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# CLASSICAL DYNAMICAL SYSTEMS WITH THE SYMMETRY OF THE KEPLER PROBLEM

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# CLASSICAL DYNAMICAL SYSTEMS WITH THE SYMMETRY OF THE KEPLER PROBLEM

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Классические динамические системы с кеплеровской симметрией

Рассматриваются динамические системы с тремя степенями свободы с гамильтонианом $H = \frac{1}{2}G_1 p^2 + \frac{1}{2}G_2(xp)^2 + G_3(xp) + U$, где $G_j$ и $U$ – функции $r = \sqrt{x^2}$. Найдены те из них, для которых "вектор Рунге–Ленца" квадратичен по импульсам. Показано, что все они обладают замкнутыми траекториями. Рассмотрен вопрос об одном способе минимального нарушения этой "строгой" кеплеровской симметрии, при котором сохраняется точная разрешимость. Показано, что при нарушении этой строгой кеплеровской симметрии возникает прецессия перигелия.

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Classical Dynamical Systems with the Symmetry of the Kepler Problem

Hamiltonian dynamical systems of the form $H = \frac{1}{2}G_1 p^2 + \frac{1}{2}G_2(xp)^2 + G_3(xp) + U$, where $G_j$ and $U$ are functions of $r = \sqrt{x^2}$, are investigated. The notion of a strict Kepler symmetry is introduced to single out the cases where there is a "Runge-Lenz vector" quadratic in the momentum. All dynamical systems with this property are found. They depend on an arbitrary function of the distance to the centrum of symmetry and two arbitrary interaction constants. The equations of motion are solved and it is shown explicitly that the orbits are closed. Cases when the strict Kepler symmetry is related to an underlying E(3) symmetry are noted. The breaking of the strict Kepler symmetry and its relation to the precession of the perihelium are discussed.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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## 1. *Introduction*

There is an extensive literature devoted to the so-called dynamical (or accidental, or hidden) symmetries and to the Kepler problem, in particular (both in classical and in quantum theory). It is usually assumed in these studies that the Hamiltonian is of the form

$$H = T + U \tag{1.1}$$

that is, a sum of a kinetic part

$$T = \frac{1}{2m} p^2, \quad m = \text{const} \tag{1.2}$$

invariant under rotations, and a momentum independent potential that is also assumed to be rotation invariant,

$$U = U(r). \tag{1.3}$$

The Kepler problem,

$$H = \frac{1}{2m} p^2 + g\frac{1}{r} \tag{1.4}$$

and the isotropic harmonic oscillator are two examples of the form (1.1)-(1.3) known since 19 century to possess hidden symmetry O(4) and SU(3), respectively, higher than the obvious rotational symmetry. It was proved by Bertrand /1/ in 1874 that these are the only interactions of the form (1.1)-(1.3) which have closed orbits and it was believed for a long time that the closed orbits are a consequence of the exceptional higher symmetry of these particular interactions.

In the case of the hydrogen atom the O(4) symmetry of the Kepler problem has a beautiful quantum-mechanical manifestation /2-11/. We are not concerned in our work, however, with the quantum-mechanical aspect of this symmetry.

In the sixties the interest in the symmetry of the Kepler problem was revived /12-21/ due to the success of the group theoretic approach in particle physics. At that time it was gradually realized /22-27/ that O(4) and SU(3) are symmetries of any interaction of the kind (1.1)-(1.3). And what is more, a general proof was given /24/ that for any dynamical system of $n$ degrees of freedom, whatever function of the canonical variables the Hamiltonian

$$H = H(q_1, ..., q_n, p_1, ..., p_n) \tag{1.5}$$

would be, there exist O(n+1) and SU(n) algebras of integrals of motion. Consequently, the symmetry of the Kepler problem (as well as that of the isotropic harmonic oscillator) is inherent to any dynamical problem. It does not depend on the interaction. It is neither dynamical nor accidental. It is rather a general characteristics of the canonical Poisson bracket Lie algebra. And the property of the orbits to be closed is not a consequence of this symmetry.

