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ANALYTICAL CONTINUATION OF PERTURBATIVE RESULTS OF THE $g\varphi^4$ MODEL INTO THE REGION $g \gtrsim 1$

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ANALYTICAL CONTINUATION OF PERTURBATIVE RESULTS OF THE $g\varphi^4$ MODEL INTO THE REGION $g \gtrsim 1$

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Казаков Д.И., Тарасов О.В., Ширков Д.В. E2 - 11571

Аналитическое продолжение результатов теории возмущений модели $g\phi^4$ в область $g \gtrsim 1$

Рассматривается вопрос о том, что нового дал прогресс в многопетлевых вычислениях и метод асимптотических оценок коэффициентов ряда ТВ для прояснения физической ситуации в поведении эффективного заряда на малых расстояниях.

Рассмотрение проводится на примере теории $\phi^4_{(4)}$. Предложена процедура построения аппроксимантов функции Гелл-Манна-Лоу на основе синтеза точных коэффициентов низших порядков и асимптотических оценок под знаком интегрального представления. Получено, что в модели $g\phi^4_{(4)}$ функция ГМЛ имеет поведение типа $0.9g^2$ при $g \gtrsim 1$.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Kazakov D.I., Shirkov D.V., Tarasov O.V. E2 - 11571

Analytical Continuation of Perturbative Results of the $g\phi^4$ Model into the Region $g \gtrsim 1$

A new information concerning the behaviour of the effective charge at small distances obtained through the progress in multiloop calculations and the method of asymptotic estimates of PT coefficients is considered.

The model $\phi^4_{(4)}$ is studied in more detail. The construction procedure of Gell-Mann-Low function approximation is proposed based on the synthesis of exact coefficients of lower orders and asymptotic estimates under the integral representation. In the $g\phi^4_{(4)}$ model the G-M-L function behaviour like $0.9g^2$ for $g \gtrsim 1$ is obtained

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. Introduction

While investigating the ultraviolet asymptotics of the Green functions it is necessary to know the behaviour of the Gell-Mann-Low function or the β-function in a theory. Usually for this purpose the standard perturbation theory (PT) in quantum field theory is used. Recently a remarkable progress has been achieved in calculation of higher order PT contributions to the β-function /1-3/ and a method of approximate estimations of large orders has been developed /4,5/.

In the present paper we consider what new information comes from the progress in calculations and theory for understanding the physical situation in the effective charge behaviour at small distances. The examination is carried out by the most studied example - scalar model $\varphi^4_{(4)}$.

It should be noted that this analysis is also valid in the theories with asymptotical freedom based on the Yang-Mills fields but in the infrared region. The search of the ultraviolet stable fixed point changes to the search of the infra-red one.

2. Perturbation Theory in $\varphi^4_{(4)}$ Model

Consider the theory with the following interaction Lagrangian

$$\mathcal{L}_{int} = -\frac{h}{4!}\varphi^4 = -\frac{16\pi^2}{4!} g \varphi^4 \qquad (1)$$

in the four dimensional space-time.

New methods of Feynman integral evaluation developed in recent years allow us to calculate the β -function in this theory up to the 4-loop level.

Here one should note, however, that in PT calculations beginning from the 3-loop level, the β -functions depends on the renormalization procedure and in different schemes the coefficients of PT series differ, beginning from the third one. Besides, the renormalization group (RG) equations can be written in different forms (in the Lie form or in the Ovsiannikov-Callan-Symanzik one) and the β -function in these formulations also differs in general, beginning from the third coefficient. (For a detailed discussion of this question see ref. /6/). This does not, however, affect the physical effects since this reduces to the redefinition of the physical charge.

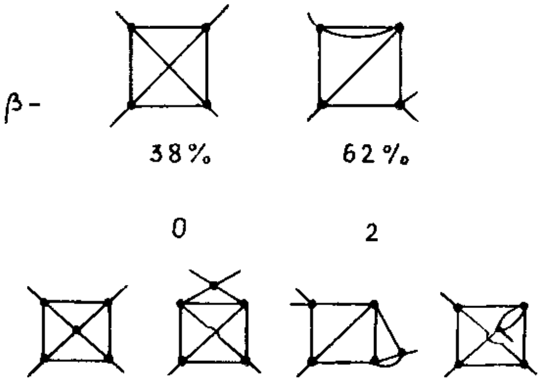
We exploit the expression for the β -function in the model (1) obtained by using Bogolubov R-operation with subtractions at symmetrical point $p_i^2 = -\lambda^2, \ s = t = u = -\frac{4}{3}\lambda^2$. In this case the β -functions of the RG equations in the Lie and Ovsiannikov-Callan-Symanzik forms coincide.

