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ANALYTIC PROPERTIES OF THE QUASIPOTENTIAL SCATTERING AMPLITUDE IN COMPLEX RAPIDITY AND ANGULAR MOMENTUM PLANES

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Аналитические свойства квазипотенциальной амплитуды рассеяния в комплексных плоскостях быстроты и углового момента

Изучены аналитические свойства релятивистской амплитуды рассеяния двух частиц как функции комплексных переменных быстроты и орбитального момента. Рассмотрение основано на дифференциальном уравнении квазипотенциального типа в релятивистском конфигурационном пространстве.

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Amirkhanov I.V., Grusha G.V., Mir-Kasimov R.M.

Analytic Properties of the Quasipotential Scattering Amplitude in Complex Rapidity and Angular Momentum Planes

The analytic properties of the relativistic two-particle scattering amplitude as a function of complex variables rapidity and orbital momentum are studied. The consideration is based on differential quasipotential type equation in relativistic configurational space.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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INTRODUCTION

In the proposed paper the analytic properties of the relativistic two-particle scattering amplitude as a function of complex variables rapidity and orbital momentum are studied in quasipotential approach [1-3] (see refs. [4,6,10,11]).

We proceed from the quasipotential type equation reducing in the relativistic configurational representation [5] to the second order differential equation [7].

The relativistic configurational space originates due to the application of the Fourier analysis on the Lorentz group [5]. Relativistic plane waves have the form [5]:

$$\xi(\vec{q},\vec{r})=(\mathrm{ch}\chi_q-\vec{n}\vec{n}_q\,\mathrm{sh}\chi_q)^{-1-ir} \tag{1.1}$$

and turn in nonrelativistic limit into the usual plane waves

$$\xi(\vec{q},\vec{r})\xrightarrow[c\to\infty]{} e^{i\vec{q}\vec{r}} . \tag{1.2}$$

Here $\vec{r}=r\vec{n}\,(\vec{n}^2=1)$, $\vec{q}=\vec{n}_q\,\mathrm{sh}\chi_q$ $(h=c=m=1)$. The quantity

$$\chi_q=\ln(E_q+\sqrt{E_q^2-1}),\quad E_q=\sqrt{\vec{q}^2+1} \tag{1.3}$$

(rapidity) defines in the case of equation [7] the structure of the Riemann energy surface of the partial amplitude (comp. [8]).

Note that in the quasipotential equations [2,6,7] the integration is carried over the Lobachevsky momentum space with element of volume $d\Omega_q = \frac{d\vec{q}}{\sqrt{1+\vec{q}^2}}$, i.e., these equations do not contain the additional kinematic factors in comparison with Lippman-Schwinger equation. This fact permits to hope that in this approach the partial amplitude does not contain nonphysical branchpoints in momentum plane q (comp. [4] and [10]). We have studied the scattering on the local, spherically symmetric quasipotentials [5], such that inequalities

$$\int_0^\infty r\,|V(r,\chi_q)|\,e^{\nu r}\,dr < \infty , \tag{1.4}$$

$$\int_0^\infty \left(\frac{r}{r+\delta}\right)\left|\frac{\partial}{\partial \chi_q} V(r,\chi_q)\right| e^{\nu r} dr < \infty \tag{1.5}$$

are valid for all ν $(0<\nu<\mu)$ and δ $(0<\delta<1)$.

2. THE SCATTERING THEORY

Let us consider the solutions of differential QPE [7]

$$\hat{h}_0 \psi_\ell(r,\chi) = \gamma_\ell(r,\chi) V(r,\chi) \psi_\ell(r,\chi) \tag{2.1}$$

with different boundary conditions in r-space.

The operator $\hat{h}_0$ has the form:

$$\hat{h}_0 = \frac{d^2}{dr^2} - U_\ell^{(1)}(r,\chi) - U_\ell^{(2)}(r,\chi)\frac{d}{dr},$$

where the coefficients $U_\ell^{(1,2)}$ and γ_ℓ [7] depend on the relativistic analogs of the Riccati-

Bessel functions $s_\ell(r,\chi)$, $c_\ell(r,\chi)$ or Ricatti-Hankel functions $e_\ell^{(12)}(r,\chi)$*.

Let us define the regular at the origin solution $\psi_\ell^{(0)}(r,\chi)$, the scattering solutions $\psi_\ell^{(\pm)}(r,\chi)$ and the analogs of the Jost solutions $\psi_\ell^{(\pm J)}(r,\chi)$ by boundary conditions:

$$\lim_{r\to 0}\left[\frac{\psi^{(0)}(r,\chi)}{r}\right] = \lim_{r\to 0}\left[\frac{s(r,\chi)}{r}\right] = e^{-\frac{i\pi\ell}{2}} Q_\ell(\mathrm{cth}\chi), \tag{2.2}$$

$$\lim_{r\to\infty}\left[e^{\mp i(r\chi-\pi\ell/2)}\,\psi_\ell^{(\pm J)}(r,\chi)\right] = 1, \tag{2.3}$$

$$\psi_\ell^{(\pm)}(r,\chi) \underset{r\to\infty}{\sim} \sin\left(r\chi - \frac{\pi\ell}{2}\right) + A_\ell^{(\pm)}(\chi)\, e^{\pm i\left(r\chi - \frac{\pi\ell}{2}\right)}, \tag{2.4}$$

where $A_\ell^{(\pm)}(\chi)$ is the scattering amplitude**.

