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Построение перенормированных коэффициентных функций фейнмановских диаграмм с помощью ЭВМ

В работе приведены алгоритм и краткое описание программы для ЭВМ, написанной на языке аналитических преобразований SCHOONSCHIP. Программа предназначена для построения подынтегральных выражений перенормированных коэффициентных функций, фейнмановских диаграмм скалярных теорий в случае произвольной точки вычитания. Для заданного графа Фейнмана ЭВМ полностью проводит R-операцию Боголюбова-Парасюка и выдает результат в виде интеграла по фейнмановским параметрам.

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The Construction of Renormalized Coefficient Functions of Feynman Diagrams by Computer

In this paper an algorithm and short description of computer program, written in SCHOONSCHIP, are given. The program is assigned for construction of integrands of renormalized coefficient functions of Feynman diagrams in scalar theories in the case of arbitrary subtraction point. For the given Feynman graph computer completely realizes the R-operation of Bogolubov-Parasjuk and gives the result as an integral over Feynman parameters.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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The Feynman diagram technique gives us a possibility of finding Green functions, scattering amplitudes, and other quantities in quantum field theory as a series in the coupling constant g. However, even in quantum electrodynamics, where the coupling constant is small, the effective parameter of the decomposition in some situation has a value close to unity, and hence there arises the necessity of calculation of contributions of arbitrary high orders. The calculation at high orders in perturbation theory is a very cumbersome problem and sometimes at this stage of development of calculation methods and computer technique cannot be achieved at all.

Among main difficulties encountered along this way one may outline the following:

1) the generation (construction) of Feynman diagrams of a given order;

2) the construction of integrands for renormalized functions of diagrams;

3) the evaluation of integrals over parameters.

Without application of computer technique, to overcome these difficulties is impossible. Untill this time, the generation of diagrams /1,2/ and evaluation of integrals /3/ were given the most attention. Least of all, there was investigated the second problem-construction of integrands /1/. Meanwhile even in scalar theories large difficulties are encountered when we evaluate high orders in perturbation theory.
In this paper we build an algorithm and give a description of the program, written in a symbolic language "SCHOONSCHIP" /4/, assigned to constract the integrands of renormalized coefficient functions for scalar theories in the case of arbitrary subtraction point.

2. Mathematical formalism

In this section we give the well known formulae for coefficient functions of Feynman diagrams, in scalar theories (see /5-7/).
For a diagram G of order N in coupling constant in α -representation there corresponds the following integral:

$$T_G = (-1)^{L+C} \pi^{2C} \int_0^\infty \prod_{\ell=1}^{L} d\alpha_\ell \, \frac{e^{iA_G(\alpha,\kappa) - i\sum_{\ell=1}^{L}\alpha_\ell(m_\ell^2 - i\varepsilon)}}{D_G^2(\alpha)} \quad . \qquad (1)$$

Here C is the number of loops of the diagram, L, the number of internal lines, K, the external momenta, m_ℓ, the masses of particles.
The structure functions $D_G(\alpha)$ and $A_G(\alpha,K)$ can be constructed by different methods (see, for example, refs. /5-7/).

In the program described below these functions are constructed as a sums over trees and 2-trees, respectively (see /5/, § 29):

$$D_G(\alpha) = \sum_{\substack{\text{on trees}\\ \text{of graph}}} \prod_{\substack{\text{on chord}\\ \text{of tree}}} \dots \alpha_j \dots \quad ; \qquad (2)$$

$$A_G(\alpha,\kappa) = \frac{Q_G(\alpha,\kappa)}{D_G(\alpha)} = \frac{1}{D_G(\alpha)} \sum_{\substack{\text{on 2-}\\ \text{trees}}} \Big[\Big(\prod_{\substack{\text{on chords}\\ \text{of 2-trees}}} \dots \alpha_j \dots\Big) \cdot \Big(\sum_{\substack{\text{on comp.}\\ \text{of 2-tree}}} \kappa_{j_*}\Big)^2\Big] \qquad (3)$$