In the 4-loop approximation the Gell-Mann-Low function of the model (1) is /2/

$$\beta(g) = \frac{3}{2}g^2 - \frac{17}{6}g^3 + 19{,}3\, g^4 - (146 \pm 2) g^5 + ?\, g^6 \qquad (2)$$

calculated (year)		1972	1973	1977	?....
number of diagrams	1	2	7	23	>50

The main diagrams which contribute to the third and fourth coefficients in eq. (2) are represented in Fig.1.



3-loop approximation		
contribution to the β-function	38%	62%
number of divergent subgraphs	0	2

4-loop approximation				
Contribution to the β-function	43%	22%	20%	12%
number of divergent subgraphs	0	1	3	1

Fig.1

The given contributions to the β -function from various diagrams depend, in general, on the calculation scheme. Thus in the renormalization scheme with ultraviolet cut-off the β -function of RG equation in the Ovsiannikov form is /3/

$$\beta(g) = \frac{3}{2}g^2 - \frac{17}{6}g^3 + 20{,}84\, g^4 - 216{,}09\, g^5 + \cdots \qquad (3)$$

and contributions from the given diagrams are 35%, 58% and 28%, < 5%, 13% and 36%, respectively. As can be seen, the fact that in the theories like $\varphi^n_{(D)}$, where $D = \frac{2n}{n-2}$ for $n \to \infty$, the main contribution to the Gell-Mann-Low function should come from the diagrams with maximal symmetry without subgraphs /7/ is not yet seen for n=4. So, what kind of information about the behaviour of function $\beta(g)$ follows from PT? We know 4 terms of PT series which is (as will be discussed below) an asymptotic series of the Poincare type[8]. Then, limiting ourselves by the first N terms of the series we have the remainder of series $R_{N+1}(g) = O(g^{N+1})$. Moreover, by using the method of asymptotic estimates of the Gell-Mann-Low function coefficients, discussed in the next section and asymptotic representation of the β -function (6) it is easy to show that the error is not larger than the first missed term of PT series, i.e.,

$$\beta(g) \simeq \sum_{n=2}^{N} \beta_n (-g)^n \pm |\beta_{N+1} g^{N+1}| . \qquad (4)$$

Consider how far in g we can move on knowing 4 terms of the series (4) with say 10% accuracy. The results are presented in Fig.2.

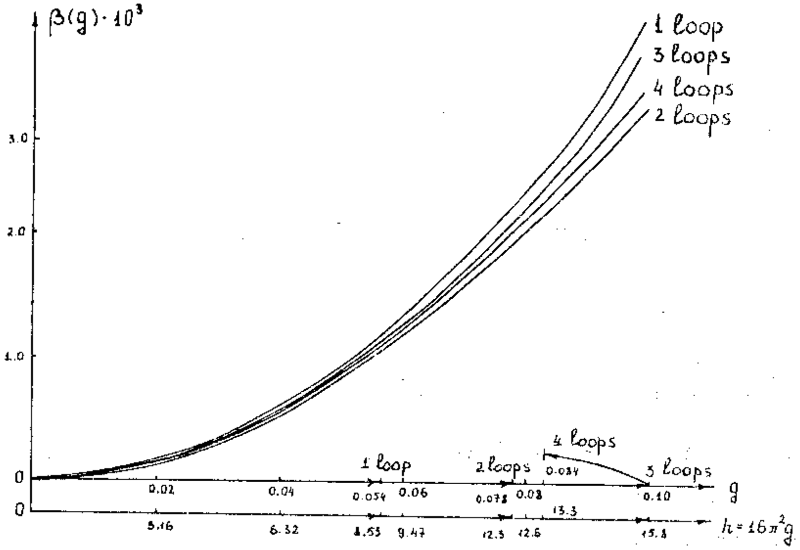


Fig.2.