The Poincare theorem reads that the solution of differential equation with entire coefficients is the entire function of the parameter, if it satisfies the boundary condition, independent of this papameter [9]. However, in contrast with the case of Schrödinger equation the coefficients of equation (2.1) are the meromorphic functions of ℓ and χ [7]. Following [9], we construct the system of integral equations for the wave functions, satisfying conditions (2.2)-(2.4):

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* The analytic and asymptotic properties of these functions are presented in Appendix A.

** Here, the norm of the wave functions is chosen so, that their nonrelativistic limit coincides with the Schrödinger equation solutions considered in ref. [12].

$$\psi_\ell^{(\pm,0)}(r,\chi) = s_\ell(r,\chi) + \int_0^\infty g_\ell^{(\pm,0)}(r,r';\chi)\, V(r',\chi)\, \psi_\ell^{(\pm,0)}(r',\chi)\, dr', \qquad (2.5)$$

$$\psi_\ell^{(\pm J)}(r,\chi) = e_\ell^{(1,2)}(r,\chi) + \int_0^\infty g_\ell^{J}(r,r';\chi)\, V(r',\chi)\, \psi_\ell^{(\pm J)}(r',\chi)\, dr', \qquad (2.6)$$

where

$$g_\ell^{(\pm)}(r,r';\chi) = -\frac{\nu_\ell(r')}{\mathrm{sh}\chi}\left[\theta(r-r')\, e_\ell^{(1,2)}(r,\chi)\, s_\ell(r',\chi) + \theta(r'-r)\, s_\ell(r,\chi)\, e_\ell^{(1,2)}(r',\chi)\right] \qquad (2.7)$$

[7],

$$g_\ell^{(0)}(r,r';\chi) = \theta(r-r')\, K_\ell(r,r';\chi), \qquad (2.8)$$

$$g_\ell^{(J)}(r,r';\chi) = -\theta(r'-r)\, K_\ell(r,r';\chi). \qquad (2.9)$$

Here, the kernel K_ℓ has the form

$$K_\ell(r,r';\chi) = \frac{\nu_\ell(r')}{\mathrm{sh}\chi}\left[s_\ell(r,\chi)\, e_\ell^{(1)}(r',\chi) - e_\ell^{(1)}(r,\chi) \times s_\ell(r',\chi)\right], \qquad (2.10)$$

$\theta(r)$ is the step function, and $\nu_\ell(r)$ is defined in (A.6).

Now we introduce the relativistic Jost functions $f_\ell^{(\pm)}(\chi)$ so that for regular solution the asymptotic behaviour

$$\psi_\ell^{(0)}(r,\chi) \underset{r\to\infty}{\sim} \frac{1}{2i}\left[f_\ell^{(-)}(\chi)\, e^{i(r\chi - \frac{\pi\ell}{2})} - f_\ell^{(+)}(\chi)\, e^{-i(r\chi - \frac{\pi\ell}{2})}\right] \qquad (2.11)$$

takes place.

The expressions for Green functions (2.7), (2.8) give the following representations:

$$f_\ell^{(\pm)}(\chi) = \frac{\psi_\ell^{(0)}(r,\chi)}{\psi_\ell^{(\pm)}(r,\chi)}, \qquad (2.12)$$

$$f_\ell^{(\pm)}(\chi) = 1 + \frac{1}{\mathrm{sh}\chi}\int_0^\infty e_\ell^{(1,2)}(r,\chi)\, V(r,\chi)\, \psi_\ell^{(0)}(r,\chi)\, \nu_\ell(r)\, dr = \qquad (2.13)$$

$$= \left[1 - \frac{1}{\mathrm{sh}\chi}\int e_\ell^{(1,2)}(r,\chi)\, V(r,\chi)\, \psi_\ell^{(\pm)}(r,\chi)\, \nu_\ell(r)\, dr\right]^{-1}.$$