Here we study a class of dynamical systems which is much larger than the class (1.1)-(1.3). We assume that the Hamiltonian is a quadratic function of the momenta, $p = (p_1, p_2, p_3)$,

$$H = \frac{1}{2}[G_1(r)p^2 + G_2(r)(xp)^2] + G_3(r)(xp) + U(r) \tag{1.6}$$

invariant under rotations in configuration space; $r = \sqrt{x^2}$ and $x = (x_1, x_2, x_3)$ are Cartesion coordinates. Since the O(4) symmetry by itself cannot distinguish a proper subclass of dynamical systems, we choose another property to single out such a subclass. Namely, some of the elements of the symmetry algebra are required to be second degree polynomials in the momenta. We say for brevity that the so-defined dynamical systems possess the symmetry of the Kepler problem in a strict sense. The subclass of interactions we obtain in this way includes the case (1.4) and depends on an arbitrary function of $r$, $f(r)$, and two arbitrary interaction constants, $g_2$ and $g_3$. It has the virtue that any of the problems in this subclass can be solved exactly. The orbits are closed for any choice of $f(r)$, $g_2$, and $g_3$.

The interaction (1.6) is not standard in the sense that it cannot be fully separated from the kinetic part. Regarding this point we recall that even in the classical theory of the electromagnetic interaction a dependence on the momentum, though linear, is contained in the interaction part. Though a linear in $p$ term is present in (1.6) actually it is only possible to include electrostatic fields in our treatment. The linear term in (1.6) is equivalent to a vector potential $A(r) = (G_3/(G_1 + r^2 G_2))x$, i.e., to a vanishing magnetic field, rot $A = 0$. Or, in other words, the choice of $G_3$ is a gauge freedom that, adding a gradient term to the momentum, does not affect the action. We choose $G_3 = 0$. In order to include a magnetic field we should have at our disposal other vectors except $x$ in configuration space or additional degrees of freedom. We note, however, that in some cases in nuclear physics the nuclear potential is assumed to depend quadratically on the momentum (see, for instance /28-32/). The particle physics gives us even more reasons not to neglect the interactions of the form (1.6) in favour of the standard form (1.1)-(1.3). As a matter of fact, quantum field theory had long ago gone out of this standard form of interaction and one of the applications of our results is related to quantum field theory . We have found new (Lagrangian) nonlinear realizations of current algebra. Finally we recall that the Hamiltonian and the time are canonically conjugate and that the change of the form of the dynamics should be dual in a certain sense to a change of the form of the metric in space-time. In this sense some of the Hamiltonians (1.6) should be dual to the Schwarzschild metric.

In Sec. 2 we find all Hamiltonians of the form (1.6) which allow a strict Kepler symmetry in the above sense. In Sec. 3 we discuss the smooth deviations from the $1/r$ law. As far as the dynamical systems with the strict symmetry of the Kepler problem form a proper subclass it makes sense to speak of breaking of the symmetry. In Sec. 3 we study a minimal way of symmetry breaking such that the exact solvability is preserved. The effect of the symmetry breaking is that the orbits are no longer closed. We demonstrate how a smooth extremely small breaking of the $1/r$ law implies a precession of the perihelium in the framework of classical (nonrelativistic) dynamics.

## 2. *A Class of Dynamical Systems with the Symmetry of the Kepler Problem*

Here we shall strongly restrict the class of the dynamical systems (1.5) by requiring that the Hamiltonian is of the form (1.6), i.e., it is quadratic in the momentum $p$ and invariant with respect to the $SO(3)$ group generated by

$$J_j = \epsilon_{jk\ell} x_k p_\ell . \qquad (2.1)$$

The $SO(3)$ Lie algebra of $J_j$ is

$$\{J_j , J_\ell\} = \epsilon_{j\ell n} J_n , \qquad (2.2)$$

where $\{,\}$ is the Poisson bracket. Let us denote by $K_j$ an $SO(3)$ vector,

$$\{J_j , K_\ell\} = \epsilon_{j\ell n} K_n . \qquad (2.3)$$

We shall assume that $K$ is orthogonal to the angular momentum $J$,

$$K \cdot J = 0 \qquad (2.4)$$



and quadratic in the momentum

$$K_j = Ap_j + Bq_j ,$$
$$A = a_2(r)(xp) + a_1(r) ,$$
$$B = b_4(r)p^2 + b_3(r)(xp)^2 + b_2(r)(xp) + b_1(r) . \qquad (2.5)$$

The problem we study is the following: which are all $H$ and $K_j$ of the form (1.6), (2.5) such that $K_j$ are integrals of motion

$$\{K_j , H\} = 0 \qquad (2.6)$$

satisfying

$$\{K_j , K_\ell\} = \alpha H \epsilon_{j\ell n} J_n \qquad (2.7)$$

with $\alpha$ a constant.