$D_G(\alpha)$ and $Q(\alpha,\kappa)$ are sums every term of which contains the product of $L-N+1$ and $L-N+2$ parameters α_j, respectively. Generally speaking, integral (1) is divergent and we must remove these divergences, i.e., apply the R-operation. If a strongly connected diagram G contains divergent subgraphs $\gamma_1, \gamma_2, \dots \gamma_p$, then the R-operation for G with subtraction at point $\kappa^2 = \lambda^2$ may be written as follows:

$$R_\lambda T_G = (1 - M_G)(1 - M_{\gamma_1}) \cdots (1 - M_{\gamma_p}) \, T_G \qquad (4)$$

(When $i<j$ γ_i do not belong to γ_j as a whole), where M_{γ_i} are operators which replace the coefficient function T_G by corresponding numbers of terms of its series in powers of scalar combinations of external momenta. If γ_i is logarithmically divergent, then operator M_{γ_i} replaces the integral corresponding to subgraph by its value at point $\kappa_i^2 = \lambda_i^2$ (contraction of a subgraph into a point). The renormalized coefficient function in α -representation

looks as follows: (see [7]):

$$R_\lambda T_G = \frac{i^{N+1}\,\delta(\sum k)}{\pi^{2(N-L-1)}} \lim_{\varepsilon\to 0} \int_0^\infty \prod_{i=1}^{L} d\alpha_i \int_0^1 \prod_{j=1}^{P} dx_j \frac{(1-x_j)^{\frac{\omega_j}{2}}}{\left(\frac{\omega_j}{2}\right)!} \cdot \left(\frac{\partial}{\partial x_j}\right)^{\frac{\omega_j}{2}+1} x_j^{2c_j} \cdot$$

$$\cdot \exp\Big[i\,\frac{Q(\beta,k)}{D(\beta)} + i\sum_{q=1}^{P}(1-x_q)\frac{Q^{(q)}(\beta^{(q)},\lambda^{(q)})}{D^{(q)}(\beta^{(q)})} - i\sum_{\ell}(m_\ell^2 - i\varepsilon)\alpha_\ell\Big] . \tag{5}$$

Here to each subgraph γ_i of the diagram G there corresponds parameter x_i ; $\beta_\ell = \alpha_\ell$ if line ℓ does not belong to any divergent subgraph and $\beta_\ell = x_{i_\ell} \cdots x_{j_\ell}\alpha_\ell$ if line ℓ enters into the divergent subgraphs $\gamma_{i_\ell} \cdots \gamma_{j_\ell}$. In the forms $Q^{(q)}(\beta^{(q)},\lambda^{(q)})$ and $D^{(q)}(\beta^{(q)})$ for subgraph γ_q , $\beta_{\ell_q} = x_{j_q}\cdots x_{i_q}\alpha_{\ell_q}$ only in that case, if line ℓ enters into divergent subgraphs $\gamma_{j_q} \cdots \gamma_{i_q}$ which completely belongs to γ_q otherwise $\beta_{\ell_q} = \alpha_{\ell_q}$. $\omega_q = \sum_{\ell=1}^{L^{(q)}}(r_\ell + 2) - 4(N^{(q)} - 1)$ is the index of divergence of subgraph γ_q , r_ℓ is determined by the type of a considered theory, $c_q = L^{(q)} - N^{(q)} + 1$ is the number of independent circles of this subgraph.

