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A NEW METHOD OF THE DERIVATION OF THE QUANTUM FIELD THEORY EQUATIONS FOR THE DETERMINATION OF GREEN FUNCTIONS. THE CONSIDERATION OF THE MODELS $g[\varphi^4]_d$, $d = 2,3$ VIA GREEN FUNCTIONS METHOD

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Новый метод вывода уравнений для функций Грина в квантовой теории поля. Рассмотрение моделей $g[\phi^4]_d$, $d=2,3$, методом функций Грина

Дан простой и общий метод вывода уравнений для функций Грина в квантовой теории поля. На основе этого метода проведено рассмотрение моделей $g[\phi^4]_2$ и $g[\phi^4]_3$.

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A New Method of the Derivation of the Quantum Field Theory Equations for the Determination of Green Functions. The Consideration of the Models $g[\phi^4]_d$, $d=2,3$ via Green Functions Method

Models $g[\phi^4]_d$, $d=2,3$, are considered.

1. A new method of the derivation of the usual Green function equations is given. (The reader familiar with Green functions method would easily derive our main eq. (37) from basic formulae (14)-(14) without our method as well).

2. We transform the usual definition of the Green function (11) into the form (17), (18). The γ-integral in (17) is shown to be defined (in some region $M^2<M_0^2$) by (infinitely narrow) neighbourhoods of two points $\gamma = \pm\gamma_0(M^2, g)$, $\gamma_0^2>0$: This fact corresponds to the vacuum degeneration in the region $M^2<M_0^2$.

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SEC. 1. INTRODUCTION

We have proved in the work [1] the following result. Let the function $G_0(p)$ be defined through the infinite-dimensional integral

$$G_0(p) = h \int \phi(p)\phi(-p)\, e^{-S_0} \prod_k d\phi(k) \Big/ \int e^{-S_0} \prod_k d\phi(k)\,, \qquad (1)$$

where

$$S_0 = \frac{1}{2} h \sum_k a(k) R(k) + h^3 \sum_{k_1 k_2} C_{k_1 k_2} R(k_1) R(k_2) +$$

$$+ h^5 \sum_{k_1 k_2 k_3} C_{k_1 k_2 k_3} R(k_1) R(k_2) R(k_3) + \dots\,, \qquad (2)$$

$$R(k) = \phi(k)\,\phi(-k)\,, \qquad (3)$$

$$\overline{\phi(k)} = \phi(-k)\,, \quad C_{k_1 k_2} = C_{k_2 k_1}\,, \dots\,; \qquad (4)$$

h is the infinitesimal volume element of the k summation (k runs over the values $k = h^{1/d}(i_1, i_2, \dots, i_d)$ with i integer). Then the function $G_0(p)$ satisfies the integral equation

$$G_0(p)^{-1} = a(p) + 4\int C_{pk}\, dk/G_0(k) + 6\int C_{pk_1k_2} \prod_{i=1}^{2} (dk_i / G_0(k_i)) + 8\int C_{pk_1k_2k_3} \prod_{i=1}^{3} (dk_i/G_0(k_i)) + \ldots \quad (5)$$

1. In the present work we show that eqs. (1)-(5) allow one to get all the quantum field theory Green function equations.

1.1. We shall consider in detail an example of the quantum field theory model defined by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left(\frac{\partial\tilde{\phi}(x)}{\partial x_a}\right)^2 - \frac{1}{2}M^2\tilde{\phi}(x)^2 - \tilde{g}\tilde{\phi}(x)^4 \quad (6)$$

in the Euclidean space of d dimensions.

1.2. The Lagrangian (6) gives the Action

$$S = \frac{1}{2}\int (p^2 + M^2)\phi(p)\,\phi(-p)\,dp + g\int \prod_1^4 (\phi(p_i)\,dp_i)\,\delta(\sum_1^4 p_j), \quad (7)$$

$$\phi(p) = (2\pi)^{-d/2}\int \tilde{\phi}(x)\,e^{ipx}\,dx, \quad (8)$$

$$g = \tilde{g}(2\pi)^{-d}. \quad (9)$$

1.3. Introducing volume element h, we shall rewrite eq. (7) in the form

$$S = \frac{1}{2}h \sum_p (p^2 + M^2)\,\phi(p)\,\phi(-p) + gh^3 \sum_{p_1+p_2+p_3+p_4=0} \prod_{i=1}^4 (\phi(p_i)), \quad (10)$$

here we assume the momentum cut off to be introduced via the conditions

$$|p| < \ell, \quad |p_1| < \ell, \ldots, |p_4| < \ell.$$

1.4. We define, as usual, the Green function $G(p; M^2, g)$ via the infinite dimensional Euclidean integral

$$G(p; M^2, g) = h\int \phi(p)\,\phi(-p)\,e^{-S} \prod_k d\phi(k) \Big/ \int e^{-S} \prod_k d\phi(k). \quad (11)$$

1.5. It turns helpful to produce in eq. (11) the change of variables

$$\phi(p) = \gamma\delta(p) + \psi(p), \quad \psi(0) = 0, \quad (12)$$

and separate in (11) the γ integration

$$\delta\phi \equiv \prod_k d\phi(k) = h^{-1} d\gamma \prod_{k\neq 0} d\psi(k) = h^{-1} d\gamma\,\delta\psi. \quad (13)$$

1.6. Then the Action takes the form

$$S(\psi, \gamma; M^2, g) = \frac{h}{2}\sum_p{}' (p^2 + M^2 + 12g\gamma^2)\psi(p)\psi(-p) + 4g\gamma h^2 \sum_{p_1+p_2+p_3=0}{}' \psi(p_1)\psi(p_2)\psi(p_3) + gh^3 \sum_{p_1+p_2+p_3+p_4=0}{}' \psi(p_1)\psi(p_2)\psi(p_3)\psi(p_4) + \frac{1}{h}\left(\frac{M^2}{2}\gamma^2 + g\gamma^4\right) \quad (14)$$

here $p \neq 0, p_1 = 0, \ldots p_4 \neq 0$.

