

ОБЪЕДИНЕННЫЙ  
ИНСТИТУТ  
ЯДЕРНЫХ  
ИССЛЕДОВАНИЙ  
ДУБНА



31/vii - 78

E2 - 11462

E-27

3152/2-78

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VACUUM ENERGY

IN  $g\varphi_d^4$  -THEORY FOR  $g \rightarrow \infty$

**1978**

E2 - 11462

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*Submitted to "Communications  
in Mathematical Physics"*

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E2 - 11462

Энергия вакуума в  $g\phi_d^4$ -теории в пределе  $g \rightarrow \infty$

В нелокальной  $g\phi_d^4$  ( $d \geq 1$ ) и локальной  $g\phi_2^4$ -теории получена S-матрица в форме функционального интеграла. Доказано, что этот интеграл существует.

Исследуется плотность энергии вакуума

$$E(g) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln \langle 0 | S_V(g) | 0 \rangle.$$

Доказано, что  $E(g)$  является аналитической функцией в комплексной  $g$ -плоскости, кроме отрицательной вещественной оси и точки  $g=0$ . Найдено асимптотическое поведение функции  $E(g)$  при  $g \rightarrow \infty$ . Показано, что в  $\phi_2^4$ -теории существует фазовый переход.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

Препринт Объединенного института ядерных исследований, Дубна 1978

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E2 - 11462

Vacuum Energy in  $g\phi_d^4$ -Theory for  $g \rightarrow \infty$

In the nonlocal  $g\phi_d^4$  ( $d \geq 1$ ) and local  $g\phi_2^4$  theory the S-matrix is obtained in the form of a functional integral which is proved to exist. The density of vacuum energy

$$E(g) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln \langle 0 | S_V(g) | 0 \rangle$$

is investigated. It is proved to be analytic through the whole complex  $g$ -plane except for the negative real axis and point  $g=0$ . Its asymptotic behaviour for  $g \rightarrow \infty$  is found. The  $g\phi_2^4$ -theory is shown to possess a phase transition.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

Preprint of the Joint Institute for Nuclear Research.

Dubna 1978

## I. STATEMENT OF THE PROBLEM

The  $\phi^4$ -theory is rather popular; and quite a number of papers are devoted to investigations of various aspects of this theoretical model. Without exaggeration, one can say that it is just this model that tests majority of theoretical methods and approaches. It is difficult to report all contributions of investigations of different aspects of the  $\phi^4$ -theory.

In the given paper, the  $g\phi^4$ -theory is applied to study the density of vacuum energy

$$E(g) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln \langle 0 | S_V(g, \phi) | 0 \rangle. \quad (1.1)$$

The function  $E(g)$  in (1.1) is finite in the local  $\phi_d^4$ -theory for space-time dimensions  $d=1$  (anharmonic oscillator) and  $d=2$  (the so-called  $\phi_2^4$ -theory) only. For  $d > 2$  the function  $E(g)$  does not exist at all in the local theory because of ultraviolet divergences. However,  $E(g)$  is finite in the nonlocal theory when the causal Green function  $\bar{D}(k^2)$  of the scalar field  $\phi$  decreases rather fast in the Euclidean direction ( $k^2 = k_0^2 - k^2 \rightarrow -\infty$ ).

In the given paper we will consider all these cases and obtain the analytical properties in the complex  $g$ -plane and the asymptotical behaviour of the function  $E(g)$  for  $g \rightarrow \infty$ .

We will examine the theory of one component scalar field  $\phi(x)$  described by the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2} \phi(x) (\square - m^2) \phi(x) - g [K(\ell^2 \square) \phi(x)]^4. \quad (1.2)$$

The operator  $K(\ell^2 \square)$  is nonlocal and satisfies the conditions:

i)  $K(z)$  is an entire analytical function of an order of  $\rho \geq \frac{1}{2}$  in the complex  $z$ -plane,

ii)  $K(z)$  decreases rapidly enough when  $z = \ell^2 k^2 = \ell^2(k_0^2 - \vec{k}^2) \rightarrow -\infty$ ,

iii)  $K(\ell^2 m^2) = 1$ ,

iv)  $\ell$  is a parameter characterizing the region of nonlocal interaction.

The local theory is the limit  $\ell \rightarrow 0$ . The finite unitary  $S$ -matrix in perturbation theory for the Lagrangian (1.2) is constructed in /1/.

In paper /2/ the representation of the  $S$ -matrix as a functional integral was given for the case of nonpolynomial interactions. In this paper we use the representation obtained in /2/ for investigation of the theory described by the Lagrangian (1.2).

The  $S$ -matrix as a functional of the scalar field  $\phi(x)$  in the Euclidean space of dimension  $d$  is defined in the form of the functional integral

$$S_V(g, \phi) = \int d\sigma_u \exp \left\{ -g \int_V d^d x [\Phi(u, x) + \phi(x)]^4 \right\}. \quad (1.3)$$

Here the following notations are introduced.

The system is supposed to be in a finite volume  $V \subset \mathbf{R}_d$ . In the volume  $V$  there is an orthonormal system of functions  $\{g_s(x)\}$  ( $s = 1, 2, 3, \dots$ ) such that

$$\int_V d^d x g_s(x) g_{s'}(x) = \delta_{ss'},$$

$$\sum_{s=1}^{\infty} g_s(x) g_s(x') = \delta^{(d)}(x - x'). \quad (1.4)$$

The volume  $V$  and the system of functions  $\{g_s\}$  can be chosen in the following way

$$V = \{x: -L \leq x_j \leq L, j = 1, 2, \dots, d\},$$

$$\{g_s(x)\} = \{g_{s_1, \dots, s_d}(x_1, \dots, x_d)\} = \left\{ \prod_{j=1}^d f_{s_j}(x_j) \right\}, \quad (1.5)$$

$$f_s(x) = \begin{cases} \frac{1}{\sqrt{2L}} \cos \frac{\pi}{L} \left( \frac{s-1}{2} \right) x, & s \text{ is odd} \\ \frac{1}{\sqrt{2L}} \sin \frac{\pi}{L} \frac{s}{2} x, & s \text{ is even.} \end{cases}$$

The measure  $d\sigma_u$  is defined as

$$d\sigma_u = \prod_{s=1}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} u_s^2 \right\}$$

so that

$$\int d\sigma_u = \prod_{s=1}^{\infty} \int_{-\infty}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp \left\{ -\frac{1}{2} u_s^2 \right\} = 1.$$

Sometimes we will use the notation

$$\delta u = \prod_{s=1}^{\infty} \frac{du_s}{\sqrt{2\pi}}.$$

The function  $\Phi(u, x)$  is defined as

$$\Phi(u, x) = \sum_{s=1}^{\infty} \mathcal{D}_s(x) u_s, \quad (1.6)$$

where

$$\mathcal{D}_s(x) = \int_V dy \mathcal{D}_V(x, y) g_s(y),$$

$$\mathcal{D}_V(x, y) = h_V(x) \mathcal{D}(x-y) h_V(y),$$

$$\mathcal{D}(x) = \int \frac{d^d k}{(2\pi)^d} \frac{K(-\ell^2 k^2)}{\sqrt{m^2 + k^2}} e^{ikx} \quad (1.7)$$

A function  $h_V(x) \in D(V)$ ; this function is infinitely differentiable and positive, has the support  $V$  and

$$\lim_{V \rightarrow \infty} h_V(x) = 1.$$

The function  $\mathcal{D}(x)$  is connected with the causal function  $D(x)$  in the following way

$$D_V(x, x') = \int_V dy \mathcal{D}_V(x, y) \mathcal{D}_V(y, x'),$$

$$\lim_{V \rightarrow \infty} D_V(x, x') = D(x - x') = \int dy \mathcal{D}(x - y) \mathcal{D}(y - x') = \quad (1.8)$$

$$= \int \frac{d^d k}{(2\pi)^d} \frac{[K(-\ell^2 k^2)]^2}{m^2 + k^2} e^{ik(x-x')}.$$

The system  $\{g_s\}$  is chosen in such a way that

$$\sum_{s=1}^{\infty} \mathcal{D}_s^2(x) = D_V(x, x) \xrightarrow{V \rightarrow \infty} D(0) = \int \frac{d^d k}{(2\pi)^d} \frac{[K(-\ell^2 k^2)]^2}{m^2 + k^2} < \infty;$$

$$\sum_{s=1}^{\infty} s^M \mathcal{D}_s^2(x) < \infty, \quad \forall M > 0, \quad V < \infty. \quad (1.9)$$

The functional integral in (1.3) is defined as

$$S_V(g, \phi) = \lim_{N \rightarrow \infty} S_V^{(N)}(g, \phi),$$

$$S_V^{(N)}(g, \phi) = \int d\sigma_u^{(N)} \exp\left\{-g \int_V d^d x [\Phi^{(N)}(u, x) + \phi(x)]^4\right\}, \quad (1.10)$$

where

$$d\sigma_u^{(N)} = \prod_{s=1}^N \frac{du_s}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} u_s^2\right\}, \quad (1.11)$$

$$\Phi^{(N)}(u, x) = \sum_{s=1}^N \mathcal{D}_s(x) u_s. \quad (1.12)$$

The choice of the interaction in (1.3) or (1.10) corresponds to the so-called usual but not normal product of operators. We can do it in the nonlocal theory and for the anharmonic oscillator. In the case of the local  $\phi_2^4$ -theory we have to take the normal product of operators (see Sec.6).

In this paper we will study the density of vacuum energy defined as (1.1), where

$$\langle 0 | S_V(g, \phi) | 0 \rangle = S_V(g, 0) = S_V(g)$$

is given by (1.10). Our investigation proceeds as follows. We prove that:

i) the functional integral (1.3) given by limit (1.10) does exist for any  $V < \infty$  and defines an analytical function in the complex  $g$ -plane singular at point  $g = 0$ ;

ii) the function  $E(g)$  in (1.1) is analytical throughout the complex  $g$ -plane except the negative real axis including the point  $g = 0$ ;

iii) the upper and lower bounds are found for the function  $E(g)$  when  $g \rightarrow \infty$  in the nonlocal theory as a function of dimension  $d$  and an ultraviolet behaviour of the causal Green function  $\bar{D}(k^2)$  in the Euclidean region;

iv) the representation of  $E(g)$  is obtained as a dispersion integral in the complex  $g$ -plane;

v) the upper and lower estimations in the local  $\phi_2^4$ -theory for the function

$$E(g, \Phi) = -\lim_{V \rightarrow \infty} \frac{1}{V} \ln S_V(g, \Phi)$$

are found, where  $\Phi = \text{const}$ , and it is shown that the minimum of  $E(g, \Phi)$  for  $g \rightarrow \infty$  is at the points

$$\Phi_{\pm} = \pm \sqrt{\frac{\eta}{2\pi}} \ln g \quad (\eta \approx 1),$$

i.e., we established the phase transition does exist in this model.