$\lambda^{(q)}$ is the subtraction point of subgraph γ_q . If all subgraphs of the diagram G diverge logarithmically, then for any term in (4) we have the following expression (see [8]):

$$M_{\gamma_1}\cdots M_{\gamma_n} T_G = T_{\gamma_n}(\lambda^2)\cdot T_{\gamma_{n-1}/\gamma_n}(\lambda^2)\cdot T_{\gamma_{n-2}/\gamma_n+\gamma_{n-1}}(\lambda^2)\cdots T_{G/\gamma_n+\cdots+\gamma_1}(k) =$$

$$= \int_0^\infty \prod d\alpha\, \frac{e^{iA_{\gamma_n}(\alpha,\lambda) + iA_{\gamma_{n-1}/\gamma_n}(\alpha,\lambda)} \cdots e^{iA_{G/\gamma_n+\cdots+\gamma_1}(\alpha,k)}}{D^2_{\gamma_n}(\alpha)\, D^2_{\gamma_{n-1}/\gamma_n}(\alpha)\, D^2_{\gamma_{n-2}/\gamma_n+\gamma_{n+1}}(\alpha) \cdots D^2_{G/\gamma_n+\cdots+\gamma_1}} . \tag{6}$$

Here $T_{G/\gamma}$ corresponds to the coefficient function for graph G , where subgraph γ is contracted into a point. If diagram contains subgraphs divergent quadratically, then (see (5)) for construction of its coefficient function it is necessary to know first derivatives of $Q(\beta,k)$ and $D(\beta)$ with respect to parameters corresponding to these subgraphs, taken at $x = 0$. The program described below constructs all $Q(\beta,k)$ and $D(\beta)$ which enter in (6) for each term from (4), and for quadratically divergent subgraphs, also their first derivatives taken at $x = 0$ with respect to parameters corresponding to these subgraphs.

3. Description of the program

Schematically, the program may be presented as follows:

a) the representation of a graph,

b) the topological analysis of the graph: finding of divergent subgraphs,

c) the construction of functions $Q(\beta,k)$ and $D(\beta)$ for the diagram and its divergent subgraphs,

d) the construction of all $Q(\beta,k)$ and $D(\beta)$ entering into (6) and also the first derivatives of $Q(\beta,k)$ and $D(\beta)$ for quadratically divergent subgraphs.

The representation of a graph means the following: to define the order of the graph, number of internal lines, external momenta, masses, and representation of the topological structure.

For the topological representation of the graph, for each vertex and each line we associate some symbols. For example, consider the two-loop diagram in the $g\varphi^4$ model.

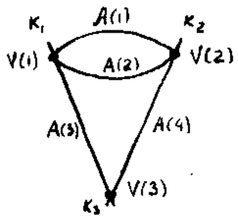

Fig.1

For the L vertex of the graph we associate symbol $V(L)$ for K line of the graph symbol $A(K)$. The topological representation of the graph means the enumeration of all possible products of two verticies and lines connecting them. For our considered example the topological representation means that there are the following substitutions in the program:

$$\begin{aligned} &V(1)*V(2) \rightarrow V(1)*V(2)*A(1)*A(2)\\ &V(1)*V(3) \rightarrow V(1)*V(3)*A(3)\\ &V(2)*V(3) \rightarrow V(2)*V(3)*A(4). \end{aligned} \tag{7}$$

To determine divergent subgraphs it is necessary to consider first of all possible splittings of the diagram in two, three-, four vertices, etc.
For the diagram represented in fig.1 we consider the following sum in the program:

$$S(2,3)=V(1)*V(2)+V(1)*V(3)+V(2)*V(3). \tag{8}$$

The construction of such splittings is performed by the following formulae:

$$S(K,N)=\sum_{j_1=K}^{N} V(j_1)*\sum_{j_2=K-1}^{j_1-1} V(j_2)\cdots\sum_{j_K=1}^{j_{K-1}-1} V(j_K). \tag{9}$$