1.7. Let us denote

$$D(p; \gamma, M^2, g) \equiv h\int \psi(p)\,\psi(-p)\,\delta\psi\, e^{-S(\psi,\gamma; M^2, g)} \times K^{-1}(\gamma; M^2, g), \quad p \neq 0, \quad (15)$$

$$K(\gamma; M^2, g) = \int \delta\psi\, e^{-S(\psi,\gamma;M^2,g)} . \quad (16)$$

Then, according to (11), the equation

$$G(p; M^2, g) = \int K(\gamma; M^2, g)\, d\gamma [D(p; \gamma, M^2, g) + \gamma^2 \delta(p)] K^{-1}, \quad (17)$$

$$K = \int K(\gamma; M^2, g)\, d\gamma \quad (18)$$

holds.

1.8. It appears that

$$K(\gamma; M^2, g) = \exp[\, f(\gamma; M^2, g)/h], \quad (19)$$

where the function f does not depend on h. We have $h \to 0$, and γ-integrals (17), (18) are defined by an infinitesimal neighbourhood of the function f maximum $\gamma = \gamma_0$; so we have

$$G(p; M^2, g) = D(p; \gamma_0, M^2, g) + \gamma_0^2\, \delta(p) . \quad (20)$$

1.9. In the model considered there exists a function $M_0^2(g,\ell)$ such that for $M^2 > M_0^2(g,\ell)$ the function (19) has maximum at $\gamma=0$, and for $M^2 < M_0^2(g,\ell)$ the function (19) has two maxima at $\gamma = \pm a(M^2, g)$, $a>0$. This result corresponds to the appearance of the ground state degeneration at $M^2 < M_0^2(g,\ell)$ (see, e.g., ref. /3/). At $M^2 = M_0^2(g,\ell)$ our system has a phase transition with the breaking of the analytical dependence of $G(p; M^2, g)$ on M^2. In eq. (20) one has to put $\gamma_0 = 0$ at $M^2 > M_0^2(g,\ell)$ and $\gamma_0 = \pm a(M^2, g)$ at $M^2 < M_0^2(g,\ell)$.

1.10. The function (16), (19) is called the effective potential /2/.

1.11. The article is planned as follows.

In *Sec. 2* we transform the functional e^{-S} to the form (26), (33) as to get from eqs. (1)-(5) the equation (37) for the Green function D. In *Sec. 3* we derive eq. (41) which defines extrema of the effective potential and the expression (46) of the function $(d/d\gamma)^2 K(\gamma; M^2, g)$ which is necessary in order to establish, whether the given extremum is maximum. Equations (37), (41), (46) are used in *Secs. 4* and *5* to confirm the results sketched in *Sec. 1*, in the case of two and three space-time dimensions.

1.12. Our derivation of the Green function equation (37) is simple and straightforward and deserves preponderance over other methods.

1.13. Equations (17), (12), (19) at $p=0$ give

$$G(0; M^2, g) = \int d\gamma \frac{\gamma^2}{h} e^{f(\gamma;M^2,g)/h} / \int d\gamma\, e^{f(\gamma;M^2,g)/h} . \quad (21)$$

Let now $\gamma_0 = 0$ and

$$f(\gamma; M^2, g) = f(0; M^2, g) - U(M^2, g)\gamma^2 + O(\gamma^4) . \quad (22)$$

Then we get the equation

$$G(0; M^2, g) = [2U(M^2, g)]^{-1} , \quad (23)$$

which, with due regard for the contibuity of function $G(p; M^2, g)$ for $M^2 > M_0^2(g)$ implies

$$\lim_{p\to 0} D(p; 0, M^2, g) = [2U(M^2, g)]^{-1} , \quad M^2 > M_0^2(g) . \quad (24)$$

Equation (24) connects the values of two quite different functions f and D and thus is of certain interest.

SEC. 2. THE TRANSFORMATION OF THE FUNCTIONAL e^{-S}

In order to be able to use eqs. (1)-(5) for the computation of integrals with Action (14), let us separate from the quartic term of eq. (14) the part

$$-P_4 = 3gh^3 \sum_{p_1, p_2}{}' R(p_1) R(p_2) , \quad R(p) = \psi(p)\psi(-p). \quad (25)$$

Then we have

$$S(\psi, \gamma; M^2, g) = -P_0 - P_2 - P_4 + S_3' + S_4' \quad (26)$$

here P_2 is the first, and P_0, the last term of r.h.s. of eq. (14), S'_3 and S'_4 are the terms of third and fourth order in ψ. Were the Action to contain only P_0, P_2 and P_4, eqs. (1)-(5) would reduce the problem of calculating the Green function to the problem of solving the integral equation (5). The terms S'_3 and S'_4 prevent, however, such a reducing.