As can be seen, the forward motion into the region of large g is slowing down with every new term of PT series. Moreover, the consideration of the 4-loop approximation does not give rize to further progress since for $g \sim 0,1$ the contribution

from the 4-loop term to the r.h.s. of eq. (2) becomes equal to the one of 3-loop term. This is a common feature of asymptotic series with zero radius of convergence.

Hence, further progress in PT calculations now does not give new information about function $\beta(g)$ and the region $g \sim 1$ cannot be achieved in principle. Near the origin where we know the function $\beta(g)$ it is positive and has no non-trivial zero.

In this slightly disappointing situation certain hopes for clearing up the behaviour of $\beta(g)$ were placed on the method of asymptotic estimates of Gell-Mann-Low function coefficients, proposed by Lipatov[4]. Let us see what new information gives us this method.

3. Correspondence to the asymptotics estimates.

The method for determination of asymptotic estimates to the coefficients of the Green functions coupling constant expansion is based on the functional integral representation and on the procedure of the steepest descent method in the functional space. The saddle-point corresponds to the Euclidean classical solutions to the equations of motion of the "instanton" type with finite action. Using this approach the asymptotic expression for the coefficients of β -function expansion in powers of g in the model (1) was obtained[4]. It is of the form

$$\beta_n \underset{n\to\infty}{\simeq} \tilde{\beta}_n = n!\, n^{7/2} a^n c \left(1 + O(\tfrac{1}{n})\right), \qquad (5)$$

where $a = 1$, $c \simeq 1,096$.

Analogous estimates for the β -function in quantum statistics (model $\varphi^4_{(3)}$) and for the ground state energy of the anharmonic oscillator in quantum mechanics (model $\varphi^4_{(1)}$) were carried out in paper /5/. A didactic example is also the usual integral

$$\int_0^\infty dx \exp\left\{-x^2 - gx^4\right\}$$

being the zero-dimensional analog $\left(\varphi^4_{(0)}\right)$ of the functional integral.

The comparison of asymptotic estimates with exact values calculated using PT is represented in Fig.3.

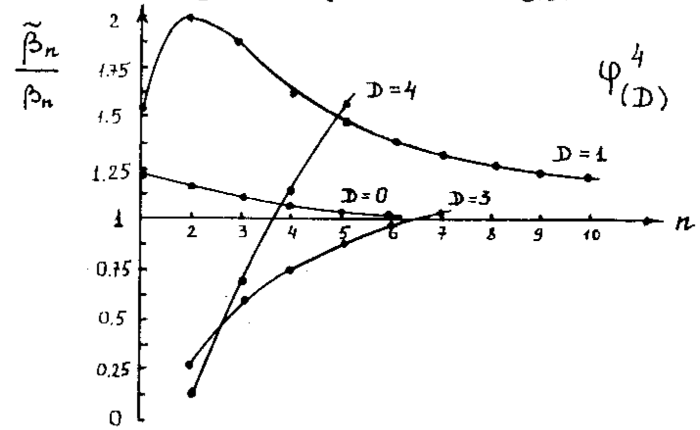


Fig.3.

It can be seen that in the case $D=0$ asymptotics sets in rapidly, in the case $D=1$, slower. As for the cases $D=3, 4$ for known values of n the asymptotics unlike the existing opinion, does not yet set in. Consider for comparison the N -component model $(\varphi^a\varphi^a)^2$, $a=1,2,\dots,N$ in four space dimensions/5/. The ratios of asymptotic coefficients to the exact ones for various values of N are plotted in Fig.4.

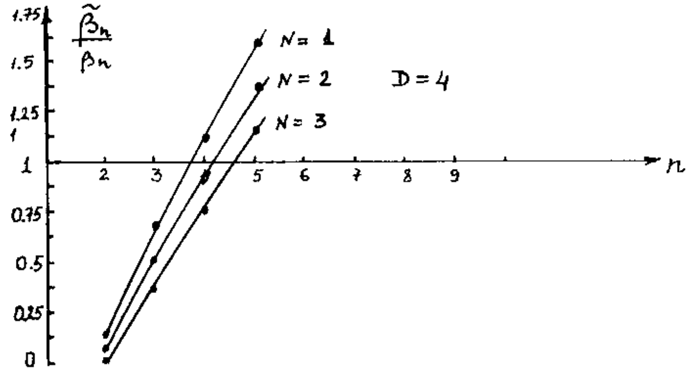


Fig.4.

