\vec{x}, t) = \exp(-\frac{1}{2}\omega^2)\cdot\exp(-i\omega t)\cdot\exp(-\frac{\vec{x}^2}{2}). \qquad (9)$$

The Lagrangian density (4), (5) and the Hamiltonian density (6), (7) are nonanalytical ones at $\phi=0$: $d^2U/d|\phi|^2(\phi=0) = 2m^2_{eff} = \infty$. The fact that the frequency ω is not restricted from above is just connected with $m^2_{eff} = \infty$. Thus, this model gives an example of soliton having the finite mass at $m_{eff} = \infty$.

It was noted in paper [9] that equation (1), or (3), admits the search for its solution in the factorized form

$$\phi(\vec{x}, t) = z(t)\cdot\exp(-\frac{\vec{x}^2}{2}), \; z(t) = y(t)\cdot\exp[-i\psi(t)], \qquad (10)$$

where $y(t)$ and $\psi(t)$ are real functions.

For such solutions it is easy to find the charge

$$Q = 2\pi^{n/2} G^2 \gamma, \quad \gamma = y^2 \psi_t = \text{const}, \qquad (11)$$

the energy of the complex field

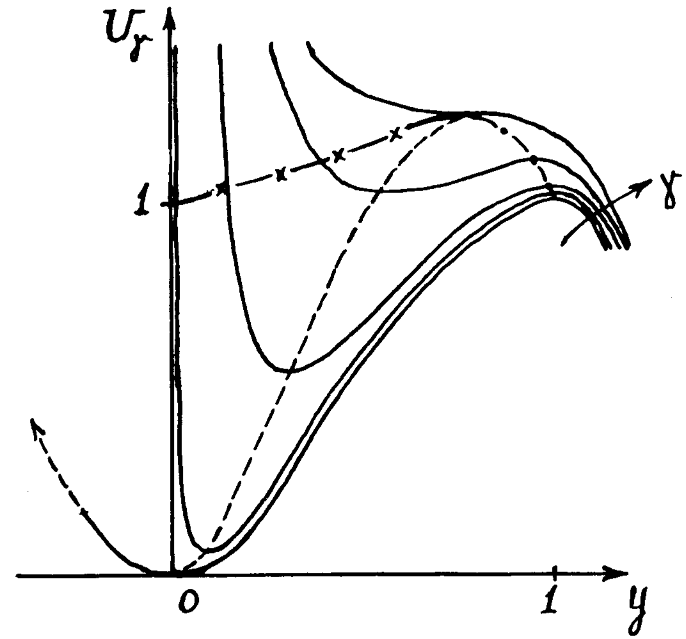

Fig. 1. The potential relief curves $U_\gamma(y)$. *The arrow shows the direction of* γ *increase;* —×—×— *is the curve connecting the points* $\{U_\gamma[y_{min}(\gamma)], y_{min}(\gamma)\}$ *at different* γ, —·—·— *is the same for* $\{U_\gamma[y_{max}(\gamma)], y_{max}(\gamma)\}$, - - - - - $\{U_\gamma[y_s(\gamma), y_s(\gamma)\}$.

$$E_c = \pi^{n/2} G^2 \ell^{-1} [\,y_t^2 + y^2 \psi_t^2 + y^2(1 - \ln y^2)], \; y^2\psi_t^2 = \gamma^2/y^2, \quad (12)$$

and the energy of the real field ($\gamma = 0$)

$$E_r = \frac{1}{2}\pi^{n/2} G^2 \ell^{-1} [\,y_t^2 + y^2(1 - \ln y^2)]. \qquad (13)$$

The formulae (12) and (13) express the energy conservation law at the motion of a point particle (p.p.) in the potential relief (see *Fig. 1*)

$$U_\gamma(y) = \frac{1}{2}[\,\gamma^2/y^2 + y^2(1 - \ln y^2)\,]. \qquad (14)$$

The dependence $y(t)$ describes the variation of the radius of the p.p. at its orbital motion and the conservation of an orbital momentum $y^2 \psi_t = \gamma = \mathrm{const}$ defines at given $y(t)$ the angular displacement of the p.p. $\psi(t)$.