The expressions for the scattering amplitude $A_\ell^{(\pm)}(\chi)$, the phase $\delta_\ell(\chi)$ and the s-matrix element $\hat{S}_\ell(\chi)$ in terms of Jost functions coincide formally with the non-relativistic ones:

$$A_\ell^{(\pm)}(\chi) = -\frac{1}{\mathrm{sh}\chi}\int_0^\infty s_\ell(r,\chi)\, V(r,\chi)\, \psi_\ell^{(\pm)}(r,\chi)\, \nu_\ell(r)\, dr =$$

$$= \frac{f_\ell^{(-)}(\chi) - f_\ell^{(+)}(\chi)}{2i\, f_\ell^{(\pm)}(\chi)}, \qquad (2.14)$$

$$\hat{S}_\ell(\chi) = \frac{f_\ell^{(-)}(\chi)}{f_\ell^{(+)}(\chi)}, \quad f_\ell^{(\pm)}(\chi) = r_\ell(\chi)\, e^{\mp i\delta_\ell(\chi)}. \qquad (2.15)$$

To study the analytic properties of the Jost functions let us represent them as Wronskians of the solutions of equation (2.1). Wronskian of the linearly-independent solutions of equation (2.1) equals:

$$J[\psi^{(1)},\psi^{(2)}]=\psi_\ell^{(1)}(r,\chi)\frac{d}{dr}\psi_\ell^{(2)}(r,\chi)-\psi_\ell^{(2)}(r,\chi)\frac{d}{dr}\psi_\ell^{(1)}(r,\chi)$$

$$=\frac{1}{\chi}w(s_\ell,c_\ell)J[\phi^{(1)},\phi^{(2)}],\quad w(s_\ell,c_\ell)=J[s_\ell,c_\ell],\tag{2.16}$$

where

$$\phi_\ell^{(1,2)}(r,\chi)=\frac{\psi_\ell^{(1,2)}(r,\chi)}{\sqrt{w(s_\ell,c_\ell)/\chi}},\quad J[\phi^{(1)},\phi^{(2)}]$$ /7/

is independent of r.

Now we express $\psi_\ell^{(0)}(r,\chi)$ in terms of the superposition:

$$\psi_\ell^{(0)}(r,\chi)=\frac{1}{2i}[f_\ell^{(-)}(\chi)\psi_\ell^{(+J)}(r,\chi)-f_\ell^{(+)}(\chi)\psi_\ell^{(-J)}(r,\chi)]\tag{2.17}$$

and using (2.16) obtain

$$f_\ell^{(\pm)}(\chi)=\frac{J[\psi^{(\pm J)},\psi^{(0)}]}{w(s_\ell,c_\ell)}=\frac{1}{\chi}\lim_{r\to\infty}J[\psi^{(\pm J)},\psi^{(0)}].\tag{2.18}$$

From (2.18) and the boundary condition (2.2) there follows the simple expression

$$f_\ell^{(\pm)}(\chi)=\psi_\ell^{(\pm J)}(0,\chi)\left[\frac{1}{w(s_\ell,c_\ell)}\frac{d}{dr}\psi_\ell^{(0)}(r,\chi)\right]_{r=0}.\tag{2.19}$$

Passing to complex rapidities and orbital momenta, we assume that (comp. /4/): 1) the quasipotential is real on the interval of real axis in χ^2-plane containing the point $\chi=0$ (i.e., in the range of elastic scattering and bound states); 2) the quasipotential is analytic in some part of χ-plane, except the branch points corresponding to the thresholds of inelastic processes.

In the range where quasipotential is real, we obtain from (A.5) the rules of complex conjugation:

$$[\psi_\ell^{(\pm J)}(r,\chi)]^*=\nu_{\ell^*}(r)\psi_{\ell^*}^{(\mp J)}(r,\chi^*),\tag{2.20}$$

$$[\psi_\ell^{(0)}(r,\chi)]^*=\nu_{\ell^*}(r)\psi_{\ell^*}^{(0)}(r,\chi^*),\tag{2.21}$$

$$[f_\ell^{(\pm)}(\chi)]^*=f_{\ell^*}^{(\mp)}(\chi^*),\ [\tau_\ell(\chi)]^*=\tau_\ell(\chi),\tag{2.22}$$

$$[\delta_\ell(\chi)]^*=\delta_\ell(\chi),$$

and the unitarity condition for $\hat{S}_\ell$-matrix

$$[\hat{S}_\ell(\chi)]^*=[\hat{S}_{\ell^*}(\chi^*)]^{-1}.\tag{2.23}$$

3. ANALYTIC PROPERTIES OF THE PARTIAL-WAVE AMPLITUDE IN THE COMPLEX RAPIDITY AND ANGULAR MOMENTUM PLANE

Let us denote $\Omega(\chi)$ the rapidity strip $|\mathrm{Im}\,\chi|<\pi,\ |\chi|\geq\epsilon>0$ corresponding to the whole nonrelativistic momentum plane q, except for the neighbourhood of the point $q=0$. We consider the part $\Omega'(\chi)$ of the region $\Omega(\chi)$, in which the quasipotential is analytic, and the half-plane $\mathrm{Re}\,\ell+1>\delta\ (1>\delta>0)$ of complex momentum.