With N growing the situation is not improved that reflects the fact that asymptotic coefficients decrease while exact ones increase with increasing N (see ref./5/,/2/).

At the same time one should realize the relative character of such a comparison. Eq.(5) has accuracy $O(1/n)$. It can be written down also in a slightly different form obtained from the dispersion integral over the coupling constant, namely /9,10/:

$$\beta(g) \sim \int_{-\infty}^{0} \frac{dz\, e^{1/az}\, \tilde{c}\,(1+O(z))}{(z-g)\,(-z)^{9/2}} \sim \sum_n \tilde{\tilde{\beta}}_n (-g)^n, \qquad (6)$$

$$\tilde{\tilde{\beta}}_n = \Gamma(n+9/2)\, a^n c\,\left(1+O\left(\tfrac{1}{n}\right)\right). \qquad (7)$$

The difference between the representations (5) and (7) is due to the order $1/n$ which is unknown. This difference becomes small for large values of n but in the region where we work it is rather substantial. The ratio $\tilde{\tilde{\beta}}_n / \tilde{\beta}_n$ has the form

$$\frac{\tilde{\tilde{\beta}}_n}{\tilde{\beta}_n} \equiv f_n = \frac{\Gamma(n+9/2)}{n!\, n^{7/2}} ;$$

n	2	3	4	5
f_n	12,7	6,7	4,5	3,6

The ratios $\tilde{\tilde{\beta}}_n/\beta_n$ and $\tilde{\beta}_n/\beta_n$ and the values of function f_n are plotted in Fig.5.

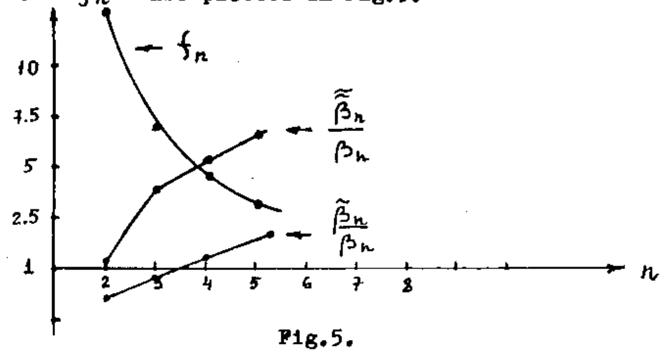


Fig.5.

Thus, for the optimal choice of asymptotic formulae we need to know the term $O(1/n)$.

However, for any choice of the term of the order $1/n$ the desired asymptotics does not yet set in and the knowledge of asymptotic coefficients does not allow us to progress in constructing the function $\beta(g)$.

Nevertheless, the knowledge of increasing character of the PT coefficients gives us some usefull information. We apply it in order to construct the partial sum of PT series.

4. Summation with the use of asymptotic estimates.

The factorial character of the increase of PT coefficients obtained by the asymptotic estimates method indicates that the series is asymptotic with the zero radius of convergence and cannot be summed by usual methods. Investigations of the theory

of asymptotic series indicate the opportunity to implement the Borel summation method/8/ to the series (5) and (7). This method is favoured by the fact that i) in two-dimensional constructive quantum field theory models/11/ and for the anharmonic oscillator /12/ the Borel summability is proven, ii) the dispersion integral (6) reduces to the Borel type via the change of integration variables and iii) the Borel "sum" is stable with respect to the addition of higher order terms of PT series.