The points y_s where the function $U_\gamma(y)$ has minimum at fixed γ (or at fixed Q, it is the same thing) correspond to solitons (9). The condition of minimum ($dU_\gamma/dy = 0$, $d^2U_\gamma/dy^2 > 0$) defines the region of the soliton stability: $\omega > \omega_{cr} = 1/\sqrt{2}$; accordingly, the maximum amplitude of stable solitons $y_{s\,max} = \exp(-1/2\,\omega^2_{cr}) = \exp(-1/4)$. The circular orbital motion ($y = \mathrm{const}$, $\psi_t = \omega = \mathrm{const}$) naturally corresponds to soliton solution (9).

The p.p. radial oscillations in the potential relief $U_\gamma(y)$ correspond to more general LS of (1) /9/, the complex pulsons. These oscillations take place with respect to the solition equilibrium position y_s and are restricted by the turning points y_ℓ and y_r. Their amplitudes are limited by the inequality $y < y_{max}(\gamma)$, where $dU_\gamma/dy(y_{max}) = 0$, $d^2U_\gamma/dy^2(y_{max}) < 0$, in other words, the p.p. should remain within the hole of the potential relief $U_\gamma(y)$.

The motion in the potential relief $U_0(y)$ restricted by the turning points $y_r = y_m$ and $y_\ell = -y_m$ corresponds to the real pulsons ($\gamma = 0$). The maximum amplitude of these oscillations $y_{max} = 1$.

Let us discuss the very important problem of the stability of these pulson solutions in the Lyapunov's sense. It is well known (see, e.g., ref. /10/) that the scalar nonlinear field cannot form stable stationary solitons $\phi_s(\vec{x})$ at $n > 2$ (Derrick's theorem). But the oscillating LS, the real pulsons of eq. (1), are at any n nonradiating and, apparently, stable field bunches. At present, there is no analytical proof of their stability (it is difficult as there is no analytical expression for $y(t)$ in a distinct form), and making this proof is an interesting mathematical problem. But one can apparently regard the factorization (10) of eq. (3) solutions and the effective reduction of the problem to the studying of the p.p. motion as arguments in favour of the stability of these pulsons.

The computer experiments definitely indicate the stability of both real and complex pulsons at all allowed amplitudes. Really in computer experiments the pulson solution $z(t)\exp(-\vec{x}^2/2)$ was conserved with a high accuracy during all the time interval which has been numerically investigated (about 10^3 of pulson oscillations have been computed). In particular, the pulson energy contained in the region of the computer experiment was conserved with the accuracy of the order of 10^{-5} (and it is the error of computation but not real energy radiation, i.e., the numerical method leads to nonconservation of the energy in computer experiments).

In other computer experiment the width of the Gaussian bell has been increased at initial moment $t = 0$ by 25 per cent: $\phi(x,0) = y(0)\exp[-(0.8\vec{x}^2)/2]$. The result of such a pulson perturbation was that the pulson oscillations were accompanied by almost periodic (with the period T_1, about 4 pulson period T) compressions-expansions of the oscillating field bunch near the middle "position", described by the distribution $\exp(-\vec{x}^2/2)$. The energy radiation to space infinity during the period T_1 of such a compression-expansion turned out to be very small, and hence it needs very much computer time to follow the formation of unperturbed pulson up to the end. But nevertheless the distinct qualitative difference of the evolution of such a "broadened" pulson within the framework of the model (1) from the evolution of *unstable* "broadened" ss-pulson in the model of the real Klein-Gordon equation with the cubic nonlinearity /6b/ is clearly seen (the size of the latter one is monotonously increased).

The above-mentioned considerations let to conclude, to the author's mind, that eq. (1) has a unique property. Namely, it apparently has the stable real and complex pulsons at any n, in particular, at $n = 3$ (ss -ones). Thus, taking into consideration the oscillating scalar LS enables us to avoid the difficulties expressed by the Derrick's theorem /10/ (cf. with the results of /6/).

