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V.S.Gerdjikov, P.P.Kulish

COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS, CONNECTED WITH THE NON-SELF-ADJOINT DIRAC OPERATOR

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COMPLETELY INTEGRABLE HAMILTONIAN SYSTEMS, CONNECTED WITH THE NON-SELF-ADJOINT DIRAC OPERATOR

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* Leningrad division of the Mathematical Institute, Academy of Sciences, USSR.

Герджиков В.С., Кулиш П.П. **E2 - 11394**

Вполне интегрируемые гамильтоновые системы, связанные с несамосопряженным оператором Дирака

В настоящей работе рассматривается одномерный несамосопряженный оператор Дирака:

$$L = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \begin{pmatrix}0 & q(x)\\ p(x) & 0\end{pmatrix},$$

где комплекснозначные функции $q(x)$ и $p(x)$ удовлетворяют условиям $q(x) \to q_{\pm}$ и $p(x) \to p_{\pm}$ при $x \to \pm\infty$, $q_+p_+ = q_-p_-$. В предположении конечности и простоты дискретного спектра оператора L решены прямая и обратная задачи рассеяния для оператора L. На основе полученных формул следа описан класс бесконечномерных гамильтоновых систем и вычислены переменные действие-угол.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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Gerdjikov V.S., Kulish P.P. **E2 - 11394**

Completely Integrable Hamiltonian Systems, Connected with the Non-Self-Adjoint Dirac Operator

We consider the non-self-adjoint one-dimensional Dirac operator

$$L = i\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\frac{d}{dx} + \begin{pmatrix}0 & q(x)\\ p(x) & 0\end{pmatrix},$$

where the complex-valued functions $q(x)$ and $p(x)$ satisfy the conditions $q(x) \to q_{\pm}$, $p(x) \to p_{\pm}$ when $x \to \pm\infty$, $q_+p_+ = q_-p_-$. The direct and the inverse scattering problem for the operator L are solved, provided that the discrete spectrum of the operator L is finite and simple. Using the corresponding trace identities a class of infinite dimensional Hamiltonian systems is described, for which the action-angle variables are calculated.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. INTRODUCTION

The inverse scattering method (ISM) for solving one-dimensional non-linear evolution equations (NLEE), invented on the example of the Kortweg-de-Vries equation [1,2], became very popular after the papers of V.E.Zakharov, A.B.Shabat, and L.D.Faddeev [3,4,5]. In refs. [3,4] the one-dimensional non-linear Schrödinger equation was solved:

$$i u_t + u_{xx} + 2\epsilon |u|^2 u = 0, \quad \epsilon = \pm 1, \quad -\infty < x,t < \infty. \tag{1.1}$$

Equation (1.1) has a wide application in plasma physics, non-linear optics, and in a number of other physical phenomena [6]. It can be considered also as a Hartree-Fock equation for one-dimensional bose-particles with a $-\epsilon\delta(x)$ interaction. As natural boundary conditions, under which non-singular soliton solutions exist, we consider: $\epsilon = 1$ (attraction): $u(x,t) \to 0$, $|x| \to \infty$; $\epsilon = -1$ (repulsion): $|u(x,t)| \to \mathrm{const} \neq 0$, $|x| \to \infty$. The former case corresponds to one-dimensional bose-gas with a finite density. These cases are interesting also because the corresponding quantum problems are exactly solved and it can be demonstrated that to the solitons (particle-like solutions) after quantization there correspond quantum particles [7].

Initially the ISM consisted in finding such a pair of operators L and M depending on the solution of the NLEE $u(x, t)$, that Lax's evolution equation

$L_t = [L,M]$ is equivalent to the corresponding NLEE. Today there exist other formulations of the ISM, where the NLEE is the integrability condition for a pair of linear operators X and T (i.e., NLEE $\leftrightarrow$ $[X,T] = 0$). For more details see papers /8-13/.

One can also use the approach of paper /5/, which is applicable provided that: i) the inverse scattering problem for the operator L is solved, and ii) that there exists a symplectic form on the manifold of coefficient functions (potentials) of the operator L . Then recalculating the symplectic form in terms of the scattering data we can determine the action-angle variables. As Hamiltonians we will choose the functionals entering into the trace identities: therefore we need not use the M operator. It is this approach we use in the present paper.

As an L operator for equation (1.1) the one-dimensional Dirac operator is used:

$$L \equiv i\sigma_3 \frac{d}{dx} + Q(x), \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.2}$$

where $Q(x)$ is a matrix potential of a special type:

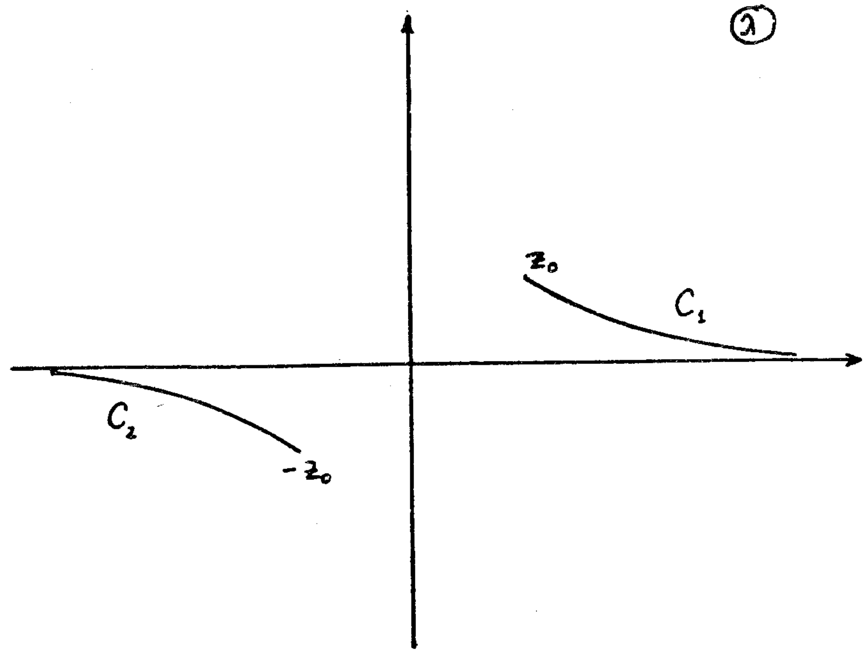
$$Q(x) = \begin{pmatrix} 0 & q(x) \\ p(x) & 0 \end{pmatrix}, \quad q(x) = -\epsilon p^*(x) = u(x). \tag{1.3}$$

The operator L with zero boundary conditions for $q(x)$ and $p(x)$ is investigated in detail, and the corresponding class of NLEE is described in refs./8,12/ The inverse scattering problem for the operator L with non-zero boundary conditions was considered in refs. /4,7,14/, but these papers do not contain the complete solution of the problem, and a number of formulae should be made more precise.

We are going to consider the general case of a non-self-adjoint operator L, restricting only the asymptotic values of the potential by:

$$\lim_{x\to\pm\infty} q(x) = q_\pm, \quad \lim_{x\to\pm\infty} p(x) = p_\pm, \quad q_+ p_+ = q_- p_- . \tag{1.4}$$

Although the continuous spectrum of the operator L does not lie any more on the real axis, it consists of two branches (see the figure) and, like in the selfadjoint case, is doubly degenerated provided the condition (4) holds. We use the standard methods of the inverse scattering theory, which are reviwed in detail in /15,16/.



The cuts C_1 and C_2 run along the arcs of the hyperbola $\xi\eta = uv$, where $\lambda = \xi + i\eta$, , $z_0 = u_0 + iv_0$. (see (2.1)).

In Sec.2 we consider the direct scattering problem for the operator L , (1.2), (1.4). Provided the discrete spectrum of the operator L is simple and finite we obtain the completeness relation, decribe the scattering data of the operator L , and derive its trace identities.

In Sec.3 we solve the inverse scattering problem for the operator L , provided its discrete spectrum satisfies the above-mentioned conditions, and discuss the features of the reflectionless potentials.

The description of the NLEE as a Hamiltonian system and the calculation of the action-angle variables are given in Sec.4.

In the appendix we derive some subsidiary formulae.

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2. THE DIRECT SCATTERING PROBLEM

Let us consider the scattering problem for the operator (1.2), (1.4) acting in the space of a quadratically integrable two-component vector-functions. Besides we suppose that the complex constants $q_\pm$, $p_\pm$ fixing the boundary conditions (1.4) satisfy the relation:

$$q_+ p_+ = q_- p_- = z_0^2 , \quad z_0 = u_0 + i v_0 , \; u_0 \neq 0. \qquad (2.1)^*$$

* When $u_0 = 0$, $v_0 \neq 0$ the continuous spectrum of the operator L divides the complex plane of the spectral parameter λ into two unconnected parts (see the figure). Therefore in this case the scattering problem should be solved separately.

This condition guarantees that the spectra of the asymptotic operators L_+ and L_- coincide, where

$$L_\pm \equiv i\sigma_3 \frac{d}{dx} + \lim_{x\to\pm\infty} Q(x). \qquad (2.2)$$

We introduce two pairs of Jost solutions for the operator L, which are determined by their asymptotical behaviour at $x\to\infty$ and $x\to-\infty$ respectively:

$$L\Psi(x,\lambda,\chi) = \lambda\Psi(x,\lambda,\chi), \quad L\Phi(x,\lambda,\chi) = \lambda\Phi(x,\lambda,\chi),$$

$$\Psi(x,\lambda,\chi) \equiv ||\psi(x,\lambda,\chi), \bar{\psi}(x,\lambda,\chi)||, \; \Phi(x,\lambda,\chi) \equiv ||\bar{\phi}(x,\lambda,\chi), \phi(x,\lambda,\chi)||, \qquad (2.3)$$

$$\Psi(x,\lambda,\chi) \xrightarrow[x\to+\infty]{} \Psi_+(x,\lambda,\chi) = \frac{1}{\sqrt{2\chi(\lambda+\chi)}} \begin{pmatrix} q_+ e^{i\chi x}, & (\lambda+\chi) e^{-i\chi x} \\ (\lambda+\chi)^{i\chi x}, & p_+ e^{-i\chi x} \end{pmatrix}, \qquad (2.4)$$