For all that the standard Borel method needs a slight modification whose main purpose is to provide the decreasing PT coefficients for the Borel transform and hence the nonzero radius of convergence. There are possible various modifications of the Borel transform, for instance,

$$\beta^{B}(g) = \begin{cases} \int_0^\infty \frac{dx}{g} e^{-x/g} \left(\frac{x}{g}\right)^b B^{(b,0)}(x), \quad B_n^{(b,0)} = \frac{\beta_n}{\Gamma(n+1+b)} & (8) \\ \int_0^\infty \frac{dx}{g} e^{-x/g} \left(x\frac{\partial}{\partial x}\right)^b B^{(b,b)}(x), \quad B_n^{(b,b)} = \frac{\beta_n}{n!\, n^b} & (9) \end{cases}$$

or their combination

$$\beta^{B}(g) = \int_0^\infty \frac{dx}{g} e^{-x/g} \left(\frac{x}{g}\right)^{b-a} \left(x\frac{\partial}{\partial x}\right)^a B^{(b,a)}(x), \quad B_n^{(b,a)} = \frac{\beta_n}{\Gamma(n+1+b-a)\, n^a}, \qquad (10)$$

where a is arbitrary. We discuss the optimal choice of a below.

For any b, according to (5), the PT series for the Borel transform has nonzero circle of convergence and can be summed inside this circle. Moreover, eq.(5) shows us the position and character of the nearest singularity in the complex plane of Borel variable x. Thus, choosing b in eqs. (8-10) to be integer we obtain that the function $B^{(b,a)}(x)$ is analytic inside the unit circle with the cut starting from the square-root branch point at $x = 1$ /13/. The complex x-plane is presented in Fig.6.

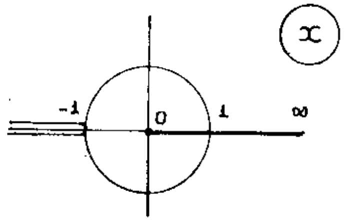


Fig.6.

Hence we obtain the series with unit circle of convergence. However, the integration region in eqs. (8-10) goes beyond this circle. We perform the analytical continuation of the $B^{(b,a)}(x)$ on the whole integration interval via the conformal mapping of the cut x-plane into interior of the unit circle in the w-plane /14/. We choose the conformal mapping also in order to provide the right singularity on the cut. The example of such a mapping is

$$w(x) = \frac{\sqrt{1+x} - 1}{\sqrt{1+x} + 1} \, . \qquad (11)$$

The complex w-plane is shown in Fig.7.

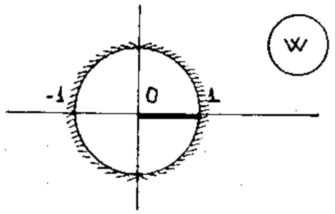


Fig.7.

After the mapping one should expand the integrand in the series over w keeping the same number of terms as in the initial function $\beta(g)$.

The conformal variable (11) possesses the following important property

$$w(x) = -\sum_{n \geq 1} \frac{(-)^n x^n \Gamma(n+1/2)}{\sqrt{\pi}\, \Gamma(n+2)} \simeq -\sum \frac{(-)^n x^n}{\sqrt{\pi}\, n^{3/2}} \left\{1 + O(1/n)\right\},$$

$$[w(x)]^2 = \sum_{n \geq 2} \frac{(-)^n x^n}{\sqrt{\pi}\, n^{3/2}} \cdot 2 \left\{1 + O(1/n)\right\},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$[w(x)]^p = \sum_{n \geq p} \frac{(-)^n x^n}{\sqrt{\pi}\, n^{3/2}} (-)^p p \left\{1 + O(1/n)\right\} . \qquad (12)$$

It follows that

$$P_N(w) = a_0 + a_1 w + \dots + a_N w^N \to$$

$$\to -\sum_n \frac{(-)^n x^n}{\sqrt{\pi}\, n^{3/2}} \left\{a_0 - a_1 + 2a_2 + \dots + N(-)^N a_N\right\} . \qquad (13)$$

Consequently, choosing in eqs. (8-10) $b = 5$, one can obtain asymptotics of β_n for large n in a strict correspondence with the Lipatov result (5) up to the numerical coefficient, and for the coefficients of the series there is a sum rule of the form

$$a_0 - a_1 + 2a_2 + \dots + (-)^N N a_N = const \, . \qquad (14)$$

The procedure described contains an ambiguity connected with the reexpansion of the Borel transform

$$B^{(b,a)}(x) = \sum_{n \geq 2}^{N} x^n B_n^{(b,a)} \qquad (15)$$

into the series over w , allowing the following arbitrariness

$$B^{(b,a)}(x) \to x^{\nu} B^{(b,a,\nu)}(w) = \frac{x^{\nu}}{w^{\nu}} \sum_{n>7/2}^{N} w^{n} B_{n}^{(b,a,\nu)}, \tag{16}$$

where ν is an arbitrary number.