The potential $U_\gamma(y) = \gamma^2/y^2 + y^2(1 - \ln y^2)$ is not positively definite, namely, $U_\gamma(y) \to -\infty$ at $y \to \infty$ When solving the stationary Schrödinger equation with this potential the stationary energy levels are absent (only quasi-stationary levels with a finite life-time might exist; in the QFT unstable particles could correspond to them /1/). One can "improve" the model by replacing the term $\ln|u|^2$ in the Lagrangian and the Hamiltonian by $-|\ln|u|^2|$. Then the Hamiltonian becomes positively definite, and the corresponding QFT problem is not a priori meaningless. The factorization (10) and the classical LS corresponding to it survives at $|\phi|<1$. The field equation (3) survives at $|\phi|<1$ and transforms at $|\phi|>1$ into (cf. with (3)):

$$\phi_{tt} - \nabla_x^2 \phi + (2-n)\phi + \phi \ln|\phi|^2 = 0. \qquad (15)$$

2. *THE BOHR-SOMMERFELD QUANTIZATION OF THE NEUTRAL n-DIMENSIONAL PULSONS*

The real pulsons are the oscillating with the period field systems having an infinite number of degrees of freedom. The BSQ condition for them is $(\tau_2 - \tau_1 = T)$

$$\int_{\tau_1}^{\tau_2} d\tau \int d^n\xi \frac{\partial \mathcal{L}}{\partial u_\tau} u_\tau = 2\pi N. \qquad (16)$$

Here N is the number of the excited energy level, $N = 1, 2, \ldots, N_{max}$. Using (5) and (10) we found

$$G^2 \int d^n x \cdot \exp\left(-\frac{\vec{x}^2}{2}\right) \int_{t_1}^{t_2} y_t^2 \, dt = 2\pi N. \qquad (17)$$

Then passing to the integration over the one-fourth of the period, substituting $\int d^n x \cdot \exp(-\vec{x}^2/2) = \pi^{n/2}$ and

expressing y_t^2 by means of the equations $y_t^2 + 2U_0(y) = y_t^2 + y^2(1-\ln y^2) = 2U_0(y_0)$ we obtain the final form of the BSQ condition for real pulsons

$$N = 2G^2\pi^{n/2-1} I(y_0)\ ;\ I(y_0) = \int_0^{y_0} dy\sqrt{2[U_0(y_0)-U_0(y)]}. \qquad (18)$$

The integral $I(y_0)$ has been numerically computed as a function of the y_0 at $0 \le y_0 \le 1$ by the Simpson method. The maximum possible N_{max} is directly proportional

to G^2. By calculating from the condition (18) the discrete values $y_{0N}(G)$ $(N=1,2,3,\ldots,N_{max})$ we found (see (13)) the energy level spectrum $E_N(G)$:

$$E_N(G) = \pi^{n/2} G^2 \ell^{-1} U_0(y_{0N}). \qquad (19)$$

From (18) and (19) it follows

$$\frac{E_N \ell}{N} = \frac{\pi U_0(y_{0N})}{2I(y_{0N})}. \qquad (20)$$

The numerical computations gave the universal within this model framework dependence $U_0(I)$, i.e., $EG^{-2}\ell\pi^{-n/2}$ on $NG^{-2}\pi^{1-n/2}/2$; N is regarded here as a continuous variable. It is shown in *Fig. 2*.

The remarkable fact is that $E \simeq \mathrm{const} \cdot N$ at small N despite the nonanalyticity of the potential $U(|\phi|)$, $U''(0) = \infty$. One can see in *Fig. 1* that $d^2E/dN^2 < 0$ at all $N \in (0, N_{max})$, thus the intervals between the neighbour levels $\Delta E_N = E_N - E_{N-1}$ become smaller when N increases. The inequality $d^2E/dN^2 < 0$ means, for example, that the N-th state cannot decay into the (N-1)-th and the first ones.