$$\Phi(x,\lambda,\chi) \xrightarrow[x\to-\infty]{} \Phi_-(x,\lambda,\chi) = \frac{1}{\sqrt{2\chi(\lambda+\chi)}} \begin{pmatrix} -q_- e^{i\chi x}, & (\lambda+\chi) e^{-i\chi x} \\ -(\lambda+\chi) e^{i\chi x}, & p_- e^{-i\chi x} \end{pmatrix}, \qquad (2.5)$$

where $\chi = \sqrt{\lambda^2 - z_0^2}$. The function χ is defined on a Riemanian surface, the first sheet of which is connected with the second one through the cuts C_1 and C_2 (see the figure); on the first sheet (the plane of the spectral parameter λ) $\mathrm{Im}\chi > 0$. Both pairs of Jost solutions form fundamental systems of solutions, since their Wronskians

$$W[\psi,\bar{\psi}] \equiv \det\Psi(x,\lambda,\chi) = -1,$$

$$W[\bar{\phi},\phi] \equiv \det\Phi(x,\lambda,\chi) = 1,$$

do not vanish. Therefore the solutions $\Psi(x,\lambda,\chi)$ are linear combinations of the solutions $\Phi(x,\lambda,\chi)$:

$$\Psi(x,\lambda,\chi) = \Phi(x,\lambda,\chi)T(\lambda,\chi), \quad T = \begin{pmatrix} -a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \tag{2.6}$$

where the functions $a, b, \bar{a}$ and $\bar{b}$ depend only on λ and χ and determine the scattering data of the operator L. They can be expressed through the Jost solutions

$$a(\lambda,\chi) = W[\phi,\psi], \quad b(\lambda,\chi) = W[\bar{\psi},\phi],$$
$$\bar{a}(\lambda,\chi) = W[\bar{\phi},\bar{\psi}], \quad \bar{b}(\lambda,\chi) = W[\bar{\phi},\psi], \tag{2.7}$$

and satisfy the "unitarity" condition:

$$\det T(\lambda,\chi) = -a\bar{a} - b\bar{b} = -1. \tag{2.8}$$

The existence and the analytical properties of the Jost solutions can be derived from the integral equation:

$$\Psi(x,\lambda,\chi) = \Psi_+(x,\lambda,\chi) - $$
$$- i\int_x^{\infty} dy\,\Psi_+(x,\lambda,\chi)\Psi_+^{-1}(y,\lambda,\chi)\sigma_3(Q(y)-Q_+)\Psi(y,\lambda,\chi), \tag{2.9}$$

and from a corresponding equation for $\Phi(x,\lambda,\chi)$ which are equivalent to the equations (2.3) with the boundary conditions (2.4) and (2.5). Supposing that the potential $Q(x)$ tends to its asymptotical values fast enough, e.g., so that

$$\int_{-\infty}^{0} dx\,|x|^k\,|q(x)-q_-| + \int_0^{\infty} dx\,|x|^k\,|q(x)-q_+| < \infty,$$
$$\int_{-\infty}^{0} dx\,|x|^k\,|p(x)-p_-| + \int_0^{\infty} dx\,|x|^k\,|p(x)-p_+| < \infty, \tag{2.10}$$

it is possible to obtain the necessary estimates and thus prove the following theorem:

Theorem: If condition (2.10) holds for all $k = 0, 1, \dots, n$, $n > 0$, , then the functions $e^{i\chi x}\phi(x,\lambda,\chi)$ and $e^{-i\chi x}\psi(x,\lambda,\chi)$ are analytic functions of λ on the first sheet of the Riemanian surface $\mathfrak{R}$, and $e^{-i\chi x}\bar{\phi}(x,\lambda,\chi)$ and $e^{i\chi x}\bar{\psi}(x,\lambda,\chi)$ are analytic functions of λ on the second sheet of $\mathfrak{R}$. On the cuts C_1 and C_2 all the four functions are continuous and n-fold differentiable with respect to λ.

Corollary: The functions $a(\lambda,\chi)$ and $\bar{a}(\lambda,\chi)$ are analytic functions of λ on the first and second sheet of $\mathfrak{R}$, respectively. On the cuts C_1 and C_2 the functions $a(\lambda,\chi)$, $b(\lambda,\chi)$, $\bar{a}(\lambda,\chi)$ and $b(\lambda,\chi)$ are continuous and n-fold differentiable with respect to λ.

Now it is possible to verify, that the expression:

$$R(x,y;\lambda,\chi) =$$
$$= \frac{i}{a(\lambda,\chi)}[\psi(x,\lambda,\chi)\phi^T(y,\lambda,\chi)\theta(x-y) + \phi(x,\lambda,\chi)\psi^T(y,\lambda,\chi)\theta(y-x)]\sigma_1 \tag{2.11}$$

is the resolvent of the operator L . Studying its singularities with respect to λ we can determine the spectrum of the operator L . From (2.11) we obtain, that the operator has a doubly degenerated continuous spectrum, coinciding with the cuts C_1 and C_2, and a discrete spectrum localized at the zeroes of $a(\lambda,\chi)$ on the first sheet of $\mathfrak{R}$. In the general case of non-self-adjoint operator L there are no limitations on the number, location and multiplicity of the zeroes of $a(\lambda,\chi)$. Here we will not consider all the possibilities, but we will limit ourselves to the simplest case of a finite number of simple zeroes located outside the continuous spectrum of L.