2.1. Consider the exponent

$$e^{-S'_3} = 1 - S'_3 + \frac{1}{2!} S'^2_3 - \frac{1}{3!} S'^3_3 + \dots \quad (27)$$

The expression S'_3 contains no pair $R(p) \equiv \psi(p)\,\psi(-p)$. The term S'^2_3, however, contains pairs

$$S'^2_3 = (4g\gamma h^2)^2 \Big[3! \sum'_{p_1+p_2+p_3=0} R(p_1)R(p_2)R(p_3) +$$

$$+ 3^2 \sum'_{\substack{p_1+p_2+p_3=0 \\ p_1-p'_2-p'_3=0}} R(p_1)\psi(p_2)\psi(p_3)\psi(p'_2)\psi(p'_3) +$$

$$+ \sum'_{\substack{p_1+p_2+p_3=0 \\ p'_1+p'_2+p'_3=0}} \psi(p_1)\psi(p_2)\psi(p_3)\psi(p'_1)\psi(p'_2)\psi(p'_3)\Big]; \quad (28)$$

here in the second and third terms the functions ψ form no pairs (in the third term $p_1+p'_1 \neq 0$, $p_1+p'_2 \neq 0$, $p_1+p'_3 \neq 0, \dots p_3+p'_3 \neq 0$, in the second term $p_2+p'_2 \neq 0$, $p_2+p'_3 \neq 0$, $p_3+p'_2 \neq 0$, $p_3+p'_3 \neq 0$).

Introduce the notations

$$-S'_3 = U_{30},$$

$$\frac{1}{2} S'^2_3 = Q_6 + U_{61} + U_{60}, \quad (29)$$

here U_{30} is the third order in ψ expression without pairs, Q_6 the sixth order in ψ expression with three pairs, U_{61} the sixth order in ψ expression with one pair, U_{60} the sixth order in ψ expression without pairs. Then it will be

$$-\frac{1}{3!} S'^3_3 = Q_6 U_{30} + \sum_{k=0}^{3} U_{9,k},$$

$$\frac{1}{4!} S'^4_3 = Q_{12} + \frac{1}{2!} Q_6^2 + Q_6 (U_{61} + U_{60}) + \sum_{k=0}^{4} U_{12,k}, \quad (30)$$

so that

$$e^{-S'_3} = (1 + \sum_{n \geq 2k+3} U_{nk})\, e^{Q_6 + Q_{12} + \dots}, \quad (31)$$

where U_{nk} are the n-th order in ψ expressions with k pairs, which are irreducible in the sense, that it is impossible to extract from U_{nk} any entirely paired multiplier Q_m, $m \leq n$:

$$Q_6 = \frac{(4g\gamma h^2)^2}{2!} 3! \sum'_{p_1+p_2+p_3=0} R(p_1)R(p_2)R(p_3),$$

$$Q_{12} = \frac{(4g\gamma h^2)^4}{4!} 6^3 \cdot 3! \sum'_{p_1+p_2+p_3=0} R(p_1)R(p_2)R(p_3) \times$$

$$\times \sum'_{p_1+p'_1+p'_2=0} R(p'_2)R(p'_3)R(p_2-p'_2). \quad (32)$$

2.2. Analogously one gets

$$e^{-S'_3 - S'_4} = (1 + \sum_{n \geq 2k+3} V_{nk}) \times$$

$$\times \exp[P_6(3^2) + P_8(4^2) + P_{10}(3^2 4) + P_{12}(3^4) + \dots]; \quad (33)$$

here

$$P_6(3^2) = Q_6 ,$$

$$P_{12}(3^4) = Q_{12} ,$$

$$P_8(4^2) = \frac{(gh^3)^2}{2!} 4! \sum_{p_1+p_2+p_3+p_4=0}' R(p_1)R(p_2)R(p_3)R(p_4) ,$$

$$P_{10}(3^2 \cdot 4) = -\frac{(4g\gamma h^3)^2 gh^3}{3!} 3^3 4! \sum_{p_1+p_2+p_3+p_4=0}' R(p_1)R(p_2)R(p_3) \times$$

$$\times R(p_4)R(p_1+p_2) , \tag{34}$$

. . . . ;

$$V_{30} = U_{30} ,$$

$$V_{40} = -gh^3 \sum_{p_1+p_2+p_3+p_4=0}' \psi(p_1)\psi(p_2)\psi(p_3)\psi(p_4) , \tag{35}$$

. . . .

see also eq. (28); in V_{40} the summation area is restricted via the conditions $p_1+p_2 \neq 0$, $p_1+p_3 \neq 0$, $p_1+p_4 \neq 0$.

2.3. Thus, eqs. (26), (33) and (15), (16) imply

$$D(p;\gamma,M^2,g) = h\int \psi(p)\,\psi(-p)\exp[\,P_2+P_4+P_6+\ldots\,] \times$$

$$\times (1+\sum_{n\geq 2k+3} V_{nk}) \prod_{k\neq 0} d\psi(k) / \int \exp[P_2+P_4+P_6+\ldots] \times$$

$$\times (1+\sum_{n\geq 2k+3} V_{nk}) \prod_{k\neq 0} d\psi(k) ; \tag{36}$$

here P_2, P_4, P_6 ,... are entirely paired expressions (25), (34).