Now let us express the regular solution in the form of the expansion:

$$\psi_\ell^{(0)}(r,\chi)=\sum_{n=0}^{\infty}\psi_{\ell,n}^{(0)}(r,\chi), \tag{3.1}$$

where

$$\psi_{\ell,0}^{(0)}(r,\chi)=s_\ell(r,\chi),$$
$$\psi_{\ell,n+1}^{(0)}(r,\chi)=\int_0^r K_\ell(r,r';\chi)V(r',\chi)\psi_{\ell,n}^{(0)}(r',\chi)dr'. \tag{3.2}$$

Using the boundness of $s_\ell(r,\chi)$ (B.6) and $K_\ell(r,r';\chi)$ (B.7), we get majorant for the series (3.1) (comp. /9/):

$$\left|\sum_{n=0}^{\infty}\psi_{\ell,n}^{(0)}(r,\chi)\right|<c(\ell,\chi)\left(\frac{r}{r+\delta}\right)e^{r|\mathrm{Im}\,\chi|}\,e^{D(\ell,\chi)P(r,\chi)}, \tag{3.3}$$

where

$$P(r,\chi)=\int_0^r\frac{|V(r',\chi)|}{r'+\delta}r'dr',$$

$$0\le c(\ell,\chi)<\infty,\qquad 0\le D(\ell,\chi)<\infty.$$

If we now assume the condition at the origin*

$$\lim_{r\to 0}[r^2V(r,\chi)]=0, \tag{3.4}$$

then $P(r,\chi)$ is uniformly bounded and expansion (3.1) is uniformly convergent in both variables ℓ and χ.

Taking into account the analytic properties of free solutions (Appendix A), we can prove that at any finite r, $\chi_{\ell,n+1}^{(0)}(r,\chi)$ are the holomorphic functions of ℓ and χ. Hence, from the Weierstrass theorem it follows that $\psi_\ell^{(0)}(r,\chi)$ is also the holomorphic function of ℓ and χ (in the considered regions). Analogously, let us put

$$\psi_\ell^{(\pm J)}(r,\chi)=\sum_{n=0}^{\infty}\psi_{\ell,n}^{(+J)}(r,\chi), \tag{3.5}$$

$$\psi_{\ell,0}^{(+J)}(r,\chi)=e_\ell^{(1)}(r,\chi),$$

$$\psi_{\ell,n+1}^{(+J)}(r,\chi)=-\int_0^{\infty}K_\ell(r,r';\chi)V(r',\chi)\times$$
$$\times\psi_{\ell,n}^{(+J)}(r',\chi)dr'. \tag{3.6}$$

Again, using the boundness of $e_\ell^{(1)}(r,\chi)$ (B.4) and $K_\ell(r,r';\chi)$ (B.8), we get majorant for series (3.5):

$$\left|\sum_{n=0}^{\infty}\psi_{\ell,n}^{(+J)}(r,\chi)\right|<\overline{C}(\ell,\chi)e^{-r\,\mathrm{Im}\chi}\,e^{\overline{D}(\ell,\chi)Q(r,\chi)}, \tag{3.7}$$

where

*This condition provides inequality (1.4).

$$Q(r,\chi) = \int_r^\infty \frac{r'|V(r',\chi)|}{r'+\delta} e^{(|\mathrm{Im}\chi| - \mathrm{Im}\chi) r'} dr', \qquad (3.8)$$

$$0 \le \bar{C}(\ell,\chi) < \infty, \quad 0 \le \bar{D}(\ell,\chi) < \infty .$$

Provided quasipotentials satisfy (1.4), the function $Q(r,\chi)$ is uniformly bounded in χ, and the series (3.6) is uniformly convergent in the strip $-\frac{\mu}{2} < \mathrm{Im}\chi < \pi$ of the region $\Omega'(\chi)$. Since the functions $\psi^{(+J)}_{\ell,n+1}(r,\chi)$ (3.6) are defined by improper integrals, we must also find the condition of convergence for their first derivatives.

Using inequalities (B.9), we can show that for quasipotentials, satisfying both the conditions (1.4) and (1.5), $\psi^{(+J)}_{\ell}(r,\chi)$ is the holomorphic function of χ in the region of uniform convergence of (3.5).

Analogously, we will be convinced of $\psi^{(-J)}_{\ell}(r,\chi)$ being holomorphic in variable χ in the strip $-\pi < \mathrm{Im}\chi < \mu/2$ of the region $\Omega'(\chi)$. From inequalities (B.10) and analytic properties of the kernel $K_\ell(r,r';\chi)$ it follows that the functions $\psi^{(\pm J)}_{\ell}(r,\chi)$ are holomorphic in the halfplane $\mathrm{Re}\,\ell+1>\delta$.