The class of solutions of this problem is certainly not empty. It contains the $1/r$ interaction (1.4) with $K_j$ the Runge-Lenz vector

$$K_j = \frac{1}{m}(p \times J)_j + g\frac{x_j}{r} = -\frac{1}{m}(xp)p_j + \left[\frac{1}{m}p^2 + g\frac{1}{r}\right] x_j . \qquad (2.8)$$

It is straightforward to find all the interactions (1.6) satisfying our restriction. Taking into account (1.6) and (2.5) we write Eqs. (2.6) and (2.7) in the form of a system of 14 nonlinear differential equations ($f' = df/dr^2$ and $C_3(r) = 0$):

$$(2r^2 b_4' - b_4)a_2 - 2b_4^2 = \tfrac{1}{2}\alpha C_1 , \qquad (2.9)$$

$$(2r^2 b_3' + 3b_3)a_2 - 2(a_2 + 2b_4 + 2r^2 b_3)a_2' - 2b_3 b_4 = \tfrac{1}{2}\alpha C_2 , \qquad (2.10)$$

$$(2r^2 b_2' + 2b_2)a_2 - 2(2r^2 a_1' - a_1)b_3 - 2(r^2 a_2' - a_2)b_2 -$$
$$- 2(a_1 a_2)' - 2b_2 b_4 = 0 , \qquad (2.11)$$

$$(2r^2 b_1' + b_1)a_2 - (2r^2 a_1' - a_1)b_2 - 2a_1 a_1' - 2b_1 b_4 = \alpha U, \quad (2.12)$$

$$(a_1 + r^2 b_2)U' = 0, \quad (2.13)$$

$$2(a_2 + 2b_4 + 2r^2 b_3)U' = 2b_1' G_1 + (2r^2 b_1' + b_1)G_2, \quad (2.14)$$

$$2r^2 a_2 U' = b_1 G_1, \quad (2.15)$$

$$(a_1 + r^2 b_2)G_2' = 2b_2' G_1 + (2r^2 b_2' + b_2)G_2, \quad (2.16)$$

$$(b_2 + 2a_1')G_1 + (2r^2 a_1' - a_1)G_2 = 0, \quad (2.17)$$

$$(a_1 + r^2 b_2)G_1' = b_2 G_1, \quad (2.18)$$

$$r^2 a_2 G_2' = (b_3 + 2a_2')G_1 + (2r^2 a_2' - a_2)G_2, \quad (2.19)$$

$$(a_2 + 2b_4 + 2r^2 b_3)G_2' = 2b_3' G_1 + (2r^2 b_3' + b_3)G_2, \quad (2.20)$$

$$r^2 a_2 G_1' = (b_4 + a_2)G_1, \quad (2.21)$$

$$(a_2 + 2b_4 + 2r^2 b_3)G_1' = 2(b_3 + b_4')G_1 + (2r^2 b_4' - b_4)G_2, \quad (2.22)$$

It is clear from (2.13) that either

$$U(r) \equiv \text{const} \quad (2.23)$$

or

$$a_1(r) = -r^2 b_2(r). \quad (2.24)$$

We have accordingly two classes of solutions.

In the case of (2.23) (choosing $U(r) \equiv 0$) we get the first class of solutions of our problem

$$H = \tfrac{1}{2}\eta r^2 f(r)\left[p^2 + \frac{1 - r^4\beta^2(r)}{r^6\beta^2(r)}(xp)^2\right], \quad (2.25)$$



$$K_j = A_2 \tilde{K}_j + A_1 T_j, \quad (2.26)$$

$$\tilde{K}_j = r\sqrt{f(r)}\left[\frac{xp}{r^2\beta(r)} p_j + \left(p^2 - \frac{1 + r^2\beta(r)}{r^4\beta(r)}(xp)^2\right)x_j\right], \quad (2.27)$$

$$T_j = r\sqrt{f(r)}\left[p_j - \frac{1 + r^2\beta(r)}{r^4\beta(r)}(xp)x_j\right]. \quad (2.28)$$