2.3.1. The parity considerations show the quantities V_{nk} to give no contribution into the integrals in eq. (36).

2.3.2. Omitting V_{nk} brings the function (36) into the form (1), (2).

So, according to eq. (5), the Green function $D(p;\gamma,M^2,g)$ can be defined from the equation

$$D(p)^{-1} = p^2 + M^2 + 12g\gamma^2 + 12g\int D(s)\,ds -$$

$$-6\frac{(4g\gamma)^2}{2!}3!\int D(s_1)D(s_2)\,ds_1 ds_2 \delta(s_1+s_2-p) -$$

$$-8\frac{g^2}{2!}4!\int \prod_1^3 (D(s_i)\,ds_i)\,\delta(\sum_1^3 s_j - p) +$$

$$+\frac{(4g\gamma)^2 g}{3!}3^3 4![\,8\int \prod_1^3 (D(s_i)\,ds_i)\,D(s_1+s_2)\,\delta(s_1+s_2+s_3-p) +$$

$$+2(\int \prod_1^2 (D(s_i)\,ds_i)\,\delta(s_1+s_2-p))^2\,] -$$

$$-\frac{(4g\gamma)^4}{4!}6^3 3!\,12\int \prod_1^4 (D(s_i)\,ds_i)\,D(s_1-s_3) \times$$

$$\times \delta(s_1+s_2-p)\,\delta(s_3+s_4-p) + \tag{37}$$

$$+ \ldots ;$$

here for brevity we omitted the arguments γ, M^2, g of the function D. All the terms of the r.h.s. of eq. (37) admit usual diagrammatic interpretation (c.f. eqs. (14), (15)).

SEC. 3. THE DEPENDENCE OF γ_0 ON M^2 (THE BEGINNING)

According to (14), one has the equation

$$(d/d\gamma)K(\gamma;M^2,g) = -\int \prod_{k\neq 0} d\psi(k)\, e^{-S(\psi,\gamma;M^2,g)}$$

$$\{h \sum_p{}' 12g\gamma\psi(p)\psi(-p) + 4gh^2 \sum_{p_1+p_2+p_3=0}{}' \psi(p_1)\psi(p_2)\psi(p_3) +$$

$$+ (M^2\gamma + 4g\gamma^3)/h\} = 0 \qquad (38)$$

defining the extrema $\gamma=\gamma_0$ of the effective potential.

3.1. Introduce the notation

$$\delta(p_1+p_2+p_3)\Gamma_3(p_1,p_2,p_3) \equiv K^{-1}(\gamma;M^2,g) \times$$

$$\times \int \psi(p_1)\psi(p_2)\psi(p_3)\, e^{-S(\psi,\gamma;M^2,g)}\,\delta\psi\,. \qquad (39)$$

Then, according to eqs. (26), (33) we get

$$\delta(p_1+p_2+p_3)\,\Gamma_3(p_1,p_2,p_3) = K^{-1}(\gamma;M^2,g)$$

$$\int \psi(p_1)\psi(p_2)\psi(p_3)\, e^{P_0+P_2+P_4+\dots}(1+\sum V_{nk})\delta\psi =$$

$$=K^{-1}(\gamma;M^2,g)\int e^{P_0+P_2+P_4+\dots}\delta\psi\,\psi(p_1)\psi(p_2)\psi(p_3)\times$$

$$\times[-4g\gamma h^2 \sum_{p_1'+p_2'+p_3'=0}{}' \psi(p_1')\psi(p_2')\psi(p_3') +$$

$$+4g\gamma h^2 gh^3 6^2 \sum_{p_1+p_2+p_3+p_4=0} \psi(p_1)\psi(p_2)R(p_3)R(p_4)\psi(-p_1-p_2)+\dots]=$$

$$=-4g\gamma\cdot 3!\,D(p_1)D(p_2)D(p_3)\,\delta(p_1+p_2+p_3) +$$

$$+ 4g^2\gamma\cdot 72\cdot D(p_1)D(p_2)D(p_3)\delta(p_1+p_2+p_3)\times$$

$$\times\int D(s_1)D(s_2)\,ds_1\,ds_2[\delta(s_1+s_2-p_1)+\delta(s_1+s_2-p_2) +$$

$$+\delta(s_1+s_2-p_3)] + \dots\,. \qquad (40)$$

(we have taken into account the fact that the integral of the form (15) of the function $\psi(p_1)\psi(p_2)$ equals zero if $p_1+p_2\neq 0$).

3.2. Equations (15) and (39) allow us to rewrite eq. (38) for the determination of γ_0 in the form

$$\gamma[M^2 + 4g\gamma^2 + 12g\int D(s;\gamma,M^2,g)\,ds +$$

$$+\frac{4g}{\gamma}\int\Gamma_3(s_1,s_2,s_3)\delta(s_1+s_2+s_3)\,ds_1ds_2ds_3] = 0\,. \qquad (41)$$

3.3. Now it is necessary to know whether the extremum $\gamma=\gamma_0$ is maximum or minimum. So, we have to consider the second derivative of the effective potential at the point of extremum.

3.4. Analogously to eq. (38) we have

$$(d/d\gamma)^2 K(\gamma;M^2,g) = \int\delta\psi\, e^{-S(\psi,\gamma;M^2,g)} \times$$

$$\times[\{\quad\}^2 - 12gh\sum_p{}'\psi(p)\psi(-p) - (M^2+12g\gamma^2)/h]; \qquad (42)$$

here curly brackets denote to the expression which is in curly brackets in eq. (38).