## 2. EXISTENCE OF THE FUNCTIONAL INTEGRAL

Here we show that the functional integral (1.3) does exist, i.e., the limit in (1.10) exists for all complex  $g \neq 0$ .

First let us consider the right half plane  $\text{Re} g > 0$ . In this region we have

$$|S_V^{(N)}(g, \phi)| \leq 1 \quad (\forall N > 1). \quad (2.1)$$

Further consider the difference

$$\begin{aligned} \Delta_N^{N+M} &= S_V^{(N+M)}(g, \phi) - S_V^{(N)}(g, \phi) = \\ &= \int d\sigma_u^{(N+M)} \left[ \exp\left\{-g \int_V dx [\phi(x) + \Phi^{(N+M)}(u, x)]^4\right\} - \right. \\ &\quad \left. - \exp\left\{-g \int_V dx [\phi(x) + \Phi^{(N)}(u, x)]^4\right\} \right]. \end{aligned} \quad (2.2)$$

Transform this difference in the following way

$$\begin{aligned} \Delta_N^{N+M} &= \int_0^1 d\xi (1-\xi) \int d\sigma_u^{(N+M)} \frac{\partial^2}{\partial \xi^2} \exp\left\{-g \int_V dx [\phi(x) + \right. \\ &\quad \left. + \Phi^{(N)}(u, x) + \xi \Psi^{(N, M)}(u, x)]^4\right\} = \end{aligned}$$

$$= \int_0^1 d\xi (1-\xi) \int d\sigma_u^{(N+M)} \exp\{-g C(u, \xi)\} \times$$

$$\times \left\{ -12 g \int_V dx [\Psi^{(N, M)}(u, x)]^2 [\phi(x) + \Phi^{(N)}(u, x) + \xi \Psi^{(N, M)}(u, x)]^2 + \right.$$

$$\left. + 16 g^2 \left[ \int_V dx \Psi^{(N, M)}(u, x) [\phi(x) + \Phi^{(N)}(u, x) + \xi \Psi^{(N, M)}(u, x)]^3 \right]^2 \right\},$$

where

$$\Psi^{(N, M)}(u, x) = \Phi^{(N, M)}(u, x) - \Phi^{(N)}(u, x),$$

$$C(u, \xi) = \int_V dx [\phi(x) + \Phi^{(N)}(u, x) + \xi \Psi^{(N, M)}(u, x)]^4.$$

Making use of the Hölder inequalities for integrals

$$\left| \int d\sigma A_1 A_2 \right| \leq \left[ \int d\sigma |A_1|^{p_1} \right]^{1/p_1} \left[ \int d\sigma |A_2|^{p_2} \right]^{1/p_2},$$

where  $p_1 + p_2 = 1$  one can obtain

$$|\Delta_N^{N+M}| \leq \int_0^1 d\xi (1-\xi) \int d\sigma_u^{(N, M)} \exp\{-\text{Re} g C(u, \xi)\} \times$$

$$\times \{12 |g| C^{\frac{1}{2}}(u, \xi) + 16 |g|^2 C^{\frac{3}{2}}(u, \xi)\} \left\{ \int_V dx (\Psi^{(N, M)}(u, x))^4 \right\}^{\frac{1}{2}}.$$

Further using the obvious inequalities

$$C^\gamma e^{-\text{Reg} \cdot C} \leq [\text{Reg}]^{-\gamma} \gamma^\gamma e^{-\gamma}.$$

$$\int d\sigma (\Psi)^a \leq \left[ \int d\sigma |\Psi|^a \right] \quad (a < 1, \int d\sigma = 1),$$

one can obtain finally

$$\begin{aligned} |\Delta_N^{N+M}| &\leq \text{const} \cdot \left\{ \int d\sigma_u^{(N+M)} \int dx (\Psi^{(N, M)}(u, x))^4 \right\}^{\frac{1}{2}} = \\ &= \text{const} \left\{ \int_V dx \left( \sum_{s=N+1}^{N+M} \mathcal{D}_s^2(x) \right)^2 \right\}^{\frac{1}{2}}. \end{aligned} \quad (2.3)$$

Because the series  $\sum_s \mathcal{D}_s^2(x)$  converges well, for any  $\epsilon > 0$  there exists  $N_0$  such that for any  $M > 0$  and  $N > N_0$  the inequality

$$|\Delta_N^{N+M}| < \epsilon \quad (2.4)$$

is valid. This means that the sequence  $S_V^{(N)}(g, \phi)$  for  $V < \infty$  and  $\text{Reg} > 0$  is fundamental and bounded.

Thus the limit in (1.10) does exist. This limit defines the functional integral (1.3) for any  $V < \infty$  and  $\text{Reg} > 0$ .

Now consider the region  $\text{Reg} \leq 0$ . The integrals in (1.11) are not defined for  $\text{Reg} < 0$ , therefore we have to do the analytical continuation of the function  $S_V^{(N)}(g, \phi)$  in the region  $\text{Reg} < 0$ , that is not a complicated problem. In order to do the continuation into the region

$$g \rightarrow ge^{i\Theta} \quad (-\pi \leq \Theta \leq \pi)$$

the contours of integrations over  $u_s$  should be transformed as follows

$$u_s \rightarrow u_s e^{-i\Theta/4}.$$

Then we get

$$\begin{aligned} S_V^{(N)}(ge^{i\Theta}, \phi) &= \prod_{s=1}^N \left( e^{-i\frac{\Theta}{4}} \int_{-\infty}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} e^{-i\frac{\Theta}{2}} u_s^2\right\} \right) \times \\ &\times \exp\left\{-g \int_V dx \left[ e^{i\frac{\Theta}{4}} \phi(x) + \Phi^{(N)}(u, x) \right]^4\right\}. \end{aligned} \quad (2.5)$$

Now it should be proved that the limit  $N \rightarrow \infty$  of this function does exist for  $-\pi \leq \Theta \leq \pi$ . To this end let us go over to the integration in (2.5) over each variable  $u_s$  along the contours  $L_s$  as is shown in Fig. 1. Then we have

$$|S_V^{(N)}(ge^{i\Theta}, \phi)| \leq \prod_{s=1}^N \int_{L_s} \left| \frac{du_s}{\sqrt{2\pi}} \right| \left| \exp\left\{-\frac{1}{2} e^{i\frac{\Theta}{2}} u_s^2\right\} \right| \times \quad (2.6)$$

$$\times \left| \exp\left\{-g \int_V dx \left[ e^{i\frac{\Theta}{4}} \phi(x) + \Phi^{(N)}(u, x) \right]^4\right\} \right|.$$

Consider the integral

$$\begin{aligned} I_s &= \int_{L_s} \left| \frac{du_s}{\sqrt{2\pi}} \right| \left| \exp\left\{-\frac{1}{2} e^{-i\frac{\Theta}{2}} u_s^2\right\} \right| \times \\ &\times \left| \exp\left\{-g \int_V dx \left[ \Phi(x) + \mathcal{D}_s(x) u_s \right]^4\right\} \right| = \\ &= I_{1s} + I_{2s} \end{aligned}$$

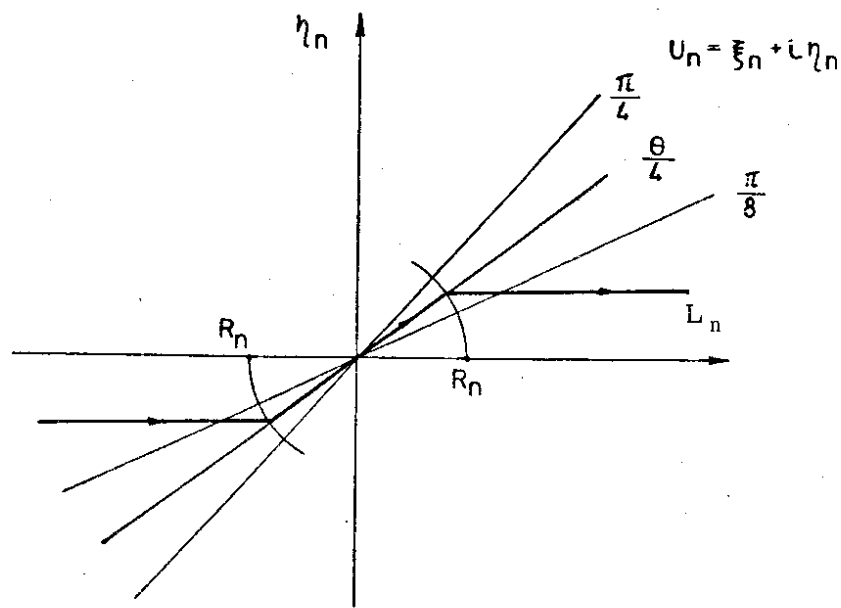


Fig. 1. Contour  $L_n$ .

$$I_{1s} = \int_{-R_s}^{R_s} \frac{d\xi_s}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi_s^2} |\exp\{-g \int dx [\Phi(x) + \mathcal{D}_s(x)\xi_{os} e^{i\frac{\theta}{4}}]\}^4| =$$

$$= |\exp\{-g \int dx [\Phi(x) + \mathcal{D}_s(x)\xi_{os} e^{i\frac{\theta}{4}}]\}^4| (1 + O(e^{-\frac{1}{2}R_s^2})),$$

where  $-R_s \leq \xi_{os} \leq R_s$ . Here the mean value theorem was used. Further

$$I_{2s} = \int_0^\infty \frac{d\eta_s}{\sqrt{2\pi}} |\exp\{-\frac{1}{2}e^{-i\frac{\theta}{2}}(R_s e^{i\frac{\theta}{4}} + \eta_s)^2\}| \times$$

$$\times \{ |\exp\{-g \int dx [\Phi(x) + \mathcal{D}_s(x)(R_s e^{i\frac{\theta}{4}} + \eta_s)]\}^4| + (\Phi \rightarrow -\Phi) \} =$$

$$= \{ |\exp\{-g \int dx [\Phi(x) + \mathcal{D}_s(x)(R_s e^{i\frac{\theta}{4}} + \eta_{1s})]\}^4| +$$

$$+ |\exp\{-g \int dx [\Phi(x) - \mathcal{D}_s(x)(R_s e^{i\frac{\theta}{4}} + \eta_{2s})]\}^4| \} O(e^{-\frac{1}{2}R_s^2}),$$

where  $\eta_{1s}, \eta_{2s} > 0$ . Finally we have

$$I_s = |\exp\{-g \int dx [\Phi(x) + \mathcal{D}_s(x)\xi_{os} e^{i\frac{\theta}{4}}]\}^4| \cdot [1 + O(e^{-\frac{1}{2}R_s^2})].$$

Making use of this estimation for each integral in (2.6) we obtain

$$|S_V^{(N)}(ge^{i\theta}, \phi)| \leq$$

$$\leq \prod_{s=1}^N |\exp\{-g \int dx [\phi(x) e^{i\frac{\theta}{4}} + \sum_{s_1=1}^N \mathcal{D}_{s_1}(x)\xi_{os_1} e^{i\frac{\theta}{4}}]\}^4| \times$$

$$\times [1 + O(e^{-\frac{1}{2}R_s^2})].$$

Choosing  $R_s = s^b$  ( $b > \frac{1}{2}$ ) we have

$$\prod_{s=1}^\infty (1 + O(e^{-\frac{1}{2}s^b})) = C < \infty$$

and

$$|\sum_{s=1}^\infty \mathcal{D}_s(x)\xi_{os}| \leq \sum_{s=1}^\infty |\mathcal{D}_s(x)| s^b < \infty.$$

Therefore

$$|S_V^{(N)}(ge^{i\theta}, \phi)| \leq C \exp\{-\text{Re} g \int dx [\phi(x) + \sum_{s=1}^N \mathcal{D}_s(x)\xi_{os}]\}^4$$

or

$$|S_V^{(N)}(ge^{i\theta}, \phi)| \leq C e^{-\text{Re} g \cdot \text{const}} \quad (2.7)$$

The analogous estimations can be applied to the difference  $\Delta_N^{N+M}$  in (2.2), and an estimation of the type (2.3) can be obtained.