So the summation procedure using the modified Borel method and conformal mapping, contains the arbitrary parameters (b, a, ν). We have fixed one of them $(b=5)$ requiring the correspondence with asymptotic expression (5). To choose the other parameters we have to involve some extra considerations.

5. Choice of transformation parameters

The choice of parameter a in eq. (10) is connected with the corrections of order $1/n$ in asymptotic estimates. In that case when they are unknown we shall follow another criterion, namely, we look for the value a when the terms of the series $B_n^{(b,a)}$ decrease more rapidly with the growth of n. The situation is that with growing a the terms of the series decrease faster and we choose the maximal value of $a=b=5$. In this case, according to eqs. (2) and (9), we have:

$$\beta^{B}(g) = \int_{0}^{\infty} \frac{dx}{g} e^{-x/g} \left(x \frac{\partial}{\partial x}\right)^{5} B^{(5,5)}(x), \tag{17}$$

$$B^{(5,5)}(x) = \frac{3}{128} x^{2} (1 - 0{,}083x + 0{,}033x^{2} + 0{,}017x^{3} + \ldots) \tag{18}$$

The value of parameter ν in eq. (16) determines the asymptotics of $\beta(g)$ for $g \to \infty$. Hence for its choice

it is necessary to go beyond the weak coupling. As far as in the $\varphi^4_{(4)}$ model such an asymptotics is unknown, we shall determine the parameter ν from the requirement of minimal contribution of higher orders of PT series over w to the $\beta^{B}(g)$ for large values of g . For this purpose we consider the following quantities

$$\eta_{N+1}(g) = 1 - \frac{\beta^{B}_{N+1}(g)}{\beta^{B}_{N}(g)}, \tag{19}$$

where the function $\beta^{B}_{N}(g)$ is obtained by using eqs. (16), (17) and summation in eq.(16) is made up to N.

As a criterion for the choice of parameter ν , we require minimum of $|\eta_{N+1}(g)|$ for large values of g . In favour of this criterion the following example testifies.

Consider the quantum mechanical problem of determination the ground state energy of anharmonic oscillator ($\varphi^4_{(1)}$ theory). In this case it is known that for g large the energy $E(g)$ behaves like $g^{1/3}$. At the same time the search of the minimum of the quantities $|\eta_{N+1}(g)|$ leads to the following results: For the value of parameter a equal to 0 and 1 the quantity $|\eta_{N+1}(g)|$ for $g=1$ has a sharp minimum for $0{,}2<\nu<0{,}4$ that is in good agreement with 1/3. The values of the quantities $|\eta_{N+1}(g)|$ for various values of N are represented in Fig.8.

Besides, the value of the ground state energy $E(g)$ at $g=1$ for these values of ν is also very close to the known value $E(1) \simeq 0{,}803$ /12/.

We have performed the analogous analysis also for the model with Gaussian propagator /15,16/ and for the relativistic Tomas-Fermi equation/17/, where the behaviour for $g \to \infty$ is known. In both cases we have observed a sharp minimum of quantities

$|\eta_{N+1}(g)|$ for the values of ν in agreement with asymptotics for $g \to \infty$.