Here, for an arbitrary function, $G_1(r)$, we have set

$$G_1 = \eta|G_1(r)|, \qquad |G_1(r)| = r^2 f(r),$$

$$\beta(r) = \frac{1}{f(r)}\frac{d}{dr^2} f(r) = \frac{d}{dr^2}\ln f(r), \quad (2.29)$$

and $A_1$ and $A_2$ are integration constants. It turns out that

$$\alpha = -2A_2^2\eta \quad (2.30)$$

so that $A_2$ is a normalization constant which should be different from zero if we want to have (2.7) with $\alpha \neq 0$. We obtain in this way an infinity of vectors $K_j$, Eq. (2.26) with $A_2 = \sqrt{-\tfrac{1}{2}\eta\alpha}$ and $A_1$ arbitrary, satisfying the requirements we have imposed. The $T_j$ form a set of three additional integrals of motion, $\{T_j, H\} = 0$ such that

$$\{J_j, T_\ell\} = \epsilon_{j\ell n} T_n, \quad \{T_j, T_\ell\} = 0. \quad (2.31)$$

We see that $J_j$ and $T_j$ close an E(3) (Euclidean group) Lie algebra. A general construction of an E(3) Lie algebra of constants of motion for any spherically symmetric Hamiltonian is given in /24/.

The $J_j$, $T_j$, $\tilde{K}_j$ and $H$ generate an algebra characterized by

$$\{\tilde{K}_j, T_\ell\} = T^2\delta_{j\ell} - T_j T_\ell \quad (2.32)$$

(along with (2.2), (2.3), (2.7) and (2.31)). The 10 constants of motion $J_j$, $T_j$, $\tilde{K}_j$ and $H$ are, of course, not independent. They are related by

$$\tilde{K}_j = (T \times J)_j \tag{2.33}$$

and

$$T^2 = 2\eta H. \tag{2.34}$$

Equations (2.7) and (2.32) as well as

$$K^2 + \alpha HJ^2 = A_1^2 2\eta H, \qquad \tilde{K}^2 + \alpha HJ^2 = 0, \tag{2.35}$$

follow from (2.33) and (2.34). A general discussion of the construction of the O(4) and some other algebras by means of analytic functions of E(3) generators is contained in /26/.

Let us now consider the more general situation when $U$ is not assumed to be a constant. Let $G_1(r)$ be an arbitrary real function, $g_1$, $g_2$, $g_3$, three arbutrary real numbers and define $\eta$, $f(r)$ and $\beta(r)$ as in Eq. (2.29). The functions

$$G_1(r) = \eta r^2 f(r),$$

$$G_2(r) = -\eta f + \eta \frac{f(1-2g_1 f)}{r^4\beta^2}[1 - 4g_1 f + g_2\sqrt{f(1-2g_1 f)}]$$

$$U(r) = g_3[\sqrt{f(1-2g_1 f)} + \tfrac{1}{2}g_2 f], \tag{2.36}$$

and

$$a_1 = 0,$$

$$a_2(r) = \frac{\sqrt{-\frac{1}{2}\eta\alpha}}{r\beta}\sqrt{f(1-2g_1 f)}, \tag{2.37}$$



$$b_1(r) = \frac{1}{r}\eta g_3\sqrt{-\tfrac{1}{2}\eta\alpha}\,[1 - 4g_1 f + g_2\sqrt{f(1-2g_1 f)}],$$

$$b_2(r) = 0,$$

$$b_3(r) = \sqrt{-\tfrac{1}{2}\eta\alpha}\,\frac{\sqrt{f(1-2g_1 f)}}{2r^5\beta^2(r)}[-8g_1 f(1-2g_1 f) + g_2(1-4g_1 f)\sqrt{f(1-2g_1 f)} - 2r^2\beta(1+r^2\beta)],$$

$$b_4(r) = \sqrt{-\tfrac{1}{2}\eta\alpha}\; r\sqrt{f(1-2g_1 f)},$$

satisfy Eqs. (2.9)-(2.19) and (2.21)- (2.22). Inserting them in Eq. (2.20), one obtains

$$\frac{1}{r^4\beta}(8g_1 + g_2^2)(1-2g_1 f)^{3/2} = 0. \tag{2.38}$$

We denote by $\sigma$ the parameter

$$\sigma = 8g_1 + g_2^2. \tag{2.39}$$

It follows from Eq. (2.38) that (2.34), (2.35) represent the solution of our problem only if