Here N is the order of the graph, K, the number of vertices considered in the decomposition. A substitution of the type (7) in the sum (9) makes correspond to each term lines connecting vertices enterning in this term. For the considered example we have:

$$S(2,3)=V(1)*V(2)*A(1)*A(2)+V(1)*V(3)*A(3)+V(2)*V(3)*A(4). \tag{10}$$

Then in the transformed sum we keep only those terms which correspond to the divergent strongly connected subgraphs. This choice is performed according to the index of divergence and a more detailed topological analysis of the graph. To do this, in the transformed sum (9) from all the lines directly connecting any two vertices it is necessary to keep two lines and to use the following inequality:

$$C-N+1\geq 0$$

which is correct for divergent subgraphs. Here C is the number of loops in the subgraph arising after we have dropped the lines according to the above prescription, N is the number of vertices of the initial subgraph. For (10) even after the choice according to the divergence index we get:

$$S(2,3) = V(1) * V(2) * A(1) * A(2). \tag{11}$$

All terms for which $\omega < 0$ are nullified. Then we put $V(J) = 1$. Thus, all divergent subgraphs are products of lines. Using these products and the sums:

$$\sum_{j_1 = k}^{L} A(j_1) * \sum_{j_2 = k-1}^{j_1 - 1} A(j_2) \cdots \sum_{j_k = 1}^{j_{k-1} - 1} A(j_k) \tag{12}$$

it is easy to construct sums of different products of lines entering into the subgraphs. In these sums we keep only those terms which correspond to trees and two-trees of subgraphs. For this aim we must drop terms containing the set of lines which form the circles.

Each line $A(k)$ is replaced by $A(k) * D(J1, J2)$ ($J1$ and $J2$ are indices of vertices connected by the given line, $D(J1,J2) \equiv g_{J1,J2}$ is the Minkowski tensor, $g_{\mu\mu} = n$). Thus, those terms which correspond to circles are proportional to some parameter n, which then is put to zero. As a result, we get functions $D(\alpha)$ and $A(k, \alpha)$ for the whole graph and its divergent subgraphs. For the example we consider the functions $D(\alpha)$ and $Q(\alpha, k)$ are:

$$D(\alpha) = \alpha_2\alpha_4 + \alpha_2\alpha_3 + \alpha_1\alpha_3 + \alpha_1\alpha_4 + \alpha_1\alpha_2 \tag{13}$$

$$Q(\alpha, k) = \alpha_2\alpha_3\alpha_4 k_3^2 + \alpha_1\alpha_3\alpha_4 k_3^2 + \alpha_1\alpha_2\alpha_4 k_2^2 + \alpha_1\alpha_2\alpha_3 k_1^2 . \tag{14}$$

Further, we consider the following sums:

$$D(\alpha) + \sum_{q=1}^{p} (1 - x_q)\, D^{(q)}$$

$$Q(\alpha, k) + \sum_{q=1}^{p} (1 - x_q)\, Q^{(q)}(\alpha, k)$$

which are multiplied by the product of lines of some subgraph $\gamma_\ell : \rho_{i_\ell} \cdots \rho_{j_\ell}$. Now, we carry out the following analysis: whether the subgraph γ_ℓ belongs to the others completely or not, and then according to the rules described in section 2, the products $\alpha_{i_\ell}\rho_{i_\ell}$ are replaced either by $x_{i_\ell}\alpha_{i_\ell}$ or by α_{i_ℓ}. If the subgraph diverges logarithmically, we take these sums for $x_{i_\ell} = 1$ and $x_{i_\ell} = 0$. If we have the quadratically divergent subgraph we construct also the first derivative of these sums at $x_{i_\ell} = 0$. The sums thus transformed are then multiplied by the product of lines of another subgraph. The above procedure is repeated as long as all subgraphs are examined. For our example the renormalized coefficient function looks as follows:

$$R_\lambda G_{\Gamma} = \int_0^\infty \Pi d\alpha \left[\frac{e^{i A_{\Gamma}(\alpha, k)}}{D_{\Gamma}^2(\alpha)} - \frac{e^{i A_{\gamma_1}(\alpha, k) + i A_{\gamma_2}(\alpha, \lambda)}}{D_{\gamma_1}^2(\alpha)\; D_{\gamma_2}^2(\alpha)} - \frac{e^{i A_{\Gamma}(\alpha, \lambda)}}{D^2(\alpha)} + \frac{e^{i A_{\gamma_1}(\alpha, \lambda) + i A_{\gamma_2}(\alpha, \lambda)}}{D_{\gamma_1}^2(\alpha)\; D_{\gamma_2}^2(\alpha)} \right]. \tag{15}$$