Integrating the resolvent (2.11) along the infinite circle in the complex λ-plane it is not difficult to obtain the completeness relation for the eigenfunctions of the operator L:

$$\frac{1}{2\pi}\int d\lambda\left[\frac{1}{a(\lambda)}\psi(x,\lambda)\phi^{T}(y,\lambda)-\frac{1}{\bar{a}(\lambda)}\bar{\psi}(x,\lambda)\bar{\phi}^{T}(y,\lambda)\right]\sigma_1 - \tag{2.12}$$

$$-i\sum_{j=1}^{N}\frac{b_j}{\dot{a}_j}\psi_j(x)\psi_j^{T}(y)\sigma_1=\delta(x-y),$$

where we have used the notations*:

$$\lambda_j:\ a(\lambda_j,\chi_j)=0,\quad \chi_j=\sqrt{\lambda_j^2-z_0^2},\ j=1,\dots,N, \tag{2.13}$$

$$\dot{a}_j=\frac{da}{d\lambda}\Big|_{\lambda=\lambda_j},\quad \psi_j(x)=\psi(x,\lambda_j,\chi_j).$$

The constants b_j can be expressed through the corresponding eigenfunctions $\psi_j(x)$ of the operator L (see the appendix):

$$b_j=-i\dot{a}_j\left(\int_{-\infty}^{\infty}dx\,\psi_j^{T}(x)\sigma_1\psi_j(x)\right)^{-1}. \tag{2.14}$$

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* Here, and in what follows the integrations over the spectral parameter λ run along the upper side of the cuts C_1 and C_2; this, together with the specifying of λ determines unambiguously the Jost solution, and therefore the χ-dependence can be dropped.

Now we proceed to derive the so-called trace identities (see, e.g., ref.[16]) for the operator L. Suppose $a(\lambda,\chi)$ has N simple zeroes located at $\lambda_1,\dots,\lambda_N$. Then, using the analytic properties of $a(\lambda,\chi)$ it is possible to derive the following dispersion relation (see the appendix):

$$\ln a(\lambda,\chi)=-\frac{1}{2\pi i}\int\frac{d\mu}{\sigma}\left(1+\frac{\chi}{\mu-\lambda}\right)\ln(1+r(\mu)\bar{r}(\mu))+ \tag{2.15}$$

$$+\sum_{j=1}^{N}\ln\frac{\lambda+\chi-\lambda_j-\chi_j}{\lambda+\chi-\lambda_j+\chi_j},$$

where by r and $\bar{r}$ we have denoted b/a and $\bar{b}/\bar{a}$, respectively, $\sigma=\sqrt{\mu^2-z_0^2}$.

Starting with the quasiclassical expansion for the Jost solution $\psi(x,\lambda,\chi)$, it is possible to obtain another representation for $\ln a(\lambda,\chi)$ in the form:

$$\ln a(\lambda,\chi)=\sum_{n=1}^{\infty}\frac{1}{(2i\chi)^n}\int_{-\infty}^{\infty}dx\,\phi^{(n)}(x), \tag{2.16}$$

where $\phi^{(n)}(x)$ satisfy the following recurrent relations:

$$\phi^{(n+1)}=-p\frac{d}{dx}\left(\frac{\phi^{(n)}}{p}\right)+\sum_{j=1}^{n-1}\phi^{(j)}\phi^{(n-j)}+c_{n+1}\frac{p_x}{p}, \tag{2.17}$$

$$\phi^{(1)}=z_0^2-pq,$$

$$c^{(2k)}=\frac{(-1)^k}{2}\binom{1/2}{k}(4z_0^2)^k,\quad c^{(2k+1)}=0.$$

Expanding (2.15) in series in the inverse powers of $2i\chi$, $\operatorname{Im}\lambda>0$ we get the following trace identities:

$$\int_{-\infty}^{\infty}dx\,\phi^{(n)}(x)=\frac{1}{2\pi i}\int d\mu\, s^{(n)}(\mu)\ln(1+r(\mu)\bar{r}(\mu))+\sum_{j=1}^{N}\frac{s_j^{(n)}}{n} \tag{2.18}$$

where

$$s^{(2k+1)}(\mu) = (-1)^{k+1}\, 2i\mu(4\sigma^2)^k ,$$

$$s^{(2k)}(\mu) = (-1)^{k+1}(4\sigma^2)^k \sum_{m=0}^{k} \binom{1/2}{m} \left(\frac{z_0}{\sigma}\right)^{2m} ,$$

$$s_j^{(2k+1)} = (-1)^{k+1}\, 2i\chi_j (4\chi_j^2)^k,$$

$$s_j^{(2k)} = (-1)^{k+1} \frac{\lambda_j}{\chi_j}(4\chi_j^2)^k \sum_{m=0}^{k-1} \binom{-1/2}{m} \left(\frac{z_0}{\chi_j}\right)^{2m}.$$