In nonrelativistic limit both solutions $\psi^{(\pm J)}_{\ell}(r,\chi)$ are the entire functions of the variable ℓ. Since the relativistic Jost solutions are not singular at $r=0$ (see App.A), expressions (2.15), (2.19) define $\hat{S}_\ell$-matrix (and the scattering amplitude) as meromorphic functions in the direct product of the halfplane $\mathrm{Re}\,\ell+1>\delta$ and the strip $|\mathrm{Im}\chi| < \frac{\mu}{2} < \pi$ of the region $\Omega'(\chi)$.

In the case of finite potentials, the proof is valid for any part of the rapidity plane, except the points $\chi = \pm i\pi n$ $(n=0,1,2,\ldots)$.

From expressions (2.3), (2.15), (2.17) it follows that zeroes of Jost function $f^{(+)}_\ell(\chi)$ (the poles of $\hat{S}_\ell(\chi)$) at $\mathrm{Im}\chi>0$ coincide with those values of the rapidity, for which the regular solution $\chi^{(0)}_\ell(r,\chi)$ is quadratic-integrable, i.e., the bound state occurs.

The equation for bound states has the form:

$$\int_0^\infty dr |\phi^{(0)}_\ell(r,\chi)|^2 \mathrm{Im}\{\chi^2 - R_\ell(r,\chi) - \gamma_\ell(r,\chi) V(r,\chi)\} = 0. \qquad (3.9)$$

Using the explicit expressions for the centrifugal barrier $R_\ell(r,\chi)$ and factor $\gamma_\ell(r,\chi)$ (ref. /7/), one can easily verify that for real quasipotentials and entire angular momenta the bound states lie on the segment $\mathrm{Re}\chi = 0,\ 0 < \mathrm{Im}\chi < \pi$.

Finally, we consider the symmetry properties of the wave functions. It is shown in App. C, that at complex ℓ the kernel $K_\ell(r,r';\chi)$ has the branch point $\chi=0$ vanishing at integer ℓ and in the nonrelativistic limit (see (C.4)). Therefore, only at entire ℓ the function $(\mathrm{sh}\chi)^{-\ell-1}\psi^{(0)}_\ell(r,\chi)$ is even, and the Jost functions are connected by analytic continuation:

$$\psi^{(\pm J)}_\ell(r,\chi e^{\pm i\pi}) = \psi^{(\mp J)}_\ell(r,\chi),$$
$$f^{(\pm)}_\ell(\chi e^{\pm i\pi}) = f^{(\mp)}_\ell(\chi). \qquad (3.10)$$

Let us now put $\chi = i\eta$, $0>\eta>\pi$. Since in the general case

$$[f^{(+)}_\ell(i\eta)]^* = f^{(-)}_{\ell^*}(-i\eta) \ne f^{(+)}_{\ell^*}(i\eta), \qquad (3.11)$$

$f_\ell^{(+)}(i\eta)$ is not a real function of ℓ, the Regge trajectory may be complex in bound state region.

Another peculiar property of our equation is asymmetry of $K_\ell(r,r';\chi)$ and $e^{\pm i\frac{\pi}{2}(\ell+\frac{1}{2})}\psi_\ell^{(\pm J)}(r,\chi)$ at reflections $\ell \to -\ell-1$ (see (C.5)).

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APPENDIX A

Let us define the analytic continuations for relativistic analogs of the Riccati-Bessel and Riccati-Hankel functions:

$$s_\ell(r,\chi) = \sqrt{\frac{\pi\,\mathrm{sh}\chi}{2}}\,\frac{\Gamma(ir+\ell+1)}{\Gamma(ir)}\,e^{-\frac{i\pi}{2}(\ell+1)}\,P^{-\ell-\frac{1}{2}}_{-\frac{1}{2}+ir}(\mathrm{ch}\chi), \quad (A.1)$$

$$e_\ell^{(1,2)}(r,\chi) = \frac{1}{i}\sqrt{\frac{2\,\mathrm{sh}\chi}{\pi}}\,\frac{\Gamma(1-ir)}{\Gamma(\ell+1-ir)}\,e^{-\frac{3}{2}i\pi\ell}\,Q^{\ell+\frac{1}{2}}_{-\frac{1}{2}\mp ir}(\mathrm{ch}\chi), \quad (A.2)$$

$$c_\ell(r,\chi) = \mathrm{ctg}\,\pi(\ell+\frac{1}{2})\left[\frac{s_{-\ell-1}(r,\chi)}{\cos\pi(\ell+\frac{1}{2})}\,a_\ell(r) - s_\ell(r,\chi)\,\mathrm{cth}\,\pi r\right] \quad (A.3)$$

(comp. [15]).

The quantities (A.2) differ from the direct product of the functions introduced in [5] by the factor $a_\ell(r) = e^{-i\pi\ell}\sin\pi(ir-\ell)/\sin i\pi r$.

At arbitrary ℓ the relations

$$e_\ell^{(1,2)}(r,\chi) = c_\ell(r,\chi) \pm i s_\ell(r,\chi), \quad (A.4)$$

following from the connection of Legendre functions, are valid.