$$\sigma = 0 \qquad \text{that is } g_1 = -g_2^2/8. \tag{2.40}$$

The seven integrals of motion $J_j$, $K_j$ and $H$ are not independent. Equations (2.2), (2.3), (2.6), (2.7) and the rotational invariance of $H$ imply the invariance of the function $K^2 + \alpha HJ^2$, , i.e., its Poisson brackets with $J_j$ and $K_j$ are zero. Calculating this function one obtains

$$K^2 + \alpha HJ^2 = -\tfrac{1}{2}\eta\alpha\left[-8g_1 H^2 + 2g_2 g_3 H + g_3^2 + \sigma\frac{a_2^4(xp)^4}{4r^4}\right] \tag{2.41}$$

which in the case of strict Kepler symmetry takes the form

$$K^2 + \alpha H J^2 = -\tfrac{1}{2}\eta\alpha(g_2 H + g_3)^2 . \tag{2.42}$$

We note that the first class of solutions of our problem, Eqs. (2.25)-(2.28) is formally contained in the second. For $\sigma = 0$ and $g_2$ and $g_3$ approaching zero we obtain the first class of solutions except for the E(3) symmetric part, $T_j$, of $K_j$.

We have assumed $G_3(r)=0$. The solution for arbitrary $G_3$, however, is readily obtained if one has already found the solution for $G_3(r)=0$. This follows from the fact that the Poisson bracket algebra is invariant under the replacements

$$p_j \to p_j + \frac{\partial}{\partial x_j} F(x) \tag{2.43}$$

for any $F(x)$. Taking into account that (2.43) with

$$F(x) = \int \frac{r G_3(r) dr}{G_1(r) + r^2 G_2(r)} \tag{2.44}$$

transforms (1.6) into a Hamiltonian of the same form (with the same $G_1$ and $G_2$) and $G_3 = 0$, it only remains to find out in what goes $K_j$ upon the inverse transformation.

It is not difficult to write the solution of any dynamical problem with strict Kepler symmetry. Specifying the values of the energy, $E$, the angular momentum, $J$, and its third projection $J_3 = 0$, we have ($\theta$ is the angle in the plane of motion).

$$\dot r = \sqrt{(G_1 + r^2 G_2)[2E - J^2 G_1/r^2 - 2U]}\,,$$
$$\dot\theta = J G_1 / r^2 . \tag{2.45}$$



Denoting by $W$ the function

$$W(r) = \sqrt{\frac{1 - 2g_1 f(r)}{f(r)}} \tag{2.46}$$

we can write for the trajectory

$$\theta - \theta_0 = -J \int_{r_0}^{r} [(W^2 + g_2 W - 2g_1)\eta(2EW^2 - 2g_3 W +$$
$$+ 4g_1 E - g_2 g_3 - \eta J^2)]^{-1/2}\, dW . \tag{2.47}$$

Taking into account (2.40) we obtain

$$\frac{2}{2W(r) + g_2} = -\eta \frac{g_3 + g_2 E}{J^2}\left[1 + \sqrt{1 + \frac{2\eta E J^2}{(g_3 + g_2 E)^2}} \cos(\theta - \theta_1)\right] \tag{2.48}$$

and it is obvious from this expression that the orbits are closed. For $G_1(r) = 1/m = \text{const}$, $g_2 = 0$, $g_3 = g\sqrt{m}$ one obtains the Hamiltonian (2.4) and the orbit equation (2.48) takes the known form

$$\frac{1}{r} = -\frac{g\sqrt{m}}{J^2}\left[1 + \sqrt{1 + \frac{2EJ^2}{mg^2}} \cos(\theta - \theta_1)\right] . \tag{2.49}$$

We note that Eq. (2.48) can be obtain algerbaically without referring to the differential equations (2.45) if we use the value

$$\frac{K}{\sqrt{-\tfrac{1}{2}\eta\alpha}} = \sqrt{(g_3 + g_2 E)^2 + 2\mu E J^2} \tag{2.50}$$

of the constant of motion $K = |\mathbf{K}|$ which comes out from (2.42). Taking into account that on the one hand

$$K_j x_j = K r \cos(\theta - \theta_1) \tag{2.51}$$

and on the other hand, according (2.5),

$$K_j x_j = r^2 b_4 p^2 + (a_2 + r^2 b_3)(xp)^2 + (a_1 + r^2 b_2)(xp) + r^2 b_1 \quad (2.52)$$