Thus the functional integral (1.3) defined as the limit (1.10) does exist for any  $V < \infty$  and defines an analytical function in the complex  $g$ -plane.

### 3. ANALYTICAL PROPERTIES OF $E(g)$ IN THE COMPLEX $g$ -PLANE

The following theorem will be a basis of further analytical investigations <sup>/3/</sup>.

#### Theorem

Let functions  $\{f_n(z)\}_{(n=1,2,\dots)}$  be regular in a region  $G \subset \mathbb{C}$  and

$$\lim_{n \rightarrow \infty} f_n(z) = f(z) \neq 0$$

uniformly in each compact  $K \subset G$ . Let  $M$  is a set of zeroes of the functions  $f_n(z)$  in  $G$ . Then the zeroes of  $f(z)$  in  $G$  coincide with the limiting points of set  $M$  lying in  $G$ .

Let us consider the function

$$B_V(g) = \ln S_V(g). \quad (3.1)$$

The singularities of  $B_V(g)$  in the complex  $g$ -plane are defined, first, by the singularities of  $S_V(g)$  and, second, by zeroes of  $S_V(g)$  in the  $g$ -plane.

We will study the position of singularities and zeroes of the function

$$S_V^{(N)}(g) = \prod_{s=1}^N \int_{-\infty}^{\infty} \frac{du_s}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \sum_{n=1}^N u_n^2 - g \int dx \left(\sum_{n=1}^N \mathcal{D}_n(x) u_n\right)^4\right\} \quad (3.2)$$

and following the above theorem determine the position of singularities of the function

$$\ln S_V(g) = \lim_{N \rightarrow \infty} \ln S_V^{(N)}(g). \quad (3.3)$$

So let us consider the function  $S_V^{(N)}(g)$  in (3.2). Introducing the new variables

$$u_s = g^{-\frac{1}{4}} v_s$$

we obtain

$$S_V^{(N)}(g) = g^{-\frac{N}{4}} \left( \prod_{s=1}^N \int_{-\infty}^{\infty} \frac{dv_s}{\sqrt{2\pi}} \right) \times \\ \times \exp\left\{-\frac{1}{2} g^{-\frac{1}{2}} \sum_{n=1}^N v_n^2 - \int dx \left(\sum_{n=1}^N \mathcal{D}_n(x) v_n\right)^4\right\}.$$

One can see from this representation that  $S_V^{(N)}(g)$  has an essential singularity at the point  $g=0$  and no other singularities in the complex  $g$ -plane.

Now let us show that the function  $S_V^{(N)}(g)$  has zeroes on the negative real axis in the  $g$ -plane only. We calculate the increment of the argument of  $S_V^{(N)}(g)$  along the contour  $C$  in Fig. 2. For this aim it is convenient to use a different representation of the function  $S_V^{(N)}(g)$ . Introducing in the integral (3.2) the new variables

$$u_s = \frac{1}{\sqrt{g}} v_s$$

and making simple transformations we obtain

$$S_V^{(N)}(g) = g^{-\frac{N}{2}} \int \frac{d^N v}{(\sqrt{2\pi})^N} \exp\left\{-\frac{1}{g} \left[\frac{1}{2} \sum_{n=1}^N v_n^2 + \right. \right. \\ \left. \left. + \int dx \left(\sum_{n=1}^N \mathcal{D}_n(x) v_n\right)^4\right]\right\} = \\ = g^{-\frac{N}{2}} \int_0^{\infty} dt e^{-\frac{1}{g} t} F_V^{(N)}(t), \quad (3.4)$$

where

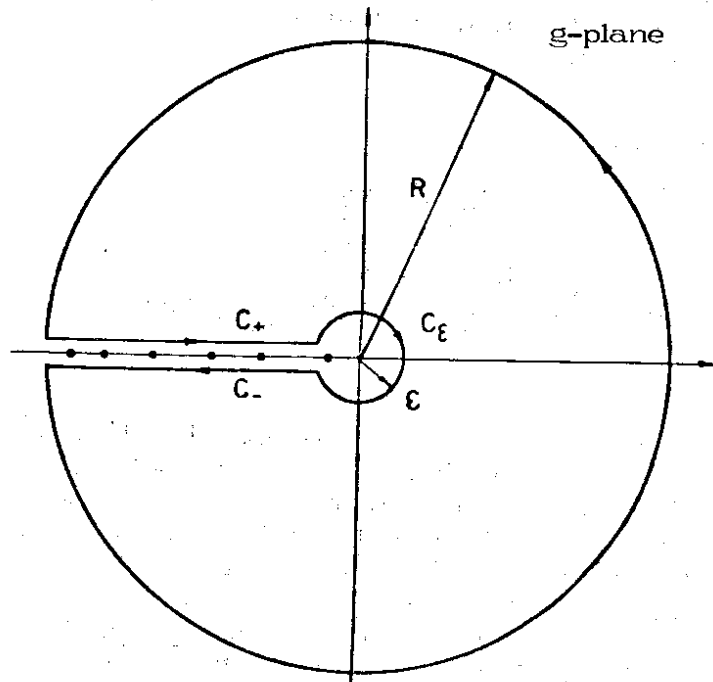


Fig. 2. Contour C.

$$F_V^{(N)}(t) = \int \frac{dN_V}{(\sqrt{2\pi})^N} \delta(t - \frac{1}{2} \sum_{n=1}^N v_n^2 - \int_V dx (\sum_{n=1}^N \mathcal{D}_n(x) v_n)^4) = \quad (3.5)$$

$$= \int_0^\infty dR R^{N-1} \int \frac{d\Omega_N}{(\sqrt{2\pi})^N} \delta(t - \frac{1}{2} R^2 - R^4 A(\Omega)).$$

Here we go over to the spherical variables in the space  $R_N$ . The function  $A(\Omega)$  in (3.5) is

$$A(\Omega) = \int_V dx (\sum_{n=1}^N \mathcal{D}_n(x) \eta_n)^4, \quad (3.6)$$

$$\sum_{n=1}^N \eta_n^2 = 1,$$

where  $\eta_n$  are the spherical variables. It is important that

$$0 < A_- \leq A(\Omega) \leq A_+ < \infty.$$

Performing the integration over  $R$  in (3.5) one can obtain after simple transformations

$$F_V^{(N)}(t) = \int_{b_-}^{b_+} \frac{db (\sqrt{b+t} - \sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+t}} Q_V^{(N)}(b), \quad (3.7)$$

where

$$Q_V^{(N)}(b) = 2^{N-2} (\sqrt{b})^{\frac{N}{2}+1} \int \frac{d\Omega_N}{(\sqrt{2\pi})^N} \delta(\frac{1}{b} - 16A(\Omega)),$$

$$b_{\pm} = (A_{\mp})^{-1}.$$

It is essential that the dependence on  $t$  in (3.7) is picked out in the explicit form, the integral over  $b$  is taken over the finite region

$$0 < b_- \leq b \leq b_+ < \infty$$

and the function  $Q_V^{(N)}(b)$  is integrable.

Thus, the function  $S_V^{(N)}(g)$  can be represented in the form

$$S_V^{(N)}(g) = g^{-\frac{N}{2}} \int_0^\infty dt e^{-\frac{1}{g} t} \int_{b_-}^{b_+} db \frac{(\sqrt{b+t} - \sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+t}} Q_V^{(N)}(b). \quad (3.8)$$

This representation is convenient for our aim.

From the representation (3.8) it is easy to find the asymptotical behaviour of the function  $S_V^{(N)}(g)$  for  $|g| \rightarrow 0$  and  $|g| \rightarrow \infty$ . Introducing the variables

$$g = r e^{i\phi} \quad \text{and} \quad t = \rho e^{i\phi}$$

one can get

$$S_V^{(N)}(re^{i\phi}) = r^{-\frac{N}{2}} e^{-i\phi(\frac{N}{2}-1)} \int_0^\infty d\rho e^{-\frac{\rho}{r}} \times$$

$$\times \int_{b_-}^{b_+} \frac{db(\sqrt{b+\rho e^{i\phi}} - \sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b+\rho e^{i\phi}}} Q_V^{(N)}(b) \quad (3.9)$$

and then

$$S_V^{(N)}(g) = \begin{cases} g^{-\frac{N}{4}} \Gamma(\frac{N}{4}) \int_{b_-}^{b_+} db Q_V^{(N)}(b) [1 + O(\frac{1}{g})], & |g| \rightarrow \infty, \\ 2\Gamma(\frac{N}{2}) \int_{b_-}^{b_+} \frac{db}{(2\sqrt{b})^{N/2}} Q_V^{(N)}(b) [1 + O(g)], & |g| \rightarrow 0. \end{cases} \quad (3.10)$$

Making use of this asymptotical behaviour (3.10) it is easy to calculate the increment of the argument of  $S_V^{(N)}(g)$  along the contours  $C_R$  and  $C_\epsilon$  in Fig. 2:

$$\Delta \arg S_V^{(N)}(g) |_{C_R} = -2\pi \frac{N}{4}, \quad \Delta \arg S_V^{(N)}(g) |_{C_\epsilon} = 0.$$

Whence

$$\Delta \arg S_V^{(N)}(g) |_{C_R + C_\epsilon} = -2\pi \frac{N}{4}.$$

Now let us consider the increment of the argument of  $S_V^{(N)}(g)$  along the contours  $C_+$  and  $C_-$  in Fig. 2. For this aim the function  $S_V^{(N)}(g)$  for  $g = re^{\pm i\pi}$  can be written in the form

$$S_V^{(N)}(re^{\pm i\pi}) = -r^{-\frac{N}{2}} e^{\mp i\pi \frac{N}{2}} \int_{b_-}^{b_+} db Q_V^{(N)}(b) \times$$

$$\times \left\{ \int_0^b d\rho e^{-\frac{\rho}{r}} \frac{(\sqrt{b-\rho} - \sqrt{b})^{\frac{N-2}{2}}}{\sqrt{b-\rho}} + \int_b^\infty d\rho e^{-\frac{\rho}{r}} \frac{(\pm i\sqrt{\rho-b} - \sqrt{b})^{\frac{N-2}{2}}}{\pm i\sqrt{\rho-b}} \right\} \quad (3.12)$$