The first three identities have the form:

$$\int_{-\infty}^{\infty} dx(z_0^2 - pq) = \frac{1}{\pi}\int \frac{d\mu}{\sigma}\mu \ln(1 + r(\mu)\bar{r}(\mu)) - \sum_{j=1}^{N} 2i\chi_j , \quad (2.19)$$

$$\frac{1}{2i}\int_{-\infty}^{\infty} dx(pq_x - p_x q) =$$

$$= -\frac{2}{\pi}\int d\mu\,\sigma\left(1 + \frac{z_0^2}{2\sigma^2}\right)\ln(1 + r(\mu)\bar{r}(\mu)) - \sum_{j=1}^{N} 2i\lambda_j\chi_j , \quad (2.20)$$

$$\int_{-\infty}^{\infty} dx[p_x q_x + (z_0^2 - pq)^2] =$$

$$= -\frac{4}{\pi}\int d\mu\,\mu\,\sigma \ln(1 + r(\mu)\bar{r}(\mu)) + \sum_{j=1}^{N} \frac{8i}{3}\chi_j^3. \quad (2.21)$$

From (2.15), using the fact that $a(\lambda,\chi) \to p_-/p_+$ when $\lambda \to \infty$, $\mathrm{Im}\,\lambda < 0$, we obtain one more trace identity:

$$\ln\frac{p_-}{p_+} = \ln\frac{q_+}{q_-} = \frac{i}{\pi}\int \frac{d\mu}{\sigma}\ln(1 + r(\mu)\bar{r}(\mu)) + \sum_{j=1}^{N} \ln\frac{\lambda_j + \chi_j}{\lambda_j - \chi_j}. \quad (2.22)$$

The importance of (2.22) will become clear when solving the inverse scattering problem, since it connects the asymptotic values of the potential at $+\infty$ and $-\infty$ with the scattering data.

3. THE INVERSE SCATTERING PROBLEMS

For the potentials, satisfying the restrictions in Sec.2 we can introduce Volterra transformation operators (see, e.g., refs. /14,15/) transforming the Jost solutions $\Psi(x, \lambda,\chi)$ of the operator $L_0 \equiv i\sigma_3 \frac{d}{dx} + Q_0(x)$ into the Jost solutions of the operator L:

$$\Psi(x,\lambda,\chi) = \Psi_0(x,\lambda,\chi) + \int_x^{\infty} dy\, K(x,y)\Psi_0(y,\lambda,\chi). \quad (3.1)$$

Inserting (3.1) into (2.3) and using the completeness of the eigenfunctions of the operator L_0 (see (2.12)), we obtain the differential equation:

$$i\sigma_3 K_x + iK_y\sigma_3 + Q(x)K(x,y) - K(x,y)Q_0(y) = 0, \quad (3.2)$$

with the initial condition:

$$[\sigma_3 , K(x,x)] = -i(Q(x) - Q_0(x)), \quad (3.3)$$

and the boundary condition $\lim_{y\to\infty} K(x,y) = 0$. Now, if we insert (3.1) into the completeness relation (2.12) for the operator L, and use the analogous to (2.12) relation for the operator L_0 together with (2.6) we get the Gel'fand-Levitan-Marchenko (GLM) equation:

$$K(x,y) + F(x,y) + \int_x^{\infty} dz\, K(x,z)F(z,y) = 0, \quad (3.4)$$

where

$$F(x,y)=\frac{1}{2\pi}\int d\lambda[(r(\lambda)-r_0(\lambda))\psi_0(x,\lambda)\psi_0^T(y,\lambda)\sigma_1-$$

$$-(\bar r(\lambda)-\bar r_0(\lambda))\bar\psi_0(x,\lambda)\bar\psi_0^T(y,\lambda)\sigma_1]-\Sigma(x,y)+\Sigma_0(x,y),$$

$$\Sigma(x,y)=i\sum_{j=1}^{N}c_j\psi_0(x,\lambda_j)\psi_0^T(y,\lambda_j)\sigma_1,\ a(\lambda_j,\chi_j)=0 \tag{3.5}$$

$$\Sigma_0(x,y)=i\sum_{j=1}^{N_0}c_{0j}\psi_0(x,\lambda_{0j})\psi_0^T(y,\lambda_{0j})\sigma_1,$$

$$a_0(\lambda_{0j},\chi_{0j})=0$$

with $c_{0j}=b_{0j}/\dot a_{0j}$. Let

$$S\equiv\{r(\lambda),\bar r(\lambda),\lambda_j,c_j;\ j=1,\dots,N\},$$
$$S_0\equiv\{r_0(\lambda),\bar r_0(\lambda),\lambda_{0j},c_{0j};\ j=1,\dots,N_0\}, \tag{3.6}$$

be the scattering data for the operators L and L_0 , where all λ_j and λ_{0j} are simple eigenvalues of the operators L and L_0 , respectively, located outside of the continuous spectrum. Then we can solve the GLM equation (3.4) and obtain the kernel $K(x,y)$ of the transformation operator; using the initial condition (3.3) we can reconstruct the potential $Q(x)$, corresponding to the scattering data S . The functions $a(\lambda,\chi)$, $b(\lambda,\chi)$, $\bar a(\lambda,\chi)$ and $\bar b(\lambda,\chi)$ entering into (2.6) can be reconstructed knowing the scattering data S_0 and using the dispersion relations for $a(\lambda,\chi)$ (see (2.15)) and $\bar a(\lambda,\chi)$. The asymptotic values of the potential $Q(x)$ at $x\to-\infty$ can be reconstructed using the trace identity (2.22) and knowing the asymptotic values of $Q_0(x)$ at $x\to\pm\infty$.