At complex conjugation:

$$[s_\ell(r,\chi)]^* = \nu_{\ell^*}(r)\, s_{\ell^*}(r,\chi^*),$$

$$[e_\ell^{(1,2)}(r,\chi)]^* = \nu_{\ell^*}(r)\,[e_{\ell^*}^{(2,1)}(r,\chi^*)], \quad (A.5)$$

where

$$\nu_\ell(r) = e^{i\pi(\ell+1)}\,\frac{\Gamma(\ell+1-ir)\Gamma(ir)}{\Gamma(\ell+1+ir)\Gamma(-ir)}. \quad (A.6)$$

Now let us consider the planes of complex rapidity χ and complex energy $E=\mathrm{ch}\chi$. The Legendre functions $P_\nu^\mu(\mathrm{ch}\chi)$ and $Q_\nu^\mu(\mathrm{ch}\chi)$ are analytic and unique in the plane E, cut from $-\infty$ to $+1$ (refs. [14,15]). In mapping onto the rapidity plane the strip

$$|\mathrm{Im}\chi| < \pi, \quad \mathrm{Re}\chi > 0 \quad (A.7)$$

corresponds to the main sheet of energy surface [14]. Circuiting the point $E=\mathrm{ch}\chi=+1$ in negative direction, we obtain a new branch of the Ledendre function, for example $Q_\nu^\mu(E, 1-)$ (see ref. [15] and App. C). In this rotation the transformation $\sqrt{E^2-1} \to -\sqrt{E^2-1}$, $\chi \to \chi e^{-i\pi}$ is performed and the strip arises

$$|\mathrm{Im}\chi| < \pi, \quad \mathrm{Re}\chi < 0, \quad (A.8)$$

corresponding to the new sheet of E-plane. We will consider further all strip

$$|\mathrm{Im}\chi| < \pi, \quad |\chi| \geq \epsilon > 0$$

(the region $\Omega(\chi)$).

Let us put $\chi \in \Omega(\chi)$. Then, in the general case, $P_\nu^\mu(\mathrm{ch}\chi)$ and $Q_\nu^\mu(\mathrm{ch}\chi)$ are the analytic functions of the variables ν and μ, except for the essential singularity at $\nu = \infty$, $\mu = \infty$ (ref./14/). Besides, the function $Q_\nu^\mu(\mathrm{ch}\chi)$ has simple poles at the points for which $\nu + \mu = -n$, $n = 1, 2, \ldots$ (ref./14/).

Taking into account the definitions (A.1), (A.2), we find the location of singularities for the functions $s_\ell(r, \chi)$, $e_\ell^{(1,2)}(r, \chi)$ (see the table).

Table

Function	The region of holomorphness	Another variables fixing
$e_\ell^{(1)}(r,\chi)$ or $\nu_\ell(r) e_\ell^{(2)}(r,\chi)$	$\chi \in \Omega(\chi)$	any ℓ, $r \neq \mp i(1+1)$
	$r \neq \mp i(1+n)$	any ℓ, $\chi \in \Omega(\chi)$
	all ℓ-plane	$r \neq \mp i(1+n)$, $\chi \in \Omega(\chi)$
$s_\ell(r,\chi)$	$\chi \in \Omega(\chi)$	$\mathrm{Re}\,\ell + 1 > \delta$, $\mathrm{Re}(ir) > -\delta$
	$\mathrm{Re}(ir) > -\delta$	$\mathrm{Re}\,\ell + 1 > \delta$, $\chi \in \Omega(\chi)$
	$\mathrm{Re}\,\ell + 1 > \delta$	$\mathrm{Re}(ir) > -\delta$, $\chi \in \Omega(\chi)$

Here $1 > \delta > 0$, $n = 0, 1, 2, \ldots;\ n \leq \ell + 1$ for entire ℓ.

We note, that infinite chain of simple poles of the functions (A.1)-(A.3) in ℓ-plane, analogous to the "Lorentz-poles" in

the expansion of the total scattering amplitude in the representation of principal series of SO(3.1) group/13/ belongs to the halfplane $\mathrm{Re}\,\ell + 1 \leq 0$ at real r.

Now let us write the boundary conditions in complex r-plane, using the asymptotic expansion of the functions $Q_\nu^\mu(\mathrm{ch}\chi)$ at $|\nu| \to \infty$, $|\arg\nu| < \pi$ /14/:

$$e_\ell^{(1)}(r,\chi) \sim e^{i(r\chi - \pi\ell/2)}, \tag{A.9}$$

$$\nu_\ell(r)\, e_\ell^{(2)}(r,\chi) \sim e^{-i(r\chi - \pi\ell/2)},$$

at $|r| \to \infty$, $|\arg r| < \pi/2$.