Let us consider the even  $N=2n$ . Without loss of generality we can consider the limit

$$S_V(g) = \lim_{n \rightarrow \infty} S_V^{(2n)}(g).$$

The function  $S_V^{(2n)}(re^{\pm i\pi})$  can be represented in the form

$$S_V^{(2n)}(re^{\pm i\pi}) = A_V^{(n)}(r) \pm i B_V^{(n)}(r) = R_V^{(n)}(r) e^{\pm i\Theta_V^{(n)}(r)}$$

Let us consider

$$B_V^{(n)}(r) = \text{Im} S_V^{(2n)}(re^{i\pi}) = R_V^{(n)}(r) \sin \Theta_V^{(n)}(r) =$$

$$= (-)^n \sqrt{\pi} r^{n-\frac{1}{2}} \int_{b_-}^{b_+} db Q_V^{(2n)}(b) (\sqrt{b})^{n-1} e^{-\frac{b}{r}} T_{[\frac{n-1}{2}]} \left( \frac{r}{4b} \right), \quad (3.13)$$

$$T_{[\frac{n-1}{2}]} \left( \frac{r}{4b} \right) = \sum_{q=0}^{[\frac{n-1}{2}]} (-)^q \left( \frac{r}{4b} \right)^q \frac{1}{q!} \frac{(n-1)!}{(n-1-2q)!}.$$

The function  $B_V^{(n)}(r)$  can change its sign not more than  $[\frac{n-1}{2}]$  times because the polynomial  $T_{[\frac{n-1}{2}]}$  in (3.13) is of degree  $[\frac{n-1}{2}]$ . It means that the function  $S_V^{(2n)}(re^{i\pi})$  along the contour  $C_+$  can increase its phase not more than by  $\pi([\frac{n-1}{2}] + 1)$ . Thus we have

$$\Delta \arg S_V^{(2n)}(g) |_{C_+ + C_-} \leq 2\pi([\frac{n-1}{2}] + 1). \quad (3.14)$$

Collecting all estimations (3.11) and (3.14) we obtain

$$\Delta \arg S_V^{(2n)}(g) \Big|_C \leq \pi.$$

The increment of the argument of  $S_V^{(2n)}(g)$  cannot be negative because  $S_V^{(2n)}(g)$  has no poles. Besides it is always a number multiple of  $2\pi$  because  $S_V^{(2n)}$  has zeroes only. Finally we get

$$\Delta \arg S_V^{(2n)}(g) \Big|_C = 0 \quad (V n > 0). \quad (3.15)$$

Consequently the function  $S_V^{(2n)}(g)$  has no zeroes in the complex  $g$ -plane except the negative real axis, and on this axis it has  $[\frac{n-1}{2}]$  zeroes. The number of zeroes increases when  $n \rightarrow \infty$ .

Thus, according to the Theorem, the function  $S_V(g)$  in (3.3) has zeroes on the negative real axis only in the complex  $g$ -plane for any  $V < \infty$ . It means that  $\ln S_V(g)$  has singularities at point  $g=0$  and on the negative real axis. Consequently according to the same Theorem the function  $E(g)$  in (1.13) has singularities at point  $g=0$  and on the negative real axis in the complex  $g$ -plane.

#### 4. UPPER AND LOWER BOUNDS OF $E(g)$ IN NONLOCAL THEORY

In this section we obtain the upper and lower estimations for real positive  $g$  on the density of the vacuum energy in nonlocal theory when the causal Green function  $\bar{D}(k^2)$  decreases rapidly enough in the Euclidean direction.

First, the lower bound of  $E(g)$  will be obtained. Let us consider the function

$$S_V(g) = \int d\sigma_u \exp\{-g \int_V dx \Phi^4(u,x)\}. \quad (4.1)$$

The inequality

$$\int dx \Phi^4(u,x) \geq \frac{1}{V} \left[ \int dx \Phi^2(u,x) \right]^2$$

gives

$$S_V(g) \leq \int d\sigma_u \exp\left\{-\frac{g}{V} \left[ \int_V dx \Phi^2(u,x) \right]^2\right\}. \quad (4.2)$$

This integral can be calculated in the following way. It can be represented in the form

$$S_V(g) \leq \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt e^{-t^2} F_V(g,t), \quad (4.3)$$

where

$$F_V(g,t) = \int d\sigma_u \exp\left\{-2it \sqrt{\frac{g}{u}} \int_V dx \Phi^2(u,x)\right\}.$$

The last integral is calculated in the Appendix. When  $V$  is large enough we get

$$F_V(g,t) = \exp\left\{-\frac{1}{2} V \int \frac{d^d k}{(2\pi)^d} \ln(1 + 4it \sqrt{\frac{g}{V}} \bar{D}(k^2))\right\}. \quad (4.4)$$

Substituting this function  $F_V(g,t)$  into (4.3) and introducing the new variable  $t = u\sqrt{V}$  one can obtain

$$S_V(g) = \sqrt{\frac{V}{\pi}} \int_{-\infty}^{\infty} du \exp\left\{-V[u^2 + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln(1 + 4iu\sqrt{g} \bar{D}(k^2))]\right\}. \quad (4.5)$$

When  $V \rightarrow \infty$  this integral can be calculated by the method of steepest descents. The saddle point is on the negative imaginary axis in the  $(u+iv)$ -plane. Putting  $u_0 = -iv_0$  we have the equation

$$\frac{d}{dv_0} \left[ v_0^2 - \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln(1 + 4v_0 \sqrt{g} \tilde{D}(k^2)) \right] = 0$$

or

$$v_0 = \sqrt{g} \int \frac{d^d k}{(2\pi)^d} \frac{\tilde{D}(k^2)}{1 + 4v_0 \sqrt{g} \tilde{D}(k^2)} \quad (4.6)$$

Thus the following estimation on  $E(g)$  is valid

$$E(g) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln S_V(g) \geq E_-(g), \quad (4.7)$$

$$E_-(g) = -v_0^2 + \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln(1 + 4v_0 \sqrt{g} \tilde{D}(k^2)),$$

where  $v_0$  is determined by (4.6). The formulas (4.7) can be written on the basis of eq. (4.6) in a different form

$$E_-(g) = v_0 \sqrt{g} \int \frac{d^d k}{(2\pi)^d} \frac{\left[ -\frac{4}{d} k^2 \tilde{D}'(k^2) - \tilde{D}(k^2) \right]}{1 + 4v_0 \sqrt{g} \tilde{D}(k^2)}. \quad (4.8)$$

Now let us obtain the behaviour of  $E_-(g)$  for  $g \rightarrow 0$  and  $g \rightarrow \infty$ . When  $g \rightarrow 0$

$$v_0 = \sqrt{g} \int \frac{d^d k}{(2\pi)^d} \tilde{D}(k^2) = \sqrt{g} D(0)$$

and

$$E_-(g) = g D^2(0). \quad (4.9)$$

When  $g \rightarrow \infty$  the behaviour of  $v_0$  as a function of  $g$  is determined by the ultraviolet behaviour of the causal function  $\tilde{D}(k^2)$ . Further let us consider particular cases. First, the case when for  $k \rightarrow \infty$

$$\tilde{D}(k^2) = O(k^2)^{-1-a}, \quad (4.10)$$

where

$$2(1+a) > d \quad (4.11)$$

as a consequence of the condition  $D(0) < \infty$  in (1.9). In this case from (4.6) for  $g \rightarrow \infty$

$$v_0 = \text{const.} \cdot g^{\frac{1}{2} \left( \frac{d}{4(1+a)-d} \right)}$$

follows, whence

$$E_-(g) = C_- \cdot g^{\frac{d}{4(1+a)-d}}, \quad (4.12)$$

where  $C_-$  is a constant which can be determined if necessary.

In the case of anharmonic oscillator ( $d=1$  and  $a=1$  in (4.10)) the causal Green function is

$$\tilde{D}(k^2) = \frac{1}{1+k^2}$$

(in units  $m=1$ ). In this case the function  $E_-(g)$  in (4.7) can be calculated in the explicit form:

$$E_-(g) = \max_u \left\{ -\frac{u^2}{g} + \frac{1}{2} (\sqrt{1+4u}-1) \right\}.$$

This expression can be written after simple transformations

$$E_-(g) = \frac{v(16+18v+9v^2)}{8(1+v)(3v+4)}$$

where

$$v(1+v)(2+v) = 2g.$$

Hence it appears that in the limit of large and small  $g$

$$E_-(g) = \begin{cases} \frac{g}{2}, & g \rightarrow 0, \\ \frac{3}{8}(2g)^{1/3} = 0.474g^{1/3}, & g \rightarrow \infty. \end{cases} \quad (4.13)$$

Second, the case when

$$\tilde{D}(k^2) = O(\exp(-k^{2\gamma})) \quad (4.14)$$

for  $k^2 \rightarrow \infty$ . The solution  $v_0$  of the equation (4.6) is

$$v_0 = \text{const}(\ln g)^{\frac{d}{4\gamma}}$$

in the limit  $g \rightarrow \infty$  so that

$$E_-(g) = C_-(\ln g)^{\frac{d}{2\gamma}+1} [1 + O(\frac{\ln \ln g}{\ln g})]. \quad (4.15)$$

Here  $C_-$  is a constant which can be found.