Analogically we can obtain the corresponding formulae for the transformation operator, transforming the Jost solutions with given asymptotics at $x\to-\infty$.

Thus, in the framework of our restrictions made for the discrete spectrum of the operator L , the inverse scattering problem for the operator L is solved.

In particular, if $L_0\equiv L_+$, then $r_0(\lambda)=\bar r_0(\lambda)=0$, $\Sigma_0=0$ and the GLM equation is simplifyed. In this case we specially note the class of the so-called reflectionless potentials, for which $r(\lambda)=\bar r(\lambda)=0$. In this case the kernel of the GLM equation equals:

$$F_d(x,y)=-i\sum_{j=1}^{N}c_j\psi_{+j}(x)\psi_{+j}^T(y)\sigma_1, \tag{3.7}$$

where $\psi_{+j}(x)=\psi_+(x,\lambda_j,\chi_j)$ is the first column of $\Psi_+(x,\lambda,\chi)$ in (2.4). We look for the solution of the GLM equation in the form.

$$K_d(x,y)=\sum_{j=1}^{N}k_j(x)\psi_{+j}^T(y)\sigma_1. \tag{3.8}$$

Inserting (3.7) and (3.8) into (3.4) we get a system of linear algebraic equations for $k_j(x)$. Thus the reflectionless potentials can be explicitly calculated. The most simple reflectionless potential, corresponding to $N=1$ has the form:

$$q(x)=q_+\frac{1+s_1V_1}{1+V_1},\qquad p(x)=p_+\frac{1+s_1^{-1}V_1}{1+V_1},$$
$$s_1=\frac{\lambda_1-\chi_1}{\lambda_1+\chi_1},\qquad V_1=\frac{c_1q_+}{2\chi_1^2}e^{2i\chi_1x}, \tag{3.9}$$

where λ_1 is the eigenvalue of the operator L . In (3.9) we suppose that $1+V_1\neq0$, which is not trivial since c_1 , χ_1 and q_+ are complex numbers, and if they are related by

$$\operatorname{Im}\frac{1}{2i\chi_1}[(2n+1)\pi i-\ln\frac{c_1q_+}{2\chi_1^2}]=0 \tag{3.10}$$

for some integer n, then $1+V_1$ may vanish. For $N>1$ the regularity condition for the corresponding reflectionless potential is equivalent to the restriction, that all the zeroes of the integral function

$$M(z)=\det\|M_{ij}(z)\|,$$

$$M_{ij}(z)=\delta_{ij}+\frac{c_j q_+(\lambda_i+\chi_i+\lambda_j+\chi_j)e^{i(\chi_i+\chi_j)z}}{(\chi_i+\chi_j)\sqrt{2\chi_i(\lambda_i+\chi_i)2\chi_j(\lambda_j+\chi_j)}}, \qquad (3.11)$$

are located outside the real axis $\mathrm{Im}\,z=0$. Since the function $M(z)$ has a denumerable set of zeroes, appropriately choosing the constants c_j, we can always satisfy the condition (3.11).

4. NLEE AS HAMILTONIAN SYSTEMS AND THE ACTION-ANGLE VARIABLES

Let us consider the manifold of pairs of complex valued functions $M\equiv\{q(x),p(x)\}$ satisfying condition (1.4). On this manifold we can introduce symplectic structure, defining the 2-form (see, e.g., ref. [18]):

$$\Omega=i\int_{-\infty}^{\infty}dx\,\delta p(x)\wedge\delta q(x). \qquad (4.1)$$

If we consider the functional $H[q,p]$ on M, then with the help of the 2-form Ω we can define Hamiltonian equations of motion:

$$\frac{d}{dt}(p,q)=\Omega^{-1}dH=i\left(\frac{\delta H}{\delta q},-\frac{\delta H}{\delta p}\right), \qquad (4.2)$$

where the 1-form dH is the external differential of the function H and Ω^{-1} is the mapping of the 1-forms on M onto the vector fileds, connected with the form Ω.

Constructing the NLEE we can choose as a Hamiltonian an arbitrary functional $H[q,p]$. For instance, if we choose $H=\int_{-\infty}^{\infty}dx\,\phi^{(3)}(x)$ (see (2.21)) then the equations of motion (4.2) will have the form:

$$\begin{aligned} &iq_t+q_{xx}-2q(pq-z_0^2)=0,\\ &ip_t-p_{xx}+2p(pq-z_0^2)=0. \end{aligned} \qquad (4.3)$$

In particular, if $q=p^*=u(x,t)$ and $z_0^2=m^2$, m -real, the system (4.3) reduces to the non-linear Schrödinger equation (1.1) with an additional term, linear with respect to $u(x,t)$.