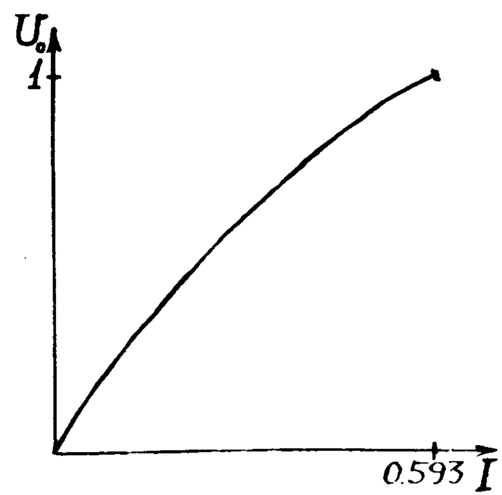

Fig. 2. The universal dependence $U_0(I)$ *at BSQ of the real pulsons of Eq. (3).*

It is interesting to note that the semiclassical quantization of the bions of the SG-equation also gives the spectrum, for which

$$d^2E/dN^2 < 0 \, [\, E_N = 2M_0 \sin(\frac{m}{2M_0} N), \; N \le \frac{M_0}{m}\pi \,] \; /1,2/ .$$

The analogous spectrum should be expected for the zero charge sector of the massive Thirring model because of its equivalence /7/ to the SG-equation.

The analogous decrease of ΔE_N at increase of N takes place as the result of the BSQ of the p.p. motion in the following potential reliefs:

1) $U_1(y) = -U_0 \, ch^{-2}(\alpha y) \; ; \; -\infty < y < +\infty$,

2) $U_2(y) = \alpha |y|^k$, $k < 2$; $-\infty < y < +\infty$,

3) $U_3(y) = y^2 - \beta^2 y^4$; $0 \le |y| \le 1/\sqrt{2|\beta|}$,

4) $U_4(y) = 1 - \cos y$; $-\pi \le y \le +\pi$.

In these models $d^2E/dI^2 < 0$ at all y's described above. Here $I(y_0) = \int_0^{y_0} \sqrt{2[\,U(y_0) - U(y)\,]}\, dy$. For the potentials U_1 and U_2 this fact is easily verified analytically, for U_3 and U_4 it is found by the numerical integration.

3. THE "DOUBLE" QUANTIZATION OF THE CHARGED PULSONS

For the complex pulsons the condition (16) should be replaced by the following one

$$\int_{\tau_1}^{\tau_2} d\tau \int d^n\xi \left(\frac{\partial \mathcal{L}}{\partial u_\tau^*} u_\tau + \frac{\partial \mathcal{L}}{\partial u_\tau} u_\tau^* \right) = 2\pi N , \qquad (16')$$

after transformations it takes the form

$$\int_{\tau_1}^{\tau_2} d\tau \int d^n\xi \, |u_\tau|^2 = \pi N . \qquad (21)$$

Passing to the integration over the half of the period of the p.p. motion in the potential relief (14), we find the BSQ condition

$$N = 2G^2 \pi^{n/2-1} \int_{y_\ell}^{y_r} (y_t^2 + \frac{\gamma^2}{y^2}) \frac{dy}{y_t}, \; y_t = \sqrt{2[\,U_\gamma(y_r) - U_\gamma(y)\,]} . \qquad (22)$$

Let us impose the condition of the "charge quantization" in addition, i.e., let us demand that the complex pulsons under consideration would have an elementary charge, Q=1 (the "double" quantization). With a given value of G, defined by the constants m, ℓ, a of the initial eq. (1) (see (2)), this condition determines the value of $\gamma = (2\pi^{n/2} G^2)^{-1}$ and distinguishes the corresponding curve $U_\gamma(y)$. By choosing from the condition (22) the points $y_r(N, \gamma)$ for which N=1,2,3,..and $E_N = E(N,\gamma) = \pi^{n/2} G^2 \times \ell^{-1} U_\gamma [y_r(N,\gamma)]$ corresponds to them, we find the energy spectrum of n-dimensional pulsons having the charge Q=1. So as not to restrict ourselves by the fixed values of m, ℓ, a, we suppose, that on every curve $U_\gamma(y)$ the condition Q=1 is valid and find the value G from formula (11a) with a given γ. Then the condition (22) may be rewritten in the form