Now let us obtain the upper bound to the function  $E(g)$  for real positive  $g$ . The following obvious inequality is valid for the function  $S_V(g)$  in (4.1):

$$S_V(g) \geq \int \delta u \exp\left\{-\frac{1}{2} \sum_{s=1}^{\infty} (1+q_s) u_s^2 - g \int dx \left(\sum_{s=1}^{\infty} \frac{\mathcal{D}_s(x)}{\sqrt{1+q_s}} u_s\right)^4\right\} =$$

$$= \prod_{n=1}^{\infty} (1+q_n)^{-\frac{1}{2}} \int \delta u \exp\left\{-\frac{1}{2} \sum_{s=1}^{\infty} u_s^2 - g \int dx \left(\sum_{s=1}^{\infty} \frac{\mathcal{D}_s(x)}{\sqrt{1+q_s}} u_s\right)^4\right\}. \quad (4.16)$$

Here  $q_n$  are positive numbers. Then it is easy to get the following upper estimation for  $E(g)$ :

$$E(g) = -\lim_{V \rightarrow \infty} \frac{1}{V} \ln S_V(g) \leq \frac{1}{2} \lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \ln(1+q_s) -$$

$$-\lim_{V \rightarrow \infty} \frac{1}{V} \ln \int d\sigma_u \exp\left\{-g \int dx \left(\sum_s \frac{\mathcal{D}_s(x)}{\sqrt{1+q_s}} u_s\right)^4\right\} \leq$$

$$\leq \frac{1}{2} \lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \ln(1+q_s) + 3g \lim_{V \rightarrow \infty} \frac{1}{V} \int dx \left(\sum_s \frac{\mathcal{D}_s(x)}{1+q_s}\right)^2.$$

The numbers  $q_s$  can be chosen in such a way that

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \ln(1+q_s) = \int \frac{d^d k}{(2\pi)^d} \ln(1+q(k^2)),$$

$$\lim_{V \rightarrow \infty} \sum_s \frac{\mathcal{D}_s^2(x)}{1+q_s} = \int \frac{d^d k}{(2\pi)^d} \frac{\tilde{D}(k^2)}{1+q(k^2)}. \quad (4.17)$$

For example in the case of the orthonormal system (1.5) these numbers should be taken as

$$q_s = q_{s_1, \dots, s_d} = q\left(\frac{\pi^2}{L^2}(s_1^2 + \dots + s_d^2)\right)$$

and

$$V = (2L)^d.$$

Because the numbers  $q_s$  and consequently the function  $q(k^2)$  are arbitrary, we have

$$E(g) \leq E_+(g).$$

$$E_+(g) = \min_{q>0} \left\{ \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln(1+q(k^2)) + 3g \left[ \int \frac{d^d k}{(2\pi)^d} \frac{\tilde{D}(k^2)}{(1+q(k^2))} \right]^2 \right\}. \quad (4.18)$$

When  $g \rightarrow 0$  the minimum is for  $q(k^2) = 0$ . In this limit we obtain

$$E_+(g) = 3g D^2(0). \quad (4.19)$$

i.e., the lowest perturbation order when the interaction Lagrangian is not taken in the normal form.

The behaviour of  $E_+(g)$  for  $g \rightarrow \infty$  is defined by the ultraviolet asymptotic of the Green function  $\tilde{D}(k^2)$ . In the case (4.10) the function

$$q(k^2) = \left( \frac{A^2}{k^2} \right)^\beta, \quad (4.20)$$

where  $A$  and  $\beta > \frac{d}{2}$  are parameters, can be chosen to determine the function  $E_+(g)$  in (4.18). Substituting (4.20) into (4.18) and putting  $g \rightarrow \infty$  one can see that the minimum is realized for large  $A$ . We have

$$E_+(g) \leq \min_{\beta > \frac{d}{2}} \min_A \left[ C_1(\beta) A^d + C_2(\beta) \frac{g}{A^{4(1+a)-2d}} \right],$$

where

$$C_1(\beta) = \frac{1}{2} \int \frac{d^d k}{(2\pi)^d} \ln \left( 1 + \frac{1}{k^2 \beta} \right),$$

$$C_2(\beta) = 3 \left[ \int \frac{d^d k}{(2\pi)^d} \frac{k^{2\beta}}{k^{2(1+a)} (k^{2\beta} + 1)} \right]^2.$$

Then it follows

$$E_+(g) \leq C_+ \cdot g^{\frac{d}{4(1+a)-d}} \quad (4.21)$$

where

$$C_+ = \min_{\beta} C_1(\beta) \left( 1 + \frac{d C_2(\beta)}{2(2(1+a)-d)} \right) \left[ \frac{2(2(1+a)-d) C_2(\beta)}{d C_1(\beta)} \right]^{\frac{d}{4(1+a)-d}}$$

This constant  $C_+$  can be calculated.

Thus two estimations (4.15) and (4.21) show that in the nonlocal theory when  $\tilde{D}(k^2) = O((k^2)^{1-a})$  for  $k^2 \rightarrow \infty$  in the limit  $g \rightarrow \infty$  the vacuum energy  $E(g)$  is

$$E(g) = C g^{\frac{d}{4(1+a)-d}} \quad (4.22)$$

where  $C$  is a constant satisfying  $C_- < C < C_+$ . The formula (4.22) can be rewritten in the form

$$E(g) = C g^{a(d,a)} \quad (4.23)$$

$$a(d,a) = \frac{d}{4(1+a)-d} = \frac{1 - [1 - \frac{d}{2(1+a)}]}{1 + [1 - \frac{d}{2(1+a)}]} < 1$$

because of  $2(1+a) > d$  according to the condition  $D(0) < \infty$ .

Note once more that the behaviour  $E(g)$  for  $g \rightarrow \infty$  is defined by the dimension of space-time  $d$  and the ultraviolet asymptotic of the causal Green function  $\tilde{D}(k^2)$  for  $k^2 \rightarrow \infty$ .

The function  $E_+(g)$  in (4.18) for the anharmonic oscillator reads:

$$E_+(g) = \min_{\beta} \min_A \left\{ \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \ln \left( 1 + \left( \frac{A^2}{k^2} \right)^\beta \right) + \right.$$

$$+ 3g \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk}{(1+k^2)(1+(\frac{A^2}{k^2})^\beta)} \right]^2 \} .$$

In the limit  $g \rightarrow \infty$  it is easy to obtain

$$E_+(g) = \min_{\beta} \min_A \left[ \frac{A}{2 \sin \frac{\pi}{2\beta}} + \frac{3g}{A^2} \cdot \frac{1}{(2\beta \sin \frac{\pi}{2\beta})^2} \right] =$$

$$= g^{1/3} \left\{ \frac{3}{2\pi [\max_u \frac{\sin^2 u}{u}]} \right\}^{2/3} = 0.757 g^{1/3} . \quad (4.24)$$

Collecting the estimations (4.13) and (4.24) we obtain

$$0.474 g^{1/3} < E(g) < 0.757 g^{1/3} . \quad (4.25)$$

The exact calculations which can be done for the anharmonic oscillator making use of the Schrödinger equation give/4/

$$E(g) = 0.6680 g^{1/3} .$$

One can see that our estimations are rather accurate.

At last let us consider  $E_+(g)$  for  $g \rightarrow \infty$  in the case (4.14). This estimation can be obtained by choosing  $q(k^2) = A \exp\{-k^{2\gamma}\}$ . After simple calculations one can get in the limit  $g \rightarrow \infty$

$$E_+(g) = C (\ln g)^{\frac{d}{2\gamma} + 1} \left[ 1 + O\left(\frac{\ln \ln g}{\ln g}\right) \right] . \quad (4.26)$$

Thus in the  $\phi_d^4$ -theory with the propagator  $\bar{D}(k^2)$  (4.14) we have in the limit of strong coupling

$$E(g) = C (\ln g)^{\frac{d}{2\gamma} + 1} \left[ 1 + O\left(\frac{\ln \ln g}{\ln g}\right) \right] . \quad (4.27)$$

In paper /5/ the  $\phi^4$ -theory with the propagator  $\bar{D}(k^2) = e^{-k^2}$  (i.e.,  $\gamma = 1$  in (4.14)) was considered. The authors have calculated eight perturbation orders for different characteristics of this model (energy levels, Green functions). The knowledge of exact asymptotics (4.27) should help for different methods of asymptotic summation.

## 5. DISPERSION REPRESENTATION OF $E(g)$

After the investigations performed in the previous sections, we prove that the vacuum energy  $E(g)$  defined by (1.1) has the following properties:

- i)  $E(g)$  is analytic in the complex  $g$ -plane except the negative real axis,
- ii) for  $|g| \rightarrow \infty$

$$|E(g)| \leq \text{const } |g|, \quad (5.1)$$

- ii) for  $g \rightarrow +\infty$

$$E(g) = \text{const } g^{\alpha(d,a)}, \quad (5.2)$$

where

$$\alpha(d,a) = \frac{d}{4(1+a)-d} < 1,$$

- iv) for  $\text{Re } g > 0$

$$E(g) \sim \sum_{n=1}^{\infty} g^n E_n . \quad (5.3)$$



where  $E_n = O(n!)$ , i.e.,  $E(g)$  can be developed into an asymptotical series.

It follows from (5.1) and (5.2) that for  $|g| \rightarrow \infty$

$$|E(g)| \leq \text{const } |g|^{a(d,a)}. \quad (5.4)$$

Further it is possible to write the following dispersion representation

$$\frac{E(g)}{g} = \frac{1}{2\pi i} \int_C \frac{d\zeta E(\zeta)}{\zeta(\zeta-g)}, \quad (5.5)$$

where  $C$  is the contour shown in Fig. 2. The integration over this contour  $C$  in (5.5) gives

$$\frac{E(g)}{g} = \frac{1}{\pi} \int_{-\infty}^0 \frac{d\zeta}{\zeta(\zeta-g)} \cdot \frac{1}{2i} [E(\zeta e^{i\pi}) - E(\zeta e^{-i\pi})]$$

whence

$$E(g) = g \int_0^{\infty} \frac{d\xi \sigma\left(\frac{1}{\xi}\right)}{\xi(\xi+g)},$$

or introducing the new variable  $u = \frac{1}{\xi}$

$$E(g) = g \int_0^{\infty} \frac{du \sigma(u)}{1+gu}. \quad (5.6)$$

According to the conditions (5.3) and (5.4) the function  $\sigma(u)$  satisfies

$$\sigma(u) = \begin{cases} O(u^{-\alpha}), & u \rightarrow 0, \\ O(e^{-u}), & u \rightarrow \infty. \end{cases} \quad (5.7)$$

The representation (5.6) can be written in the other form

$$\begin{aligned} E(g) &= g \int_0^{\infty} ds \int_0^{\infty} du \sigma(u) e^{-s-gsu} = \\ &= g \int_0^{\infty} ds e^{-s} B(gs), \end{aligned} \quad (5.8)$$

where

$$B(t) = \int_0^{\infty} du \sigma(u) e^{-tu}.$$

This function  $B(t)$  according to the estimations (5.7) is analytical for  $\text{Re}t > -1$  and for  $t \rightarrow \infty$

$$B(t) = O(t^{-1+a}). \quad (5.9)$$

Thus the perturbation series in the nonlocal  $\phi^4$ -theory is summarizable by the Borel method to the exact expression.

## 6. $E(g)$ IN THE LOCAL $\phi^4$ -THEORY

A great number of papers (see /6/) are devoted to the local  $\phi^4$ -theory. The upper and lower estimations were obtained for  $E(g)$ /6,7,8/ but these estimations are quite rough. Here, by using our representation (1.3) for  $S_V(g, \Phi)$ , we will get more exact estimations for

$$E(g, \Phi) = - \lim_{V \rightarrow \infty} \frac{1}{V} \ln S_V(g, \Phi) \quad (6.1)$$

in the limit  $g \rightarrow \infty$  and for an arbitrary constant field  $\Phi = \text{const}$ .