3.4.1. Let us use the definition of the irreducible Green functions Γ_4, Γ_5, Γ_6

$$K^{-1}(\gamma;M^2,g)\int\prod_1^4(\psi(p_i))\delta\psi\, e^{-S(\psi,\gamma;M^2,g)}$$

$$=\frac{1}{2}\sum_{\substack{i<j,\,k<\ell\\ i\neq k,\, i\neq\ell\\ j\neq k,\, j\neq\ell}}\delta(p_i+p_j)\delta(p_k+p_\ell)D(p_i)D(p_k)$$

$$+\delta(p_1+p_2+p_3+p_4)\,\Gamma_4(p_1,p_2,p_3,p_4)\,; \qquad (43)$$

$$K^{-1}(\gamma;M^2,g)\int\prod_1^5(\psi(p_i))\,\delta\psi\, e^{-S(\psi,\gamma;M^2,g)}$$

$$= \sum_{i<j} \delta(p_i + p_j)\,\delta\Big(\sum_{k=1}^{5} p_k - p_i - p_j\Big) D(p_i)\Gamma_3$$

$$+\delta\Big(\sum_{i=1}^{5} p_i\Big)\Gamma_5(p_1, p_2, p_3, p_4, p_5);$$

$$K^{-1}(\gamma; M^2, g) \int \prod_1^6 (\psi(p_i))\delta\psi\, e^{-S(\psi,\gamma;M^2,g)} \tag{44}$$

$$= \delta(p_1+p_2)\delta(p_3+p_4)\delta(p_5+p_6)D(p_1)D(p_3)D(p_5)+ \ldots$$

$$+\delta(p_1+p_2+p_3)\delta(p_4+p_5+p_6)\Gamma_3(p_1,p_2,p_3)\Gamma_3(p_4,p_5,p_6)+\ldots$$

$$+\sum_{i<j}\delta(p_i+p_j)\delta\Big(-p_i-p_j+\sum_{k=1}^{6} p_k\Big)D(p_i)\Gamma_4$$

$$+\delta\Big(\sum_1^6 p_k\Big)\Gamma_6(p_1,p_2,p_3,p_4,p_5,p_6). \tag{45}$$

In eq. (44) the arguments of Γ_3 are three momenta of the set $p_1,\ldots p_5$ except for p_i and p_j, in eq. (45) the arguments of Γ_4 are four momenta of the set $p_1,\ldots p_6$ except for p_i, p_j.

3.4.2. Equations (38), (41)-(45) give

$$hK^{-1}(\gamma;M^2,g)(d/d\gamma)^2 K(\gamma;M^2,g)\Big|_{\gamma=\gamma_0} =$$

$$= -[M^2 + 12g\gamma_0^2 + 12g\int D(s)ds]$$

$$+ (12g\gamma_0)^2\,[2\int D^2(s)ds + \int\Gamma_4(s_1,-s_1,s_2,-s_2)ds_1 ds_2]$$

$$+ 12g\gamma_0 \cdot 8g\int[6D(s_1)\Gamma_3(s_1,s_2,s_3) + \int\Gamma_5(s,-s,s_1,s_2,s_3)ds]$$

$$\delta(s_1+s_2+s_3)ds_1 ds_2 ds_3$$

$$+(4g)^2\{3!\int\prod_1^3 (D(s_i)ds_i)\delta\Big(\sum_1^3 s_i\Big) \tag{46}$$

$$+3^2\int\Gamma_4(s_1,s_2,s_3,s_4)D(s_1+s_2)\delta\Big(\sum_1^4 s_i\Big)ds_1\ldots ds_4$$

$$+3^2\int\delta(s_1+s_2+s_3)\delta(s_1'+s_2'+s_3)\Gamma_3(s_1,s_2,s_3)\Gamma_3(s_1's_2's_3)$$

$$ds_1\, ds_2\, ds_3\, ds_1'\, ds_2'$$

$$+\int\delta(s_1+s_2+s_3)\delta(s_1'+s_2'+s_3')\Gamma_6(s_1,s_2,s_3,s_1',s_2',s_3')ds_1\ldots ds_3'\}.$$

3.5. Now we are able to search for extrema of the effective potential and to define whether the given extremum is maximum or minimum (eqs. (41), (46)).

3.6. One has the following expression for the Green function Γ_4:

$$\Gamma_4(p_1,p_2,p_3,p_4) = -4!\,g\prod_1^4 D(p_i) +$$

$$+ (4g\gamma)^2\,\frac{3}{2}\prod_1^4 (D(p_i))\sum_{\alpha<\beta}^{4} D(p_\alpha+p_\beta)+\ldots \tag{47}$$

SEC. 4. THE DEPENDENCE OF γ_0 ON M^2 (d = 2)

Let us consider the simplest case of two-dimensional space-time.