Here A_{γ} . and D_{γ} are the forms corresponding to the graph obtained after "the construction of the subgraph into a point " A_{α} and D_{α} are the forms for that subgraph. On the output of the program we have two sums:

$$TREE = (\alpha_1\alpha_3 + \alpha_2\alpha_3 + \alpha_1\alpha_4 + \alpha_2\alpha_4 + \alpha_1\alpha_2)\cdot P(0)\cdot L(1) -$$

$$-(\alpha_1+\alpha_2)(\alpha_3+\alpha_4)\,P(0)R(1) - (\alpha_1+\alpha_2)P(1)\,R(1) \qquad (16)$$

$$TREE\,2 = [(\alpha_1+\alpha_2)\alpha_3\alpha_4 K_3^2 + \alpha_1\alpha_2(\alpha_4 K_2^2 + \alpha_3 K_1^2)]\cdot P(0)\cdot L(1) -$$

$$-(\alpha_1+\alpha_2)\alpha_3\alpha_4 K_3^2\cdot P(0)\,R(1) - \alpha_1\alpha_2\,P(1)R(1)\,. \qquad (17)$$

In the program, symbol $P(0)$ corresponds to the whole graph; $P(1), P(2), \ldots$ to subgraphs, and that one staying before $P(0)*P(1)\ldots$ is obtained from the forms $D(\alpha), Q(\alpha,\kappa)$; $D^{(1)}(\alpha)$, $Q^{(1)}(\alpha,\kappa); \ldots$, respectively. Symbols $L(1), L(2)\ldots$ mean that forms $D(\alpha), Q(\alpha,\kappa), D^{(1)}(\alpha), Q^{(1)}(\alpha,\kappa)$ were taken at $x_1 = 1, x_2 = 1, \ldots$; and $R(1), R(2), \ldots$ mean that those forms were taken at $x_1 = 0$, $x_2 = 0, \ldots$. Formulae (16) and (17) contain all Q and D required to construct (15). The form $A_{\varphi}(\alpha,\kappa)$ is the ratio of the sum in front of $P(0)*L(1)$ in $TREE\,2$ to the sum in front of $P(0)*L(1)$ in TREE. $D_{\varphi}(\alpha)$ is the sum in front of $P(0)\cdot L(1)$ in TREE. Forms A_{γ} and A_{α} enterning into the second term of (15) are constructed as a ratio of sums in front of $P(0)R(1)$ and $P(1)R(1)$ in TREE2 to the sums in front of P(0)*R(1) and P(1)*R(1) in TREE. The denominator of the second term in (15), $D_{\gamma}^2(\alpha)\cdot D_{\alpha}^2(\alpha)$ is the square of the sum in front of $P(0)R(1)$. To get the

third and fourth terms, it is necessary to put $\kappa^2 = \lambda^2$ in the first and the second terms.
In the same way, we can construct the renormalized coefficient functions for arbitrary graphs. A general method of that construction is as follows: the denumerators of terms in (6) are considered to be squared of sums in front of coefficients. $P(0), \ldots, L(J), \ldots R(K), \ldots$ in TREE; forms $A(\alpha,\kappa)$ are the ratios of sums in front of the same products of symbols $P(I), L(J), R(K)$ in TREE2 and TREE. The first derivatives of $D(\alpha)$ and $Q(\alpha,\kappa)$ for quadratically divergent subgraphs are sums in front of the symbol $SQDIV(J)$. (J is the number of quadratically divergent subgraphs).
With the help of the program described above some diagrams of 5th order in the $g\varphi^4$ model were calculated (see /9/). The construction of the whole renormalized coefficient function takes about 30 seconds of computing time on the CDC-6500 computer. After some modification, this program may be used for constructing the forms $Q(\alpha,\kappa)$, $D(\alpha)$ for nonscalar theories as well.

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