The function  $S_V(g, \Phi)$  in the local limit can be represented as

$$S_V(g, \Phi) = \lim_{\ell \rightarrow 0} \int d\sigma_u \exp\{g W_4^{(\ell)}(u, \Phi)\}, \quad (6.2)$$

where the interaction Lagrangian is taken in the normal form

$$W_{\ell}^{(0)}(u, \Phi) = - \int_V d^2x : [\Phi_{\ell}(u, x) + \Phi]^4 : =$$

$$= - \int_V d^2x [(\Phi_{\ell}(u, x) + \Phi)^4 - 6D_{\ell}(0)(\Phi_{\ell}(u, x) + \Phi)^2 + 3D_{\ell}^2(0)]. \quad (6.3)$$

In the two-dimensional space-time the limit  $\ell \rightarrow 0$  in (6.3) does exist for any elements of the  $S$ -matrix. We will study the vacuum energy  $E(g, \Phi)$  in (6.1).

First, we obtain the upper estimation  $E(g, \Phi)$  in (6.1). Let us write down the following obvious inequality

$$S_V(g, \Phi) \geq \lim_{\ell \rightarrow 0} \int \delta u \exp \left\{ -\frac{1}{2} \sum_s u_s^2 - \frac{1}{2} \sum_s q_s (u_s - v_s)^2 - \right.$$

$$\left. -g \int dx : \left[ \sum_s \frac{\mathcal{F}_s(x) u_s}{\sqrt{1+q_s}} + \Phi \right]^4 : \right\}, \quad (6.4)$$

where  $q_s$  are positive and  $v_s$  are real numbers. After simple transformations one can get

$$S_V(g, \Phi) \geq \lim_{\ell \rightarrow 0} \prod_s (1+q_s)^{-1/2} \cdot \exp \left\{ -\frac{1}{2} \sum_s \frac{q_s}{1+q_s} v_s^2 \right\} \times$$

$$\times \int \delta u \exp \left\{ -\frac{1}{2} \sum_s u_s^2 \right\} \times$$

$$\times \exp \left\{ -g \int dx \left[ \left( \sum_s \frac{\mathcal{F}_s(x) u_s}{\sqrt{1+q_s}} + \Phi \right)^4 - \right. \right.$$

$$\left. -6D_{\ell}(0) \left( \sum_s \frac{\mathcal{F}_s(x) u_s}{\sqrt{1+q_s}} + \Phi \right)^2 + 3D_{\ell}^2(0) \right] \right\},$$

where

$$\Phi_1 = \Phi + \sum_s \frac{\mathcal{F}_s(x) q_s}{1+q_s} v_s.$$

Then the following inequality is valid for  $E(g, \Phi)$

$$E(g, \Phi) \leq \frac{1}{2} \lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \ln(1+q_s) + \frac{1}{2} \lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \frac{q_s}{1+q_s} v_s^2 +$$

$$+ \lim_{\ell \rightarrow 0} \lim_{V \rightarrow \infty} \frac{g}{V} \int d^2x \int d\sigma_u \left[ \left( \sum_s \frac{\mathcal{F}_s(x) u_s}{\sqrt{1+q_s}} + \Phi_1 \right)^4 - \right. \quad (6.5)$$

$$\left. -4D_{\ell}(0) \left( \sum_s \frac{\mathcal{F}_s(x) u_s}{\sqrt{1+q_s}} + \Phi_1 \right)^2 + 3D_{\ell}^2(0) \right].$$

The numbers  $q_s$  and  $v_s$  can be chosen in such a way that

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \ln(1+q_s) = \int \frac{d^2k}{(2\pi)^2} \ln(1+q(k^2)),$$

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \frac{q_s}{1+q_s} v_s^2 = \frac{q(0)}{1+q(0)} v^2,$$

$$\lim_{V \rightarrow \infty} \sum_s \frac{\mathcal{F}_s(x) v_s}{1+q_s} = \frac{\tilde{\mathcal{F}}(0)q(0)}{1+q(0)} v = \frac{q(0)}{1+q(0)} v,$$

$$\lim_{V \rightarrow \infty} \frac{1}{V} \sum_s \frac{\mathcal{F}_s^2(x)}{1+q_s} = \int \frac{d^2k}{(2\pi)^2} \frac{\tilde{D}_{\ell}(k^2)}{1+q(k^2)} =$$

$$= \int \frac{d^2k}{(2\pi)^2} \frac{[K(-\ell^2 k^2)]^2}{(1+k^2)(1+q(k^2))} = D_{\ell,1}.$$

Here  $q(k^2)$  and  $v$  are arbitrary,  $m=1$ .

For example, in the case of the system (1.5) it can be done in the following way

$$q_s = q_{s_1 s_2} = q \left( \frac{\pi^2}{L^2} (s_1^2 + s_2^2) \right),$$

$$v_s = v_{s_1 s_2} = \sqrt{(2L)^2} \delta_{s_1 1} \delta_{s_2 1} \cdot v.$$

Then the estimation (6.5) can be written

$$E(g, \Phi) \leq \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \ln(1+q(k^2)) + \frac{1}{2} \frac{q(0)}{1+q(0)} v^2 +$$

$$+ g \lim_{\ell \rightarrow 0} [3\Delta_\ell^2 - 6\Delta_\ell^2 \Phi_1^2 + \Phi_1^4],$$

where

$$\Phi_1 = \Phi + \frac{q(0)}{1+q(0)} v,$$

$$\Delta_\ell = D_\ell - D_{\ell,1} = \int \frac{d^2 k}{(2\pi)^2} \frac{[K(-\ell^2 k^2)]^2 q(k^2)}{(1+k^2)(1+q(k^2))}.$$

One can see that

$$\lim_{\ell \rightarrow 0} \Delta_\ell = \Delta[q] = \int \frac{d^2 k}{(2\pi)^2} \frac{q(k^2)}{(1+k^2)(1+q(k^2))}.$$

Introducing the new variable  $\Psi = \frac{q(0)}{1+q(0)} v - \Phi$  we obtain finally

$$E(g, \Phi) \leq E_+(g, \Phi),$$

$$E_+(g, \Phi) = \min_{q>0, \Psi} \left\{ \frac{1}{2} \int \frac{d^2 k}{(2\pi)^2} \ln(1+q(k^2)) + \right. \quad (6.6)$$

$$\left. + \frac{1+q(0)}{q(0)} (\Psi - \Phi)^2 + g[3\Delta^2[q] - 6\Psi^2\Delta[q] + \Psi^4] \right\}.$$

Let us choose

$$q(k^2) = \left( \frac{A}{k^2} \right)^2,$$

then

$$\int \frac{d^2 k}{(2\pi)^2} \ln \left( 1 + \left( \frac{A}{k^2} \right)^2 \right) = \frac{A}{4},$$

$$\Delta[q] = \Delta(A) = \int \frac{d^2 k}{(2\pi)^2} \cdot \frac{1}{1+k^2} \cdot \frac{A^2}{A^2 + (k^2)^2} =$$

$$= \frac{1}{4\pi} \cdot \frac{A^2}{1+A^2} \left[ \frac{\pi}{2A} + \ln A \right] = \begin{cases} \frac{A}{8}, & A \rightarrow 0; \\ \frac{1}{4\pi} \ln A, & A \rightarrow \infty. \end{cases}$$

We have for (6.6)

$$E_+(g, \Phi) = \min_{A, \Psi} \left\{ \frac{A}{8} + \frac{1}{2} (\Psi - \Phi)^2 + \right. \quad (6.7)$$

$$\left. + g[3\Delta^2(A) - 6\Psi^2\Delta(A) + \Psi^4] \right\}.$$

In the limit  $g \rightarrow \infty$  one can obtain putting  $\Psi = \text{sign } \Phi \cdot \sqrt{3\Delta(A)}$  and calculating the minimum with respect to  $A$

$$E_+(g, \Phi) = - \frac{3}{8\pi^2} g \left[ \left( \ln g + \ln \frac{3}{\pi} B \right)^2 - B \right] + \quad (6.8)$$

$$+ \frac{1}{2} \left( |\Phi| - \sqrt{\frac{3}{4\pi} \left( \ln g + \ln \frac{3}{\pi} B \right)} \right)^2,$$

where

$$B = 2 \left[ \ln g + \ln \frac{3}{\pi} B \right] = 2 \ln g \left[ 1 + O\left( \frac{\ln \ln g}{\ln g} \right) \right].$$

One can see from this estimation that the minimum vacuum energy  $E_+(g, \Phi)$  in (6.7) is at the points

$$\Phi_{\pm} = \pm \sqrt{\frac{3}{4\pi} \ln \left[ 1 + O\left(\frac{\ln \ln g}{\ln g}\right) \right]} \quad (6.9)$$

but not for  $\Phi = 0$ .

Now let us study the lower estimation  $E_-(g, \Phi)$  in (6.1). This estimation is obtained in a more complicated way.

First of all let us introduce the regularized field

$$\Phi^{(\Lambda)}(u, x) = \sum_s \mathcal{F}_s^{(\Lambda)}(x) u_s$$

for which

$$\mathcal{F}_s^{(\Lambda)}(x) = \frac{1}{(2\pi)^2} \int \frac{d^2 k}{\sqrt{1+k^2}} \frac{\Lambda^2 - 1}{\Lambda^2 + k^2} e^{ikx}$$

where  $\Lambda \geq 1$ .