Specific are the boundary conditions at the origin. The functions $e_\ell^{(1,2)}(r,\chi)$ for all ℓ are finite at $r = 0$:

$$e_\ell^{(1,2)}(0,\chi) = e^{-\frac{i\pi\ell}{2}} P_\ell(\mathrm{cth}\chi), \tag{A.10}$$

whereas $s_\ell(r,\chi)$ vanishes linearly:

$$\lim_{r \to 0} \left[\frac{s_\ell(r,\chi)}{r}\right] = e^{-\frac{i\pi\ell}{2}} Q_\ell(\mathrm{cth}\chi). \tag{A.11}$$

APPENDIX B

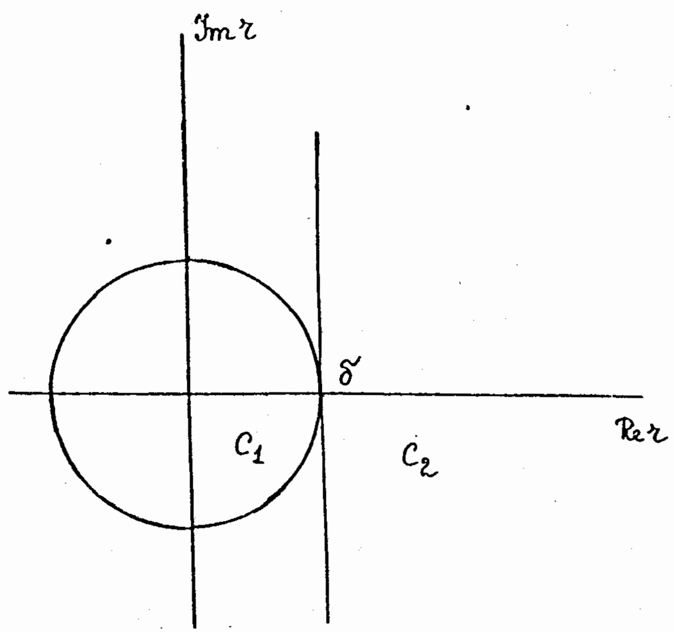
In this appendix we obtain majorants for the absolute values of free solutions and the kernels $K_\ell(r, r'; \chi)$ of equations (2.5), (2.6). Proving inequalities, we use essentially the analytic properties of majorized functions (see the table).

Let us consider the functions $e_\ell^{(1,2)}(r,\chi)$ in the plane of complex radius-vector (ℓ and χ are fixed in the holomorphness region).

Now we choose the region, which consists of a circle C_1 of radius $\delta (0 < \delta < 1)$ and the half-plane C_2 ($\mathrm{Re}\, r > \delta$) (Figure). Since the function $e_\ell^{(1)}(r,\chi)$ is analytic in C_2 and satisfies the boundary condition at infinity (A.9), the positive quantity $A(\ell,\chi)$ exists for which

$$|e_\ell^{(1)}(r,\chi)\, e^{-ir\chi}| \le A(\ell,\chi) < \infty, \quad r \in C_2 \qquad (B.1)$$

is valid for any finite ℓ and χ* (comp.[11]).



Figure

*This follows, for example, from the Phragmen-Lindelöf principle.

Obviously, the function $e_\ell^{(1)}(r,\chi)$ is also bounded (by maximum of modulus) in a circle C_1.

Analogously, we obtain:

$$|\nu_\ell(r)\, e_\ell^{(2)}(r,\chi)\, e^{ir\chi}| \le A'(\ell,\chi) < \infty, \qquad (B.2)$$

$$r \in C_1, C_2.$$

Inequalities (B.1), (B.2) coincide with those obtained in ref.[11] for a case of entire ℓ.

We cannot, however, prove the uniform boundness of the functions $s_\ell(r,\chi)$ (and the kernels $K_\ell(r,r';\chi)$) in the whole considered region of r-plane (comp.[11]) since the gamma-function $\Gamma(\pm ir + \ell + 1)$ entering into $\nu_\ell(r)$, is meromorphic.

Fortunately, to prove the existence and analyticity of the solutions of eq. (2.5), (2.6), we need not use the complex r-plane (contrary to finite-difference equations[5,11]).

For real r from (B.1), (B.2) the inequalities follow:

$$|s_\ell(r,\chi)| \le B(\ell,\chi)\, e^{r|\mathrm{Im}\,\chi|}, \qquad (B.3)$$

$$|e_\ell^{(1,2)}(r,\chi)| \le A''(\ell,\chi)\, e^{\mp r\, \mathrm{Im}\,\chi}. \qquad (B.4)$$

Let us specify inequality (B.3) for small r, taking into account the boundary condition at $r=0$ (A.10). Considering $s_\ell(r,\chi)$ inside a circle C_1, we apply the Schwartz lemme:

$$|s_\ell(r,\chi)| \le B'(\ell,\chi)\, \frac{|r|}{\delta}\, e^{r|\mathrm{Im}\,\chi|}, \qquad (B.5)$$

that leads for all real r (with account of (B.3)) to inequality

$$|s_\ell(r,\chi)| \le B''(\ell,\chi)\left(\frac{r}{r+\delta}\right)e^{r|\mathrm{Im}\chi|}. \qquad (B.6)$$