Equation (4.3) and all the other systems of equations which can be obtained from (4.2) inserting linear combinations of $\int_{-\infty}^{\infty}dx\,\phi^{(m)}(x)$ instead of $H[q,p]$, can be solved by the ISM. We will consider this in detail in a next paper, and now we proceed to prove the complete integrability of the canonical systems, described by these NLEE. We will not need the explicit form of the corresponding NLEE.

With every point $\{q,p\}$ in M we can connect an operator L and, therefore, scattering data $S\equiv\{r(\lambda),\bar{r}(\lambda),\lambda_j,c_j;\,j=1,\dots,N\}$. The properties of the functions in S were described in detail in Secs.2 and 3, where the direct and the inverse scattering problem for the operator L were solved. Thus the direct scattering problem determines a mapping of M onto S, and the inverse scattering problem, a mapping of S onto M. There naturally arises the problem of recalculation of the symplectic form in terms of the scattering data.

For solving this problem we use the GLM equation (3.4) and (3.5), which allows us to reconstruct the operator L from the operator L_0. Suppose that the scattering data S and S_0 are slightly different, i.e., that $r(\lambda)-r_0(\lambda)$ and $\bar{r}(\lambda)-\bar{r}_0(\lambda)$ can be replaced by $\delta r(\lambda)$ and $\delta\bar{r}(\lambda)$. Then from equations (3.5) and (3.3) neglecting the term $K\circ F$ in (3.5), we can express the variations of the potentials through the variations of the scattering data:

$$\delta q(x) = -\frac{i}{\pi} \int d\lambda\, [\delta r(\lambda)(\psi_0^{(1)}(x,\lambda))^2 - \delta\bar{r}(\lambda)(\bar{\psi}_0^{(1)}(x,\lambda))^2] - -2\sum_{j=1}^{N} [\delta c_j (\psi_{0j}^{(1)}(x))^2 + c_j \delta\lambda_j \frac{d}{d\lambda_j} (\psi_{0j}^{(1)}(x))^2],$$

$$\delta p(x) = \frac{i}{\pi} \int d\lambda [\delta r(\lambda)(\psi_0^{(2)}(x,\lambda))^2 - \delta\bar{r}(\lambda)(\bar{\psi}_0^{(2)}(x,\lambda))^2] + +2\sum_{j=1}^{N} [\delta c_j (\psi_{0j}^{(2)}(x))^2 + c_j \delta\lambda_j \frac{d}{d\lambda_j} (\psi_{0j}^{(2)}(x))^2]. \tag{4.4}$$

Inserting these expressions into formula (1) we obtain bilinear combinations of the scattering data variations, which coefficients can be expressed through integrals of the type:

$$\int_{-\infty}^{\infty} dx\, [(\psi_0^{(1)}(x,\lambda))^2 (\psi_0^{(2)}(x,\mu))^2 - (\psi_0^{(2)}(x,\lambda))^2 (\psi_0^{(1)}(x,\mu))^2], \tag{4.5}$$

and their derivatives with respect to λ and μ. It is well known how to calculate such expressions, see, e.g., ref. /17/. Finally we obtain the following expression for the symplectic form in terms of the scattering data variations:

$$\Omega = \frac{i}{\pi} \int d\lambda\, a(\lambda)\bar{a}(\lambda)\delta r(\lambda) \wedge \delta\bar{r}(\lambda) - - \frac{1}{2\pi^2}\int d\lambda \int d\mu\, Z(\lambda,\mu)\delta \ln(1+r(\lambda)\bar{r}(\lambda)) \wedge \delta \ln(1+r(\mu)\bar{r}(\mu)) + + \frac{2i}{\pi}\sum_{j=1}^{N} \int d\lambda\, Z(\lambda,\lambda_j)\delta \ln(1+r(\lambda)\bar{r}(\lambda)) \wedge \delta\lambda_j + + 2\sum_{j=1}^{N} \delta \ln c_j \wedge \delta\lambda_j + 2\sum_{j\neq s=1}^{N} Z(\lambda_j,\lambda_s)\delta\lambda_j \wedge \delta\lambda_s, \tag{4.6}$$

where $Z(\lambda,\mu) = \dfrac{\lambda\mu - z_0^2}{\chi\sigma(\lambda-\mu)}$.

Let us choose as independent the functions:

$$P(\lambda) = -\frac{1}{\pi}\ln(1+r(\lambda)\bar{r}(\lambda)), \qquad P_j = 2i\lambda_j,$$

$$Q(\lambda) = \frac{i}{2}\ln b(\lambda)/\bar{b}(\lambda) + \frac{i}{2}(1-\frac{\lambda}{\chi})\ln p_+/p_-, \tag{4.7}$$

$$Q_j = i\ln b_j + \frac{i}{2}(1-\frac{\lambda_j}{\chi_j})\ln p_+/p_-.$$

Then, using the dispersion relation (2.15) for $a(\lambda,\chi)$ we can reconstruct all the scattering data from $\{Q(\lambda), P(\lambda), Q_j, P_j\}$ and moreover, the symplectic form Ω in terms of the variables (7) can be cast in the canonical form:

$$\Omega = \int d\lambda\, \delta P(\lambda) \wedge \delta Q(\lambda) + \sum_{j=1}^{N} \delta P_j \wedge \delta Q_j. \tag{4.8}$$

Thus the transition from the variables $\{q,p\}$ to the variables $\{Q(\lambda), P(\lambda), Q_j, P_j\}$ is a canonical transformation.