4.1. Equations (40), (41) imply eq. (41) to have a root

$$\gamma = \gamma_0 = 0 \tag{48}$$

for all values of M^2. The equation (37) at $\gamma = 0$ takes the form

$$D(p;0,M^2,g)^{-1} = p^2 + M^2 + 12g\int D(s;0,M^2,g)ds -$$

$$-96g^2\int\prod_1^3 (D(s_i,0,M^2,g)ds_i)\delta\Big(\sum_1^3 s_i - p\Big) \tag{37a}$$

$$+ \ldots .$$

Let us introduce the notation

$$m^2 = M^2 + 12g \int D(s;0,M^2,g)ds \tag{49}$$

and produce in (37a) the change of variables

$$p = mq, \; D(p;0,M^2,g) = B(q, g/m^2)/m^2 . \tag{50}$$

Then equation (37a) takes the form

$$B(q,\epsilon)^{-1} = q^2 + 1 - 96\epsilon^2 \int \phi(q_1,\epsilon)B(q-q_1,\epsilon)dq_1$$

$$+ 2\cdot 12^3 \epsilon^3 \int \phi^2(q_1,\epsilon)B(q-q_1,\epsilon)dq_1 + O(\epsilon^4) \tag{37b}$$

here

$$\phi(q,\epsilon) = \int B(t,\epsilon)B(q-t,\epsilon)dt , \tag{51}$$

$$\epsilon = g/m^2 . \tag{52}$$

4.2.1. Equation (37b) implies the expansion

$$B(q,\epsilon) = \sum_k B_k(q)\epsilon^k \tag{53}$$

at $\epsilon \ll 1$, where

$$B_0(q) = (q^2+1)^{-1} ,$$

$$B_1(q) = 0,$$

$$B_2(q) = 96 B_0^2(q) \int \prod_{i=1}^{3} (B_0(s_i)ds_i)\, \delta(\sum_{j=1}^{3} s_j - q), \tag{54}$$

. . . .

4.2. Substituting eqs. (50)-(54) into eq. (49) gives the connection

$$M^2 = m^2 - 24\pi g[\ln(\frac{\ell}{m}) + \sum_{k=2}^{\infty} a_k \epsilon^k] \tag{55}$$

between the parameters M^2 and m^2; here ℓ is the cut off momentum, see sec. 1.3.

4.3. Equation (46) at $\gamma_0 = 0$ reduces to

$$hK^{-1}(\gamma;M^2,g)(d/d\gamma)^2 K(\gamma;M^2,g) = -m^2 + (4g)^2 \times$$

$$\times [3^2 \int D(s_1+s_2)\Gamma_4(s_1,s_2,s_3,s_4)\delta(s_1+s_2+s_3+s_4)ds_1ds_2ds_3ds_4$$

$$+3! \int \prod_1^3 (D(s_i)ds_i)\,\delta(\sum_{j=1}^{3} s_j)$$

$$+ \int \Gamma_6(s_1,\dots,s_6)\delta(\sum_1^3 s_i)\delta(\sum_4^6 s_i)ds_1,\dots,ds_6] . \tag{56}$$

Equation (47) shows that the term with Γ_4 adds here to $-m^2$ the value $\sim m^2\epsilon^3$; analogous consideration implies the term with Γ_6 to add to $-m^2$ the value $\sim m^2\epsilon^4$ and the term with $\prod_1^3 D(s_i)$ the value $\sim m^2\epsilon^2$.

4.3.1. Thus we arrive at the conclusion that the effective potential at $\gamma = \gamma_0 = 0$ has maximum, if the value of ϵ is small enough; i.e., that the value of m^2 (for given g) is large enough.

4.4. Let now eq. (41) have nonzero root γ_0, $\gamma_0^2 > 0$. Produce in eqs. (37), (41), (46) the change of variables

$$p = q\sqrt{g\gamma_0^2}\, , \quad D(p;\gamma_0,M^2,g) = C_2(q;\gamma_0)/(\gamma_0^2 g),$$

$$\Gamma_3(p_1,p_2,p_3) = C_3(q_1,q_2,q_3;\gamma_0)/(\gamma_0^5 g^2),$$

$$\Gamma_4(p_1,p_2,p_3,p_4) = C_4(q_1,\dots,q_4;\gamma_0)/(\gamma_0^8 g^3), \tag{57}$$

$$\Gamma_5(p_1,\dots,p_5) = C_5(q_1,\dots,q_5;\gamma_0)/(\gamma_0^{11} g^4),$$

$$\Gamma_6(p_1,\dots,p_6) = C_6(q_1,\dots,q_6;\gamma_0)/(\gamma_0^{14} g^5).$$

4.4.1. Then instead of (37) we get

$$C_2(q;\gamma_0)^{-1} = q^2 + 8 - (4/\gamma_0^4)\int C_3(q_1,q_2,q_3;\gamma_0)\delta(\sum_1^3 q_i)dq_1dq_2dq_3$$

$$-\frac{16\cdot 18}{\gamma_0^2}\int\prod_1^2(C_2(q_i;\gamma_0)dq_i)\delta(q_1+q_2-q)$$

$$-\frac{96}{\gamma_0^4}\int\prod_1^3(C_2(q_i;\gamma_0)dq_i)\delta(\sum_1^3 q_i-q)$$

$$+\frac{12^3}{\gamma_0^4}[8\int\prod_1^3(C_2(q_i;\gamma_0)dq_i)C_2(q_1+q_2;\gamma_0)\delta(\sum_{j=1}^3 q_j-q)$$

$$+2(\int\prod_1^2(C_2(q_i;\gamma_0)dq_i)\delta(q_1+q_2-q))^2]$$

$$-\frac{8\cdot 12^4}{\gamma_0^4}\int\prod_1^4(C_2(q_i;\gamma_0)dq_i)C_2(q_1-q_3,\gamma_0)\delta(q_1+q_2-q)\times$$

$$\times\,\delta(q_3+q_4-q)+O(\gamma_0^{-6}). \qquad (37c)$$

Here we have excluded the parameter M^2 via eq. (41).