From (B.4), (B.6) there follows the uniform boundness of the kernels K_ℓ:

$$|K_\ell(r,r';\chi)| \le D(\ell,\chi)\left(\frac{r}{r+\delta}\right)e^{|\mathrm{Im}\chi|(r-r')} \quad \text{at } r>r', \qquad (B.7)$$

$$|K_\ell(r,r';\chi)| \le D(\ell,\chi)\left(\frac{r'}{r'+\delta}\right)e^{|\mathrm{Im}\chi|r'-\mathrm{Im}\chi\cdot r} \text{at } r'>r. \qquad (B.8)$$

Taking into account the recurrence relations for the Legendre functions [15], we obtain analogous estimates for the derivatives in variable χ:

$$\left|\frac{\partial}{\partial\chi}e_\ell^{(1)}(r,\chi)\right| \le A''(\ell,\chi)e^{-r\,\mathrm{Im}\chi}(|\ell+1|(\mathrm{cth}\chi+1)+r),$$

$$\left|\frac{\partial}{\partial\chi}K_\ell(r,r';\chi)\right| \le D(\ell,\chi)e^{|\mathrm{Im}\chi|r'-\mathrm{Im}\chi\cdot r} \times \qquad (B.9)$$

$$\times\left(\frac{r'}{r'+\delta}\right)(|\ell+1)(\mathrm{cth}\chi+1)+r') \quad \text{at } r'>r.$$

The analysis of analytic and asymptotic properties of the derivatives in ℓ shows that

$$\left|\frac{\partial}{\partial\ell}e_\ell^{(1)}(r,\chi)\right| \le A'''(\ell,\chi)e^{-r\,\mathrm{Im}\chi},$$

$$\left|\frac{\partial}{\partial\ell}K_\ell(r',r';\chi)\right| \le D'(\ell,\chi)e^{|\mathrm{Im}\chi|r'-\mathrm{Im}\chi\cdot r}\left(\frac{r'}{r'+\delta}\right) \text{at } r'>r. \qquad (B.10)$$

APPENDIX C

Now we discuss the behaviour of free solutions of eq. (2.1) at the point $\chi=0$. Using the expression for a branch $Q_\nu^\mu(z,1^-)$ [15], we obtain:

$$e_\ell^{(1)}(r,\chi e^{-i\pi}) = i\sin\pi\ell\,(1+\mathrm{cth}\,\pi r)\,e_\ell^{(1)}(r,\chi) + \qquad (C.1)$$
$$+(\cos\pi\ell - i\,\mathrm{cth}\,\pi r\cdot\sin\pi\ell)e_\ell^{(2)}(r,\chi),$$

$$e_\ell^{(2)}(r,\chi e^{-i\pi}) = i\sin\pi\ell\,(1-\mathrm{cth}\,\pi r)\,e_\ell^{(2)}(r,\chi) + \qquad (C.2)$$
$$+(\cos\pi\ell + i\,\mathrm{cth}\,\pi r\cdot\sin\pi\ell)e_\ell^{(1)}(r,\chi),$$

$$s_\ell(r,\chi e^{-i\pi}) = e^{-i\pi(\ell+1)}s_\ell(r,\chi). \qquad (C.3)$$

From (C.1)-(C.3) it follows:

$$K_\ell(r,r';\chi e^{-i\pi}) = K_\ell(r,r';\chi) + 2\sin\pi\ell\cdot s_\ell(r,\chi)\times \qquad (C.4)$$
$$\times\, s_\ell^*(r',\chi)(\mathrm{cth}\,\pi r - \mathrm{cth}\,\pi r').$$

It is easy to verify that: 1) $(\mathrm{sh}\chi)^{-\ell-1}s_\ell(r,\chi)$ is holomorphic function of χ^2 in the neighbourhood of the point $\chi=0$ (comp. [9]); 2) $K_\ell(r,r';\chi)$ has branchpoint at $\chi=0$.

We conclude with the discussion of transformation $\ell\to-\ell-1$. From (A.8a,b) it follows that

$$e_{-\ell-1}^{(1)}(r,\chi) = e^{i\pi(\ell+\frac{1}{2})}e_\ell^{(1)}(r,\chi),$$

$$\nu_{-\ell-1}(r)\, e^{(2)}_{-\ell-1}(r,\chi) = e^{-i\pi(\ell+\frac{1}{2})}\,\nu_{\ell}(r)\, e^{(2)}_{\ell}(r,\chi). \quad (C.5)$$

Thus, $K_{-\ell-1}(r,r';\chi) \neq K_{\ell}(r,r';\chi)$ if ℓ is inequal to entire number.

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