and elliminating $p^2$ from

$$Kr\cos(\theta-\theta_1) = r^2(a_2 + b_4 + r^2 b_3)p^2 - (a_2 + r^2 b_3)J^2 + r^2 b_1 ,$$

$$2E = (G_1 + r^2 G_2)p^2 - G_2 J^2 + 2U \quad (2.53)$$

we obtain

$$2\eta \frac{K}{\sqrt{-\frac{1}{2}\eta\alpha}} \cos(\theta-\theta_1) - 2g_2 E = \eta[-g_2 f + 2\sqrt{f(1-2g_1 f)}]J^2 + 2g_3 \quad (2.54)$$

in the case of strict Kepler symmetry which coincides with (2.48).

To obtain the time dependence on the trajectory we should solve the integral

$$t-t_0 = -\int_{r_0}^{r} (W^2+2g_1)[\eta(W^2+g_2 W-2g_1) \times$$
$$\times (2EW^2 - 2g_3 W + 4g_1 E - g_2 g_3 - \eta J^2)]^{-\frac{1}{2}} dW . \quad (2.55)$$

Taking, for example, $\eta E = -|E| < 0$ the result is $(\sigma = 0)$

$$t - t_0 = \frac{g_3 - g_2 E}{2E\sqrt{2|E|}} [\tau + \sqrt{1-2|E| \frac{J^2 + 2\eta g_2 g_3}{(g_3-g_2 E)^2}} \sin\tau],$$

$$\sqrt{\frac{1-2g_1 f(r)}{f(r)}} = \frac{1}{2} g_2 + \frac{g_3 - g_2 E}{2E} [1 + \sqrt{1-2|E| \frac{J^2 + 2\eta g_2 g_3}{(g_3-g_2 E)^2}} \cos\tau]. \quad (2.56)$$



At the end of this section we find the functions $f(r)$ for which $G_2(r)$ given by Eq. (2.36) vanishes. These $f(r)$ are given by the solution of the differential equation

$$\frac{r^2}{f} \frac{df}{dr^2} = [(1 - 2g_1 f)(1 - 4g_1 f + g_2\sqrt{f(1-2g_1 f)}]^{\frac{1}{2}} , \quad (2.57)$$

which is

$$\frac{1}{f(r)} = 2g_1 + \frac{1}{4}[-g_2 + \sigma \frac{r}{4C} + \frac{C}{r}]^2, \quad (2.58)$$

with $C$ an integration constant. In the case of the strict Kepler symmetry this becomes

$$\frac{1}{f(r)} = \frac{C^2}{4r^2} - \frac{1}{2} g_2 \frac{C}{r} . \quad (2.59)$$

provided $C$ remains fixed when $\sigma$ approaches zero and

$$\frac{1}{f(r)} = mr^2 - g_2\sqrt{m}\, r \quad (2.60)$$

if $C$ also approaches zero in such a way that

$$\lim_{\sigma\to 0} \frac{\sigma}{4C} = 2\sqrt{m} .$$

## 3. *Symmetry Breaking and Precession of the Orbits*

The notion of a strict Kepler symmetry was defined in Sec. 2.It is clear that there are infinitely many ways to break the requirements of a Kepler symmetry in the strict sense. There is, however, a natural possibility to break this symmetry in a minimal way by means of the parameter $\sigma$, Eq. (2.39). This way of symmetry breaking has the virtue that preserves the exact solvability of the dynamical system.

Let us assume for definitness that the following quantity is non-negative

$$\nu^2(E,J) \equiv \frac{1}{4(g_3+g_2E)^2}[\sigma(8g_1E^2 - 2\eta EJ^2 - 2g_2g_3E - g_3^2) + J^4] \geq 0 \qquad (3.1)$$

as it is in the case of strict Kepler symmetry and denote

$$\lambda(E,J) = \frac{1}{2(g_3+g_2E)}[8g_1E - \eta J^2 - g_2g_3] \qquad (3.2)$$

and

$$\lambda_1 = \lambda + \nu, \quad \lambda_2 = \lambda - \nu. \qquad (3.3)$$

Then for the equation of the orbit it follows from Eq.(2.47)