4.4.2. According to (40), (57) we have

$$C_3(q_1,q_2,q_3;\gamma_0) = -24\prod_1^3(C_2(q_i;\gamma_0))$$

$$\{1-\frac{12}{\gamma_0^2}\int\prod_1^2(C_2(s_i;\gamma_0)ds_i)[\delta(s_1+s_2-q_1)+$$

$$+\delta(s_1+s_2-q_2)+\delta(s_1+s_2-q_3)]+O(\gamma_0^{-4})\}. \qquad (40a)$$

4.4.3. All the integrals in eqs. (37c), (40a) converge (we have $d=2$); so these equations imply the Green function $C_2(q;\gamma_0)$ to expand in powers of γ_0^{-2} :

$$C_2(q;\gamma_0) = \sum_{k=0}^{\infty} C_{2(k)}(q)\gamma_0^{-2k}; \qquad (58)$$

$$C_{2(0)}(q) = (q^2+8)^{-1},$$

$$C_{2(1)}(q) = 16\cdot 18 C_{2(0)}^2(q)\int\prod_1^2(C_{20}(q_i)dq_i)\delta(q_1+q_2-q)$$

$$\ldots\ldots\ldots \qquad (59)$$

4.4.4. Analogously, one has the expansions

$$C_n(q;\gamma_0) = \sum_{k=0}^{\infty} C_{n(k)}(q)\gamma_0^{-2k} \qquad (60)$$

for all Green functions C_n .

4.4.4. After the change (57) the expression (46) takes the form

$$g^{-1}hK^{-1}(\gamma;M^2,g)(d/d\gamma)^2K(\gamma;M^2,g)|_{\gamma=\gamma_0} = -8\gamma_0^2$$

$$+\frac{4}{\gamma_0^2}\int C_3(q_1,q_2,q_3;\gamma_0)\delta(q_1+q_2+q_3)dq_1dq_2dq_3$$

$$+12^2[2\int C_2^2(q;\gamma_0)dq+\gamma_0^{-2}\int C_4(q_1,-q_1,q_2,-q_2;\gamma_0)dq_1dq_2]$$

$$+\frac{12\cdot 8}{\gamma_0^2}\int[6C_2(q_1;\gamma_0)C_3(q_1,q_2,q_3;\gamma_0)+$$

$$+\gamma_0^{-2}\int C_5(q,-q,q_1,q_2,q_3;\gamma_0)dq] \qquad (46a)$$

$$\delta(q_1+q_2+q_3)dq_1dq_2dq_3$$

$$+\frac{4^2}{\gamma_0^4}[3^2\int C_2(q_1+q_2;\gamma_0)C_4(q_1,q_2,q_3,q_4;\gamma_0)\delta(\sum_1^4 q_i)dq_1..dq_4$$

$$+3!\,\gamma_0^2\int\prod_1^3(C_2(q_i;\gamma_0)dq_i)\delta(\Sigma q_i)$$

$$+3^2\int C_3(q_1,q_2,q_3;\gamma_0)C_3(q_1',q_2',q_3;\gamma_0)\delta(\sum_1^3 q_i)\delta(q_1'+q_2'+q_3)$$

$dq_1 dq_2 dq_3 dq_1' dq_2'$

$$+\gamma_0^{-2}\int \delta(\sum_1^3 q_i)\delta(\sum_4^6 q_i)C_6(q_1,\dots q_6;\gamma_0)dq_1\dots dq_6]$$

here we have again excluded the parameter M^2 via eq. (41). Equation (46a) shows that the extremum $\gamma=\gamma_0\neq 0$ of the effective potential is a maximum, if the value of γ_0^{-2} is small enough.

4.4.6. Equations (58), (41), (60) connect the values of M^2 and γ_0^2:

$$M^2=-4g\gamma_0^2-24\pi g(\ln\frac{\ell}{\sqrt{8g\gamma_0^2}}+\sum_{k=1}^{\infty}b_k\gamma_0^{-2k}). \tag{61}$$

4.4.7. So, one sees that the effective potential $K(\gamma;M^2,g)$ has a maximum at $\gamma=\gamma_0=0$ $(\gamma_0^2>0)$ for large positive (large negative) values of the parameter t,

$$t=(M^2+24\pi g\ \ln\frac{\ell}{\sqrt{g}})/g \tag{62}$$

4.5. So, we have finished the investigation of the case $d=2$.

SEC. 5. THREE DIMENSIONAL SPACE-TIME

It is easy to extend our consideration to the case $d=3$.

5.1. The problem is to transform the basic formulae (37), (41), (46) to the form similar to (37b), (37c), (55), (61), (56), (46a) via the changes of variables of the type (50), (57).