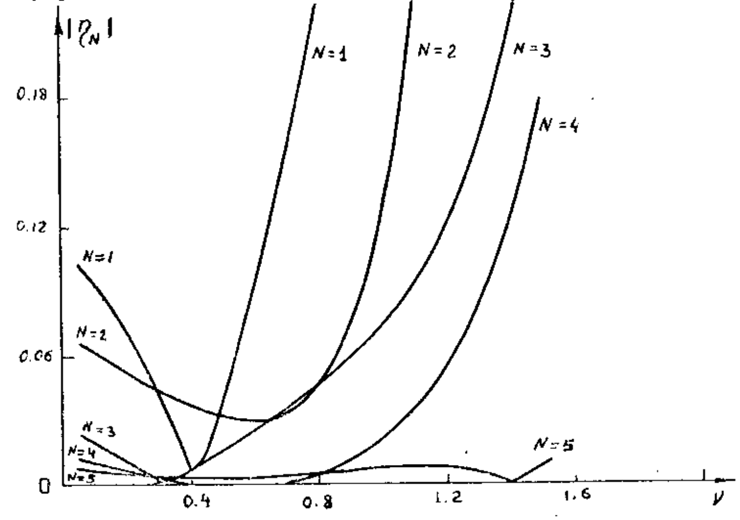


Fig.8.

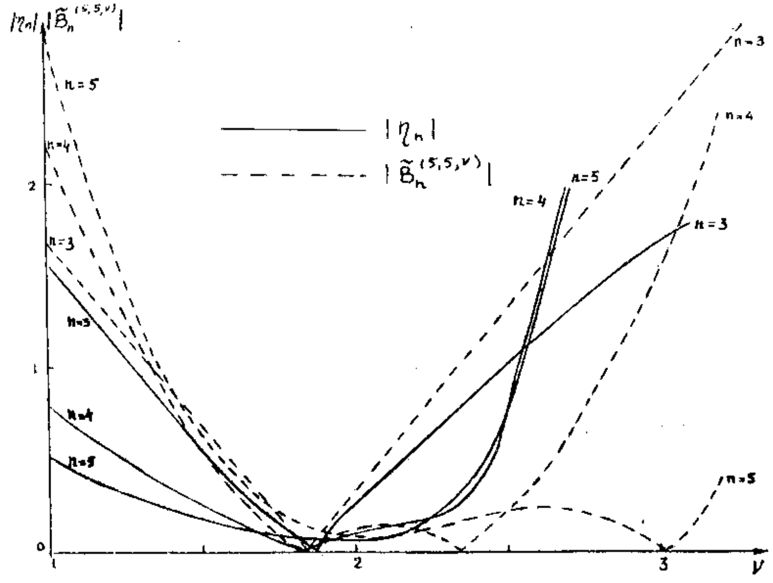


Fig.9.

The same situation takes place in the model (1),(2).

In Fig. 9 there are plotted the values of the quantities $|\eta_3|$, $|\eta_4|$ and $|\eta_5|$ for ν in the interval $(0 \div 4)$ at $g = 2$. The dotted lines denote the values of the coefficients (16) normalized to the first one, i.e., $\tilde{B}_n^{(5,5,\nu)} = B_n^{(5,5,\nu)}/B_2^{(5,5,\nu)}$.

As can be seen from the picture, the quantities $|\eta_{N+1}(g)|$ as well as $|\tilde{B}_n^{(5,5,\nu)}|$ have a sharp minimum at $1,7 < \nu < 2,2$. We have made analogous calculations for other values of parameter a $(a = 0, 1, 2, 3, 4)$ and have obtained the same results. The change of value of g does not effect the mentioned property, as well.

Choosing the value $\nu = 2$ we come to the following final representation of the partial sum for the β-function

$$\beta^B(g) = \int_0^\infty \frac{dx}{g} e^{-x/g} \left(x\frac{\partial}{\partial x}\right)^5 B(x),$$

$$B(x) = \frac{3}{128} x^2 \left(1 - 0,332w - 0,127w^2 + 0,084w^3 + \dots\right). \qquad (20)$$

The values of $\beta(g)$ with due regard for the one-, two-, three- and four-loop approximations are plotted in Fig.10. All of them lie in a narrow beam below the initial parabola and for the accepted 10% accuracy allow us to come further into the region $g \sim 50$.

The obtained β-function has asymptotics of the coefficients in the series over g coinciding with that of /4/ and we can get the right numerical coefficient, saturating the sum rule (14) by the first missed term of PT series. For $g \to \infty$ the β-function behaves like g^2.