$$F(r) = \mathrm{sn}[\Omega(E,J)(\theta - \theta_0); k] \qquad (3.4)$$

with sn the Jacobi function and

$$F(r) = \frac{W(r) - \lambda_2}{\lambda_1 - W(r)} \sqrt{\frac{-\Lambda(\lambda_1)}{\Lambda(\lambda_2)}}, \qquad (3.5)$$

$$\Omega(E,J) = \frac{-\eta}{(\lambda_1-\lambda_2)\,J} \sqrt{|(\lambda_2^2 + g_2\lambda_2 - 2g_1)\Lambda(\lambda_1)|}, \qquad (3.6)$$

$$k^2 = \frac{(\lambda_1^2 + g_2\lambda_1 - 2g_1)\Lambda(\lambda_2)}{(\lambda_2^2 + g_2\lambda_2 - 2g_1)\Lambda(\lambda_1)}, \qquad (3.7)$$

where

$$\Lambda(\lambda) \equiv 2E\lambda^2 - 2g_3\lambda + 4g_1E - g_2g_3 - \eta J^2. \qquad (3.8)$$



It is clear that the orbits are no longer closed. Let us denote by $r_+$ and $r_-$ two consecutive points on the trajectory, for which $dr/d\theta = 0$. Then we can write for the period of the motion

$$2|\theta(r_+) - \theta(r_-)| = \frac{4K}{\Omega(E,J)}, \qquad (3.9)$$

and calculate the precession $4K/\Omega - 2\pi$. Here

$$K = \int_0^{\pi/2} \frac{d\psi}{\sqrt{1 - k^2 \sin^2\psi}}. \qquad (3.10)$$

It is easily verified that when the symmetry breaking parameter $\sigma$ approaches zero then

$$\lim_{\sigma\to 0} 2|\theta(r_+) - \theta(r_-)| = 2\pi. \qquad (3.11)$$

Let us consider, for instance, small deviations of the $1/r$ law such that $G_2(r) = 0$. Using Eq. (2.58) with $4C = \sigma/2\sqrt{m}$,

$$f(r) = [mr^2 - g_2\sqrt{m}\,r + \frac{3}{8}\sigma - \frac{\sigma g_2}{8\sqrt{m}}\frac{1}{r} + \frac{\sigma^2}{256\,m}\frac{1}{r^2}]^{-1}, \qquad (3.12)$$

we obtain

$$G_1(r) = [m - \frac{g_2\sqrt{m}}{r} + \frac{3\sigma}{8r^2} - \frac{\sigma g_2}{8\sqrt{m}\,r^3} + \frac{\sigma^2}{256\,m\,r^4}]^{-1}. \qquad (3.13)$$

We see that for large enough $r$ the correction to the mass, $m$, is small for small values $g_2$ and $\sigma$. For $r\to\infty$ the mass correction becomes zero,

$$G_1(r) \underset{r\to\infty}{\sim} \frac{1}{m} \qquad (3.14)$$

the potential part of the interaction takes its usual form

$$U(r) \underset{r\to\infty}{\sim} \frac{g_3}{\sqrt{m}}\,\frac{1}{r} \tag{3.15}$$

and the precession vanishes. At extremely small distance we have

$$G_1(r) \underset{r\to 0}{\sim} \frac{256m}{\sigma^2} r^4, \quad U(r) \underset{r\to 0}{\sim} \frac{16 g_3 \sqrt{m}}{\sigma} r \tag{3.16}$$

One may also consider the case when $\sigma \to 0$ with $C = 2b$ fixed in Eq. (2.58). Then $f(r)$ is given by Eq. (2.59) and

$$H = \tfrac{1}{2}\,\frac{r^4}{b^2 - g_2 br} p^2 + \frac{g_3 r}{b^2 - g_2 br} \tag{3.17}$$

The equation of the orbits is

$$\frac{r}{b} = \frac{g_3 + g_2 E}{J^2}\left[1 + \sqrt{1 + \frac{2EJ^2}{(g_3 + g_2 E)^2}}\cos(\theta - \theta_0)\right]. \tag{3.18}$$

We see that for no value of $E$ and $J$ the particle can go to infinity,

$$0 \leq r \leq \frac{b}{J^2}\left[|g_3 + g_2 E| + \sqrt{(g_3 + g_2 E)^2 + 2EJ^2}\right] \tag{3.19}$$

It is confined to a sphera of a radius that can be done arbitrarily small (for $E$, $J$ in any finite interval) by a proper choice of $b$.

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