Then introducing the notation

$$\langle \Phi^{(\Lambda)}(u, x) \Phi^{(\Lambda')}(\bar{u}, x') \rangle = \int d\sigma_u \Phi^{(\Lambda)}(u, x) \Phi^{(\Lambda')}(\bar{u}, x')$$

we get in the limit  $V \rightarrow \infty$  and  $\ell \rightarrow 0$

$$D(x) = \langle \Phi(u, x) \Phi(u, 0) \rangle = \int \frac{d^2 k}{(2\pi)^2} \frac{e^{ikx}}{1+k^2} = \frac{1}{2\pi} K_0(t),$$

$$\begin{aligned} D_{1\Lambda}(x) &= \langle \Phi^{(\Lambda)}(u, x) \Phi(u, x) \rangle = \int dy \mathcal{F}^{(\Lambda)}(x-y) \mathcal{F}^{(\Lambda)}(y) = \\ &= \int \frac{d^2 k}{(2\pi)^2} \frac{(\Lambda^2 - 1) e^{ikx}}{(1+k^2)(\Lambda^2 + k^2)} = \frac{1}{2\pi} [K_0(t) - K_0(\Lambda t)], \end{aligned}$$

$$\begin{aligned} D_{\Lambda}(x) &= \langle \Phi^{(\Lambda)}(u, x) \Phi^{(\Lambda)}(u, 0) \rangle = \int dy \mathcal{F}^{(\Lambda)}(x-y) \mathcal{F}^{(\Lambda)}(y) = \\ &= \int \frac{d^2 k}{(2\pi)^2} \frac{(\Lambda^2 - 1)^2 e^{ikx}}{(1+k^2)(\Lambda^2 + k^2)^2} = \end{aligned}$$

$$= \frac{1}{2\pi} [K_0(t) - K_0(\Lambda t) - (1 - \frac{1}{\Lambda^2}) \frac{\Lambda t}{2} K_1(\Lambda t)], \quad (6.10)$$

where  $K_\nu(z)$  is the Mac-Donald function,  $t = \sqrt{x^2}$ .  
Let us introduce the interaction function

$$\begin{aligned} H_4^{(\Lambda)}(u, \Phi) &= - \int_V d^2 x : [\Phi^{(\Lambda)}(u, x) + \Phi]^4 : = \\ &= - \int_V d^2 x \{ [\Phi^{(\Lambda)}(u, x) + \Phi]^4 - 6D_{\Lambda}(0) [\Phi^{(\Lambda)}(u, x) + \Phi]^2 + 3D_{\Lambda}^2(0) \}, \end{aligned}$$

where

$$D_{\Lambda}(0) = \frac{1}{4\pi} (\ln \Lambda^2 - 1 + \frac{1}{\Lambda^2}) = \frac{1}{4\pi} r(\Lambda^2). \quad (6.11)$$

Making use of the following inequality

$$-(\phi^4 - 6D\phi^2 + 3D^2) \leq -2\alpha D(\phi^2 - D^2) + C_{\alpha} D^2, \quad (6.12)$$

where  $C_{\alpha} = 6 + 4\alpha + \alpha^2$ , we have

$$- : [\Phi^{(\Lambda)}]^4 : \leq -2\alpha D_{\Lambda}(0) : [\Phi^{(\Lambda)}]^2 : + C_{\alpha} D_{\Lambda}^2(0), \quad (6.13)$$

$$: [\Phi^{(\Lambda)}]^2 : = : [\Phi^{(\Lambda)}]^2 : - D_{\Lambda}(0).$$

Now let us transform  $S_V(g, \Phi)$  in (6.2)

$$\begin{aligned} S_V(g, \Phi) &= \lim_{\ell \rightarrow 0} \int d\sigma_u \exp \{ g [ W_4^{(\ell)} - 2\alpha D_{\Lambda}(0) H_2^{(\Lambda)}(u, \Phi) ] \} \times \\ &\quad \times \exp \{ 2g\alpha D_{\Lambda}(0) H_2^{(\Lambda)}(u, \Phi) \} = \\ &= \int_{-\infty}^{\infty} dt e^{gt} \left( - \frac{\partial}{\partial t} \right) \sigma(t) = g \int_{-\infty}^{\infty} dt e^{gt} \sigma(t), \end{aligned}$$

where

$$H_2^{(\Lambda)}(u, \Phi) = - \int_V d^2 x : [\Phi^{(\Lambda)}(u, x) + \Phi]^2 ;$$

$$\sigma(t) = \lim_{\ell \rightarrow 0} \int d\sigma_u \exp \{2\alpha g D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi)\} \times \\ \times \theta(W_4^{(\ell)}(u, \Phi) - 2\alpha D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi) - t) . \quad (6.14)$$

Let us consider the representation

$$S_V(g, \Phi) = T_{1V}(g, \Phi) + T_{2V}(g, \Phi) , \quad (6.15)$$

where

$$T_{1V}(g, \Phi) = g \int_{-\infty}^0 dt e^{gt} \sigma(t) , \quad (6.16)$$

$$T_{2V}(g, \Phi) = g \int_0^{\infty} dt e^{gt} \sigma(t) .$$

For the function  $T_{1V}(g, \Phi)$  the following estimation is valid

$$T_{1V}(g, \Phi) \leq \lim_{\ell \rightarrow 0} \int d\sigma_u \exp \{2\alpha g D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi)\} .$$

This integral can be calculated according to (A.1):

$$T_{1V}(g, \Phi) \leq \exp \left\{ - \frac{V}{2} \int \frac{d^2 k}{(2\pi)^2} [\ln(1 + \gamma \tilde{D}_\Lambda(k^2)) - \gamma \tilde{D}_\Lambda(k^2)] - \right. \\ \left. - \frac{V}{2} \Phi^2 \frac{\gamma}{1 + \gamma \tilde{D}_\Lambda(0)} \right\} , \quad (6.17)$$

where  $\gamma = 4gaD_\Lambda(0)$  is an arbitrary number. In the limit  $\Lambda \rightarrow \infty$  when  $\gamma$  is fixed we have

$$T_{1V}(g, \Phi) \leq \exp \left\{ - \frac{V}{8\pi} [(1+\gamma)\ln(1+\gamma) - \gamma] - \frac{V}{2} \Phi^2 \frac{\gamma}{1+\gamma} \right\} . \quad (6.18)$$

Now consider the function  $T_{2V}(g, \Phi)$  in (6.16). We want to get an estimation for the function  $\sigma(t)$  in (6.16). For this aim let us consider the region

$$W_4^{(\ell)}(u, \Phi) - 2\alpha D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi) \geq V(b + C_\alpha D_\Lambda^2(0)) ,$$

where  $b > 0$ . It is easy to see that this region is smaller than the region

$$W_4^{(\ell)}(u, \Phi) - H_4^{(\Lambda)}(u, \Phi) > Vb$$

because of

$$W_4^{(\ell)}(u, \Phi) - H_4^{(\Lambda)}(u, \Phi) \geq Vb + \quad (6.19)$$

$$+ [VC_\alpha D_\Lambda^2(0) + 2\alpha D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi) - H_4^{(\Lambda)}(u, \Phi)] \geq Vb$$

according to (6.13).

Therefore putting

$$t = V(b + C_\alpha D_\Lambda^2(0))$$

one can obtain the following inequality for  $\sigma(t)$  in (6.15) using (6.16) in the limit  $\ell \rightarrow 0$

$$\sigma(t) \leq \min_{p > 1} \int d\sigma_u \exp \{2\alpha g D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi)\} \times \\ \times \left[ \frac{1}{Vb} (W_4^{(\ell)}(u, \Phi) - H_4^{(\Lambda)}(u, \Phi)) \right]^{2p} .$$

Further using the Hölder inequality we obtain

$$\sigma(t) \leq \min_{p>1} \left\{ \int d\sigma_u \left[ \frac{1}{Vb} (W_4(u, \Phi) - H_4^{(\Lambda)}(u, \Phi)) \right]^{2q} p \right\}^{1/q'} \times$$

$$(6.20)$$

$$\times \left\{ \int d\sigma_u \exp \{ 2qga D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi) \} \right\}^{1/q}$$

where  $\frac{1}{q} + \frac{1}{q'} = 1$ ,  $q > 1$ ,  $q' > 1$ .

The integral

$$I_V = \int d\sigma_u \exp \{ 2qga D_\Lambda(0) H_2^{(\Lambda)}(u, \Phi) \}$$

can be calculated in the explicit form. According to (A.1) we get

$$I_V = \exp \left\{ V \left[ \frac{1}{8\pi} ((1+\beta) \ln(1+\beta) - \beta) - \frac{1}{2} \frac{\beta}{1+\beta} \Phi^2 \right] \right\},$$

where

$$\beta = 4qga D_\Lambda(0) = q\gamma$$

according to (6.17).

The integral

$$J_{V,N} = \int d\sigma_u \left[ \frac{1}{Vb} (W_4(u, \Phi) - H_4^{(\Lambda)}(u, \Phi)) \right]^{2N} \quad (6.21)$$

is the  $2N$ -th perturbation order for the interaction

$$W_4(u, \Phi) - H_4^{(\Lambda)}(u, \Phi) = - \int_V d^2x \left[ :(\Phi(u, x) + \Phi)^4: - :(\Phi^{(\Lambda)}(u, x) + \Phi)^4: \right].$$

When  $V \rightarrow \infty$  the main contribution to the integral (6.21) comes from the two-points coupling

$$\langle (W_4 - H_4^{(\Lambda)})^2 \rangle =$$

$$= \int d\sigma_u \left[ W_4(u, \Phi) - H_4^{(\Lambda)}(u, \Phi) \right]^2 = VC(\Lambda, \Phi),$$

where in the limit  $V \rightarrow \infty$

$$C(\Lambda, \Phi) = 24 \int d^2x \left[ \Delta_4(x) + 4\Delta_3(x) \Phi^2 + 3\Delta_2(x) \Phi^4 \right],$$

$$\Delta_s(x) = D^s(x) - 2D_{1\Lambda}^s(x) + D_\Lambda^s(x).$$

Making use of the formulas (6.10) one can get for  $\Lambda \rightarrow \infty$

$$G(\Lambda, \Phi) = \frac{21}{4\pi^3} \cdot \frac{(\ln \Lambda^2)^2}{\Lambda^2} \cdot \left[ 1 + 4 \frac{2\pi\Phi^2}{\ln \Lambda^2} + 2 \left( \frac{2\pi\Phi^2}{\ln \Lambda^2} \right)^2 \right]. \quad (6.22)$$

Turning to the integral (6.21) we obtain for large  $V$  and  $N$

$$J_{V,N} = (2N-1)!! \left[ \frac{G(\Lambda, \Phi)}{Vb^2} \right]^N \left( 1 + O\left(\frac{1}{NV}\right) \right).$$

Since we study the behaviour of all functions in the limit  $V \rightarrow \infty$  then we have (from the Stirling formula)

$$\min_{p>1} \left\{ J_{V,N} p \right\}^{1/q'} =$$

$$= \min_{p>1} \exp \left\{ p \ln q' p - p \left[ \frac{2G(\Lambda, \Phi)}{Vb^2} \right]^p \right\} = \exp \left\{ -V \frac{b^2}{2q'G(\Lambda, \Phi)} \right\}.$$

Thus the following inequality for  $\sigma(t)$  is valid

$$\sigma(t) = \sigma \left( V(b + C \frac{D_\Lambda^2(0)}{a}) \right) \leq \sigma_+(t) =$$

$$= \exp \left\{ - \frac{V}{2q} \Phi^2 \frac{\beta}{1+\beta} + \frac{V}{8\pi q} \left[ (1+\beta) \ln(1+\beta) - \beta \right] - \frac{Vb^2}{2q'G(\Lambda, \Phi)} \right\}.$$

In the limit  $g \rightarrow \infty$  the main contribution to the vacuum energy gives  $T_{2V}(g, \Phi)$  because  $\beta = q\gamma$  is arbitrary. Then we have for  $g \rightarrow \infty$

$$\begin{aligned}
-E(g, \Phi) &= \lim_{V \rightarrow \infty} \frac{1}{V} \ln S_V(g, \Phi) = \\
&= \lim_{V \rightarrow \infty} \frac{1}{V} \ln (T_{1V}(g, \Phi) + T_{2V}(g, \Phi)) = \\
&= O(1) + \lim_{V \rightarrow \infty} \frac{1}{V} \ln T_{2V}(g, \Phi) \leq \\
&\leq \max_t \left[ \frac{g}{V} t + \frac{1}{V} \ln \sigma_t(t) \right] = \\
&= \max_{\Lambda > 1} \left[ g(b + C D_{\Lambda}^2(0)) - \frac{b^2}{2q'G(\Lambda, \Phi)} - \right. \\
&\quad \left. - \frac{1}{2q} \cdot \frac{\beta}{1+\beta} \Phi^2 + \frac{1}{8\pi q} [(1+\beta)\ln(1+\beta) - \beta] \right].
\end{aligned} \tag{6.23}$$