$$N = \frac{1}{\gamma\pi} \int_{y_\ell}^{y_r} (y_t^2 + \frac{\gamma^2}{y^2}) \frac{dy}{y_t} =$$

$$= \frac{1}{\gamma\pi} \int_{y_\ell}^{y_r} \frac{[\,2U_\gamma(y_r) - y^2(1 - \ln y^2)\,]}{\sqrt{2[\,U_\gamma(y_r) - U_\gamma(y)\,]}} dy . \qquad (23)$$

With small deviations $y_r - y_s$ the curve $U_\gamma(y)$ may be approximated by the parabola $U_\gamma(y) = U_\gamma(y_s) + \frac{1}{2}U_\gamma''(y_s)(y-y_s)^2$; it is easy to obtain, that $U_\gamma''(y_s)=8(\omega^2-½)$. Then the integral in (23) can be found analytically, and regarding N as a continuous variable we obtain that $N \to N_s(\omega) = \frac{\omega}{2}(\omega^2 - \frac{1}{2})^{-½}$ at $y_r \to y_s(\omega)$. Thus, the limit value $N_s \neq Q = 1$; this result contradicts to the assertion of paper /12/. In this paper the equivalence of the "charge quantization", Q=N, and the BSQ for solitons $R(\vec{x})\exp(-i\omega t)$ was stated. The point is that the classical densities (of the Hamiltonian, charge, etc.) for such solitons are constant in time as a sequence of the U(1) -invariance of the theory. Hence it is unclear how to define for them the period having a physical meaning. The value of the period T becomes definite when one considers the pulson oscillations of arbitrary small amplitude with respect to the soliton equilibrium position y_s. When the particles move in the central fields (excepting the potentials $U=C_1y^2$ and $U=-C_2y^{-1}$, $C_1, C_2 > 0$) their orbits are not closed /13/; hence the angle variation $\Delta\psi$ during the pulson period T does not tend to 2π when the pulson solution (10) tends to soliton one (9), and consequently $T=2\pi/\omega$. That is why $N_s \neq Q = 1$. When $y_s \to 0$, $N_s \to 1/2$, at $y_s \to y_{s\,max}$ $N_s \to \infty$, $N_s = 1$ at $y_s = \exp(-1/3)$.

In the general case the problem was solved numerically. We chose the soliton amplitude $y_s(\omega) = \exp(-½\omega^2)$, $0 < y_s < y_{s\,max} = \exp(-1/4)$, computed the value $\gamma = y_s^2(\omega)\cdot\omega$, and found the point $y_{max}(\gamma)$, corresponding to it, and found the point $y_{max}(\gamma)$, corresponding to the maximum amplitude of the pulson oscillations at given γ. Then we took the point y_r such that $y_s < y_r < y_{max}(\gamma)$, and found the point y_ℓ, corresponding to it from the condition $U_\gamma(y_\ell) = U_\gamma(y_r)$. The integral (23) was computed by the Simpson method; the interval of the integration was divided into 2M parts (M=500).

The dependence of $U = \pi^{-n/2}G^{-2}\ell\cdot E$ on the continuous variable N for different curves $U_\gamma(y)$ is given in *Fig. 3*. We characterize these curves by the value of