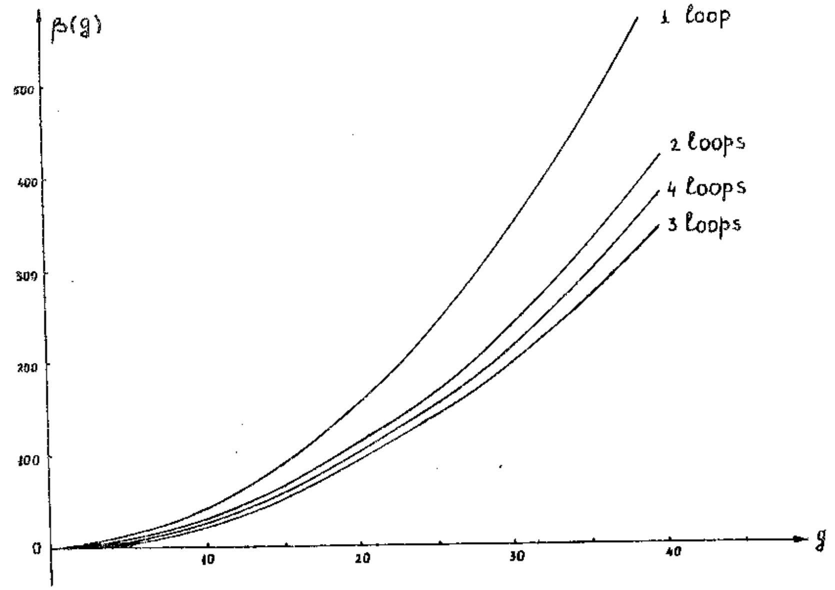


Fig.10.

We would like to note once more, that the conformal mapping is not unique and our choice (11) is guided mainly by the correspondence with asymptotic estimates (5). Eq.(5) prompts another form of conformal mapping, namely $w = \frac{x}{1+x}$ *). In this case the higher term of PT over w destroys the initial asymptotics (5). However, if the Borel transform parameters are $b = 7/2$, $a = 0$, $\nu = 2$, then the coefficients

*) The authors express their gratitude to F.M.Dittes for supplying with the result of investigation of this oppotunity.

of series rapidly decrease and the Gell-Mann-Low function up to 1% accuracy coincides with the obtained above (see Fig.10).

6. Conclusion

The results of the investigation performed can be summarized as follows:

(i) Calculations of the next orders of perturbation theory themselves give no further advance to the larger values of the coupling constant. This statement grounded in detail below for the φ^4 model, is valid also in other, more realistic models-spinor electrodynamics and Yang-Mills theory.

(ii) The knowledge of expansion coefficient asymptotics allows one to establish the character of singularity at the origin of coupling constant /10/. However, it does not allow one to obtain quantitative conclusions about the behaviour of $\beta(g)$, for $g \gtrsim 1$, i.e., in the region of physical interest /18/.

A combination of asymptotics coefficients $\tilde{\beta}_n$ with exact values β_n of low orders is a tempting possibility studied by a number of authors /17,18,19/. The performed investigation shows that such a synthesis allows one to move forward more than by two orders to the area of larger coupling constants (from the value $g \simeq 0.1$ to $g \simeq 50$ at 10% accuracy).

(iii) It should be emphasized that there is a possibility to "guess" the type of asymptotic for $g \to \infty$ by appropriately choosing the parameter ν under the Borel integral.

The obtained asymptotics

$$\beta(g) \sim 0,9\, g^2 \quad \text{for} \quad g \gg 1 \; (g \simeq 10-50) \tag{21}$$

in the model $\varphi^4_{(4)}$ leads to the zero-charge situation, i.e., to the intrinsic inconsistency in the theory. Note that this conclusion quantitatively agrees with the result of one-loop calculation $\beta_1(g) = 1,5\, g^2$, but in contrast to the latter it takes into account the effects of higher loops and eq. (21) is valid for $g > 1$. Hence we have got a signal of inherent inconsistency of the model (1).

Of course, asymptotics obtained using the information from the origin $g = 0$ may not correspond to the real one if there is phase transition for the large coupling constant in the theory or if there is unexpanded in the PT contribution growing faster than g^2 for $g \to \infty$.

(iv) The unique answer to the question about the behaviour of the function $\beta(g)$ needs certain knowledge of its analytical properties and constructive going beyond the weak coupling constant.

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