This expression should be minimized over the parameters  $b$  and  $\beta$ . Making use of (6.12) and (6.18) we obtain

$$g C D_{\Lambda}^2(0) = 6g D_{\Lambda}^2(0) + \frac{\beta}{q} D_{\Lambda}(0) + \frac{\beta^2}{16q^2g}.$$

Finally we can write changing the sign in (6.23)

$$E(g, \Phi) \geq E_-(g, \Phi),$$

$$\begin{aligned}
E_-(g, \Phi) &= \max_{\beta, b} \min_{\Lambda} \left\{ -g(b + 6D_{\Lambda}^2(0)) + \right. \\
&\quad \left. + \frac{b^2}{2q'G(\Lambda, \Phi)} + \frac{1}{2q} \frac{\beta}{1+\beta} \Phi^2 - \frac{\beta}{q} D_{\Lambda}(0) - \right. \\
&\quad \left. - \frac{1}{8\pi q} [(1+\beta)\ln(1+\beta) - \beta] - \frac{\beta^2}{16q^2g} \right\}.
\end{aligned} \tag{6.24}$$

Now let us get the asymptotical behaviour  $E_-(g, \Phi)$  for  $g \rightarrow \infty$ . Using the formulas (6.11) and (6.22), changing the variables  $b = \frac{3}{8\pi^2} \rho$  and  $\Lambda^2 = A$  and limiting ourselves to logarithmic terms  $\ln A$  we obtain

$$\begin{aligned}
E_-(g, \Phi) &= \max_{\beta, \rho} \min_A \left\{ -\frac{3}{8\pi^2} [g(\rho + (\ln A)^2) - 2 \ln A] - \right. \\
&\quad \left. - \frac{\pi \rho^2 A}{28q'(\ln A)^2 F(A, \Phi)} \right\} + \\
&\quad + \left[ \frac{\beta}{1+\beta} \frac{\Phi^2}{2q} - \frac{\beta}{q} \frac{\ln A}{4\pi} - \frac{1}{8\pi q} [(1+\beta)\ln(1+\beta) - \beta] - \frac{\beta^2}{16q^2g} \right],
\end{aligned} \tag{6.25}$$

where

$$F = F(A, \Phi) = 1 + 4 \left( \frac{2\pi\Phi^2}{\ln A} \right) + 2 \left( \frac{2\pi\Phi^2}{\ln A} \right)^2.$$

The supremums with respect to  $A, \rho$  and  $\beta$  can be performed and the result in our approximation is

$$E_-(g, \Phi) = -\frac{3}{8\pi^2} g [\ln g + \ln B]^2 + \mathcal{E}(g, \Phi), \tag{6.26}$$

where

$$B = \frac{7}{2\pi} q' F \cdot (\ln g + \ln B),$$

$$\mathcal{E}(g, \Phi) = \frac{1}{2\pi} \min_{\beta} \left[ \frac{\beta}{1+\beta} \Phi^2 - \frac{\beta}{2\pi} \ln g \right] = \tag{6.27}$$

$$= \frac{1}{2q} \left( |\Phi| - \sqrt{\frac{1}{2\pi} \ln g} \right)^2,$$

where  $\beta_{\min}^2 \sim \frac{\beta^2}{\ln g}$ . Therefore the inequality (6.23) proves its value in the limit  $g \rightarrow \infty$ .

Finally we obtain in the limit  $g \rightarrow \infty$

$$E_-(g, \Phi) = -\frac{3}{8\pi^2} g[\ln g + \ln B]^2 + \quad (6.28)$$

$$+ \frac{1}{2q} (|\Phi| - \sqrt{\frac{1}{2\pi} (\ln g + \ln B)^2})^2,$$

where  $q > 1$ .

We can see that the estimations (6.7) and (6.28) are close one to another in the limit  $g \rightarrow \infty$ . They can be joined, the result for  $E(g, \Phi)$  in (6.1) is

$$E(g, \Phi) = -\frac{3}{8\pi^2} g(\ln g)^2 [1 + O(\frac{\ln \ln g}{\ln g})] + \quad (6.29)$$

$$+ \frac{1}{2} (|\Phi| - \sqrt{\frac{\eta}{2\pi} \ln g (1 + O(\frac{\ln \ln g}{\ln g}))})^2,$$

where  $\eta \approx 1$ .

It follows from this formula that the minimum of the vacuum energy in the  $g\phi_2^4$ -theory in the limit  $g \rightarrow \infty$  is at points

$$\Phi_{\pm} = \pm \sqrt{\frac{\eta}{2\pi} \ln g} : \quad (6.30)$$

It means that in this system there is a phase transition for  $g \rightarrow \infty$ . The existence of the phase transition in the  $\phi_2^4$ -theory was proved in [7] (see also [9, 10]).

#### ACKNOWLEDGEMENTS

I would like to thank Prof. D.I. Blokhintsev, B.N. Valuev, V.A. Zagrebnov, O.A. Mogilevskij and V.V. Priezjev for discussions.

#### APPENDIX

Let us show that for  $V \rightarrow \infty$

$$Q_V = \int \delta \Lambda \exp \left\{ -\frac{1}{2} \int_V d^d x \Lambda^2(x) - ia \int_V d^d x \left[ \int_V d^d y \mathcal{L}_V(x, y) \Lambda(y) \right]^2 + \right.$$

$$\left. + i \int_V d^d x a(x) \Lambda(x) \right\} =$$

$$= \int \delta u \exp \left\{ -\frac{1}{2} \sum_s u_s^2 - ia \int_V d^d x \left[ \sum_s \mathcal{L}_s(x) u_s \right]^2 + i \sum_s a_s u_s \right\} = \quad (A.1)$$

$$= \exp \left\{ -\frac{V}{2} \int \frac{d^d k}{(2\pi)^d} \ln(1 + 2ia\tilde{D}(k^2)) \right\} -$$

$$- \frac{1}{2} \int d^d x \int d^d x' a(x) G(x-x') a(x'),$$

where

$$G(x-x') = \frac{1}{(2\pi)^d} \int \frac{d^d k e^{ik(x-x')}}{1 + 2ia\tilde{D}(k^2)}. \quad (A.2)$$

In the integral (A.1) let us change the variable

$$u_s = \sum_n U_{sn} v_n, \quad (A.3)$$

where

$$\sum_n U_{sn}^T U_{ns} = \delta_{ss},$$

$$\sum_{nn'} U_{sn}^T D_{nn'} U_{n's} = d_s \delta_{ss},$$

$$D_{nn'} = \int \int_V dy dy' g_n(y) D_V(y, y') g_{n'}(y') =$$



$$= \sum_s U_{ns} d_s U_{sn}^T \quad (\text{A.4})$$

Then we have

$$\sum_s u_s^2 = \sum_n v_n^2,$$

$$\sum_{ss'} u_s D_{ss'} u_{s'} = \sum_n d_n v_n^2,$$

$$\sum_s \alpha_s u_s = \sum_n \beta_n v_n, \quad \beta_n = \sum_s U_{ns}^T \alpha_s.$$

The integral in the new variables can be calculated:

$$\begin{aligned} Q_V &= \int \delta v \exp \left\{ -\frac{1}{2} \sum_n (1 + 2ia d_n) v_n^2 + i \sum_n \beta_n v_n \right\} = \\ &= \exp \left\{ -\frac{1}{2} \sum_n \ln(1 + 2ia d_n) - \frac{1}{2} \sum_n \frac{\beta_n^2}{1 + 2ia d_n} \right\}. \end{aligned} \quad (\text{A.5})$$

Now let us consider

$$\sum_n \ln(1 + 2ia d_n) = \sum_{k=1}^{\infty} \frac{(-)^{k+1}}{k} (2ia)^k \sum_n d_n^k. \quad (\text{A.6})$$

Making use of the formulas (A.4) one can get

$$\begin{aligned} \sum_n d_n^k &= \sum_{n_1 \dots n_k} D_{n_1 n_2} D_{n_2 n_3} \dots D_{n_k n_1} = \\ &= \int_V dy_1 \dots \int_V dy_k D_V(y_1, y_2) D_V(y_2, y_3) \dots D_V(y_n, y_1). \end{aligned}$$

When  $V$  is large enough this function is

$$\sum_n d_n^k = V \int \frac{d^d k}{(2\pi)^d} [\tilde{D}(k^2)]^k + O(\exp\{-\text{const } V^{1/d}\})$$

whence

$$\sum_n \ln(1 + 2ia d_n) = V \int \frac{d^d k}{(2\pi)^d} \ln(1 + 2ia \tilde{D}(k^2)). \quad (\text{A.7})$$

Further consider

$$\sum_n \frac{\beta_n^2}{1 + 2ia d_n} = \sum_{k=0}^{\infty} (-2ia)^k \sum_n \beta_n d_n^k \beta_n, \quad (\text{A.8})$$

$$\begin{aligned} \sum_n \beta_n d_n^k \beta_n &= \sum_{n_1 \dots n_{k+1}} \alpha_{n_1} D_{n_1 n_2} \dots D_{n_k n_{k+1}} \alpha_{n_{k+1}} = \\ &= \int_V dy_1 \dots \int_V dy_{k+1} \alpha(y_1) D_V(y_1, y_2) \dots D_V(y_k, y_{k+1}) \alpha(y_{k+1}). \end{aligned}$$

In this expression it is possible to take the limit  $V \rightarrow \infty$

$$\begin{aligned} \lim_{V \rightarrow \infty} \sum_n \beta_n d_n^k \beta_n &= \\ &= \int dy_1 \dots \int dy_{k+1} \alpha(y_1) D(y_1 - y_2) \dots D(y_k - y_{k+1}) \alpha(y_{k+1}) = \\ &= \int \int dx dx' \int \frac{d^d k}{(2\pi)^d} [\tilde{D}(k^2)]^k e^{ik(x-x')}. \end{aligned}$$

Substituting this formula into (A.8) we obtain

$$\lim_{V \rightarrow \infty} \sum_n \frac{\beta_n^2}{1 + 2ia d_n} = \int dx \int dx' a(x) G(x-x') a(x'), \quad (\text{A.9})$$

where  $G(x-x')$  is defined by the formula (A.2).

Thus substituting (A.7) and (A.9) into (A.5) one can obtain (A.1).

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Received by Publishing Department  
on April 10, 1978,