5.2. We begin with the case $\gamma_0=0$. The integral-coefficient for g^2 in eq. (37a) - diverges (logarithmically). Therefore one has to choose the value of parameter m^2 so that it would cancel the divergence:

$$m^2=\mu^2+96g^2\int\prod_1^3(D(s_i)ds_i)\delta(\Sigma s_i) \tag{49a}$$

here the value of μ^2 is finite. Instead of (50) we produce the change of variables

$$p=\mu q,\quad D(p;0,M^2,g)=\epsilon(q;\lambda)/\mu^2 \tag{50a}$$

5.2.1. Then we get instead of eq. (37b) the (divergence free, $d=3$) equation

$$\epsilon(q;\lambda)^{-1}=q^2+1+96\lambda^2\int\prod_1^3(\epsilon(q_i;\lambda)dq_i)$$

$$[\delta(\sum_1^3 q_i)-\delta(\sum_1^3 q_i-q)]$$

$$+2\cdot 12^3\lambda^3\int\epsilon(q-q_1;\lambda)dq_1(\int\epsilon(\alpha;\lambda)\epsilon(q_1-\alpha;\lambda)d\alpha)^2$$

$$+O(\lambda^4). \tag{37d}$$

Here λ is dimensionless (at $d=3$) constant

$$\lambda=g/\mu. \tag{52a}$$

Similarly to eq. (53), eq. (37d) inplies the expansion

$$\epsilon(q;\lambda)=\sum_0^\infty\epsilon_k(q)\lambda^k,\quad \epsilon_0(q)=(q^2+1)^{-1}. \tag{53a}$$

5.2.2. It would be instead of eq. (55)

$$M^2=\mu^2+96g^2\int\prod_1^3(D(s_i;0,M^2,g)ds_i)\delta(\sum_1^3 s_i)$$

$$-12g\int D(s;0,M^2,g)ds=$$

$$=\mu^2-48\pi g\ell+g\mu\sum_0^\infty\alpha_k\lambda^k$$

$$+96g^2(\int_{|q_i|<\ell/\mu}\prod_1^3(\epsilon_0(q_i)dq_i)\delta(\sum_1^3 q_j)+\sum_1^\infty\beta_k\lambda^k). \tag{55a}$$

5.2.3. Analogously to sec. 4.3, the terms with Γ_4 and Γ_6 add to the r.h.s. of eq. (56) the quantities of an order of magnitude $\mu^2\lambda^3$ and $\mu^2\lambda^4$, respectively; the term with $\prod_1^3 D(s_i)$ gives a logarithmically divergent contribution (with a factor $\mu^2\lambda^2$) which cancels with the divergent part of m^2 (see eq. (49a)). So, the r.h.s. of eq.

(56) is negative and extremum $\gamma_0 = 0$ is a maximum of the effective potential for small enough values of $|\lambda|$.

5.3. Now we turn to the case $\gamma_0^2 > 0$. Let us denote by $F_n(q;\delta)$ (instead of $C_n(q;\gamma_0)$) the functions which we get after the transformation (57). One gets the equation for F_2 substituting δ,

$$\delta = \sqrt{g}/\gamma_0 \qquad (52b)$$

for γ_0^{-2} in eq. (37c). We denote symbolically this equation (37e). The first and the third integrals in the r.h.s. of eq. (37e) diverge.These divergences, however, cancel with each other, so that eq. (37e) contains no divergences. Equation (37e) implies the expansions

$$F_n(q;\delta) = \sum_{k=0}^{\infty} F_{n(k)}(q)\delta^k . \qquad (60a)$$

5.3.1. Similarly to eqs. (55a), (61), we get

$$M^2 = -4g\gamma_0^2 - 12g^{3/2}|\gamma_0| \int_{|q|<\ell/\sqrt{g\gamma_0^2}} F_2(q;\delta)dq$$

$$-4g^2 \int F_3(q_1,q_2,q_3;\delta)dq_1 dq_2 dq_3 \delta(q_1+q_2+q_3)$$

$$= -4g\gamma_0^2 - 48\pi g\ell + g^{3/2}|\gamma_0| \sum_0^{\infty} \alpha_k' \delta^k$$

$$+ 96g^2 \Big(\int_{|q_i|<\ell/\sqrt{8g\gamma_0^2}} \prod_1^3 (F_{2(0)}(q_i)\, dq_i)\delta(\sum_1^3 q_j) + \sum_1^{\infty} \beta_k' \delta^k \Big). \qquad (61a)$$

5.3.2. At $d=3$ we have to substitute in eq. (46a) the functions $F_2(q;\delta), F_3(q;\delta),\ldots$ for the functions $C_2(q;\gamma_0)$, $C_3(q;\gamma_0),\ldots$ and the values $\gamma_0^2, \gamma_0^2\delta, \gamma_0^2\delta^2, \ldots$ for the degrees $\gamma_0^2, \gamma_0^0, \gamma_0^{-2}$ of the parameter γ_0^2; the formula thus obtained is denoted by eq. (46b). The terms with $F_3(q_1,q_2,q_3;\delta)$ and $\prod_1^3 F_2(q_i;\delta)$ give logarithmically divergent contributions to the r.h.s. of eq. (46b), which cancel with each other. So we get

$$g^{-1}hK^{-1}(\gamma;M^2,g)(d/d\gamma)^2 K(\gamma;M^2,g)|_{\gamma=\gamma_0} =$$

$$= -8\gamma_0^2(1+ O(\delta^2)):$$

extremum $\gamma = \gamma_0$, $\gamma_0^2 > 0$ is a maximum of the effective potential for small enough values of $|\delta|$.

5.4. So, at $d=2$ and $d=3$ the model considered needs only mass renormalization, this is the well known result.

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