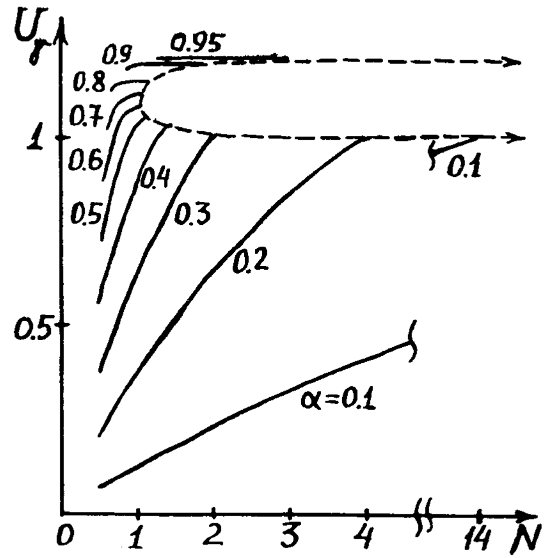

Fig. 3. The curves $U_\gamma(N)$ *at the "double" quantization of the charged pulsons for different values* $\alpha = y_s(\gamma)/y_{s\,max}$:

$\alpha = y_s/y_{s\,max} = y_s\exp(1/4)$ (the dependences $\gamma(\alpha)$ and $G^2(\alpha)$, the latter at n=3, are plotted in *Fig. 4*).

One can see, that at "double" quantization, when small values of α are considered there are the energy levels with large numbers N; $N_{max} \to \infty$ when $\alpha \to 0$. When α increases the value N_{max} gradually decreases; at $\alpha > \alpha_1 \simeq 0.31$ the only level with N=1 remains. It survives as was shown above up to $\alpha_2 = \exp(-1/3)/\exp(-1/4) = \exp(-1/12)$, at $\alpha > \alpha_2$ we have $N_s > 1$. At $\alpha = 0.92$ the level with N=2 appears again, at $\alpha = 0.96$ level with N=3 appears and so on; N_{min} and $N_{max} \to \infty$ at $\alpha \to 1$. At $\alpha \to 1$ these levels have almost equal energy (see *Fig. 3*), $U_N \to U_{max}$, where $U_{max} = U_\gamma(y_{s\,max})$ at $\gamma = y^2_{s\,max}\cdot\omega_{cr}$. Notice, that at all $\alpha \in (0,1)$ there is at least one level (with an integer N). The inequality $d^2E/dN^2 < 0$ is valid (the same as in the case of the real field, see Sec. 2) for all curves $U_\gamma(N)$.

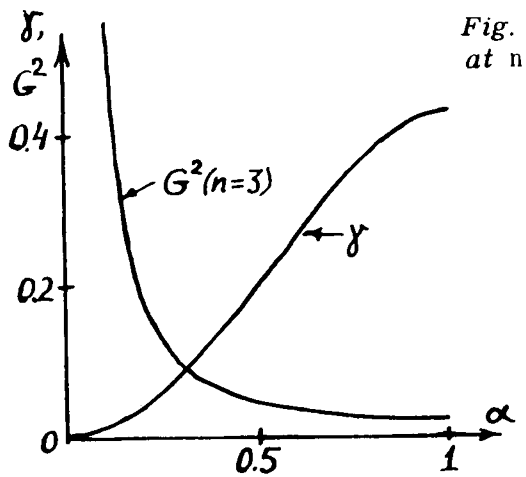

Fig. 4. Dependences $G^2(\alpha)$ *at* $n=3$ *and* $\gamma(\alpha)$.

With a small α the "mass ratio" $E_2/E_1 \simeq 2$; when α increases this ratio decreases ($E_2/E_1 \simeq 1.43$ at $\alpha=0.3$). When $\alpha \to 0$ then $U_{N_{max}} \to 1$ and $U_1 \to 0$, therefore $U_{N_{max}}/U_1 \to \infty$ when $\alpha \to 0$.

Thus, the BSQ of this model gives the semiclassical discrete mass spectrum of the "particles", having the same charge ($Q=1$) and spin($S=0$). The "mechanism" of arising of the charged "particles" mass spectrum studied in this paper may be useful as one of the possible QFT approaches to the solution of the "μ-e problem". In this sense the LIN spinorial models are of a particular interest (in particular, it is very interesting to carry out the "double" quantization of the massive Thirring model).

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