Now it is easy to see from the trace identities (2.18) that if $H[q,p]$ is choosen to be a linear combination of $\int_{-\infty}^{\infty} dx\,\phi^{(n)}(x)$ then in terms of $\{Q(\lambda), P(\lambda), Q_j, P_j\}$ the Hamiltonian will depend only on half of the canonical variables, namely, on the generalized momenta $\{P(\lambda), P_j\}$. Therefore, the NLEE corresponding to such Hamiltonians are completely integrable Hamiltonian systems.

We note, that in the literature (see, e.g., refs. /5,7,17/) there exist two ways of calculating the action-angle variables: one through the Poisson brackets, and the other-through the symplectic form. In our case these two ways lead to different answers. The difference, is, however, unessential; the generalized coordinates in the two cases differ by a function, depending on the momenta only. This is due to the fact, that using the Poisson brackets the scattering data variations are expressed through the variations of the potentials. Therefore the asymptotic values of the potentials are preserved:

$$\delta \ln \frac{p_-}{p_+} = \frac{i}{\pi} \int \frac{d\mu}{\sigma} \delta \ln(1+r(\mu)\bar{r}(\mu)) + 2 \sum_{j=1}^{N} \frac{\delta \lambda_j}{\chi_j} = 0, \qquad (4.9)$$

and the corresponding scattering data variations satisfy the condition (4.9). Using the symplectic form we express the variations of the potentials through the scattering data variations, which are arbitrary now. Thus the asymptotic values of the potentials may change (of course, the condition $q_+ p_+ = q_- p_- = z_0^2$ always holds).

APPENDIX

Here we will derive the dispersion relation (2.15) for $a(\lambda, \chi)$. Let the operator L_0 be such that it has no discrete eigenvalues, and let its scattering data be determined by the functions a_0, b_0, $\bar{a}_0$ and $\bar{b}_0$. Then, writting down the Caushy integral formula for the function $\chi^{-1} \ln(a_0(\lambda, \chi)\sqrt{p_+/p_-})$ and using (2.8) we get the dispersion relation (2.15) for $\ln a_0(\lambda, \chi)$, in which only the integral term will be present.

Using the solution of the inverse scattering problem given in Sec.3, we will calculate the functions a_1, b_1, $\bar{a}_1$ and $\bar{b}_1$ which determine the scattering data of the operator L_1; L_1 has, as compared to L_0, an additional discrete eigenvalue located at λ_1. The corresponding solution of the GLM equation is

$$K_1(x,y) = -\frac{ic_1}{1 + ic_1 J_0(\lambda_1, \lambda_1)} \psi_0(x, \lambda_1) \psi_0^T(y, \lambda_1) \sigma_1, \qquad (A.1)$$

where we used the notation:

$$J_0(\lambda, \mu) = \int_x^\infty dy\, \psi_0^T(y, \lambda) \sigma_1 \psi_0(y, \mu) = -\frac{iW[\psi_0(y,\lambda), \psi_0(y, \mu)]}{\lambda - \mu}\Big|_{y=x}^{\infty} \qquad (A.2)$$

Inserting now the kernel $K_1(x,y)$ (A.1) into equation (3.1) and letting $x \to -\infty$ we get:

$$a_1(\lambda, \chi) = m_1 a_0(\lambda, \chi), \quad b_1(\lambda, \chi) = m_1 b_0(\lambda, \chi),$$
$$\bar{a}_1(\lambda, \chi) = \frac{1}{m_1} \bar{a}_1(\lambda, \chi), \quad \bar{b}_1(\lambda, \chi) = \frac{1}{m_1} b_1(\lambda, \chi), \qquad (A.3)$$

where $m_1 = (\lambda + \chi - \lambda_1 - \chi_1)/(\lambda + \chi - \lambda_1 + \chi_1)$.
From (A.3) we see, that the scattering data of L_0 and L_1 corresponding to the continuous spectrum coincide as it should be expected, i.e., $r_1(\lambda) = r_0(\lambda)$ and $\bar{r}_1(\lambda) = \bar{r}_0(\lambda)$.

Repeating this procedure N times we get the canonical factor $\sum_{j=1}^{N} \ln(\lambda + \chi - \lambda_j - \chi_j)/(\lambda + \chi - \lambda_j + \chi_j)$ in (2.15).

The relation (2.14) can be easily obtained from the relation (A.2) letting $x \to -\infty$ and $\lambda, \mu \to \lambda_j$; here we should also use the simplicity of the corresponding eigenvalues, i.e., that in a small region around $\lambda = \lambda_j$, $a(\lambda) = (\lambda - \lambda_j)\dot{a}_j$.

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