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OPERATOR SYMBOLS IN THE DESCRIPTION OF OBSERVABLE-STATE SYSTEMS

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Символы операторов в описании систем состояний-наблюдаемых

Исследованы топологические свойства ядер и символов операторов системы состояний-наблюдаемых с конечной степенью свободы N. Для операторов из $\mathfrak{L}^+(\mathcal{S})$ эти ядра и символы оказываются распределениями, а для матриц плотности ρ они являются гладкими функциями.

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Operator Symbols in the Description of Observable-State Systems

For the observable-state system of finite degree of freedom N topological properties of the kernels and symbols belonging to the considered operators are investigated. For the operators of $\mathfrak{L}^+(\mathcal{S})$ kernels and symbols are distributions and for density matrices ρ they are smooth functions.

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I. INTRODUCTION

In the quantum mechanics a physical N-particle system is described by the wave function $\psi(x_1, x_2, \dots, x_{3N}, t) \in L(R^{3N})$, or in short $\psi(x,t)$, which satisfies the Schrödinger equation. The wave function $\psi(x,t)$ is able to describe a quantum mechanical ensemble completely at any time t /8/, if $\psi(x,0)$ is known. With the help of the wave function $\psi(x,t)$ we can determine the expectation value of a physical observable A in the state ψ at the time t by

$$\langle A \rangle = \langle \psi, A\psi \rangle .$$

The normalization of the wave function is given by $\int \psi^* \psi \, dx = 1$. If ψ is the eigenfunction of the operator A, then we get as expectation value of the operator A in state ψ just the eigenvalue. A physical state, which can be described by the wave function ψ, is called pure state. Accordingly, a statistical ensemble consisting of N particles in state ψ is called pure ensemble. Systems with great number of particles N, which can be found in different states $\psi_1, \psi_2, \dots$ are called mixed ensembles. The mean value of an operator A, considered in a mixed state is given by /8/

$$\langle A \rangle = \Sigma \omega_k \langle \psi_k, A\psi_k \rangle ,$$

whereby ω_k is the probability for the system to be in state ψ_k and $\Sigma \omega_k = 1$, $\omega_k \geq 0$.

The pure ensemble is a special case of the mixed ensemble, which occurs if all possibilities

$\omega_k = 0$ but one $\neq 0$, which has to become one because of $\Sigma \omega_k = 1$.

It is common practice to express the mixed states by the so-called density operators ρ /2,8/. So as expectation value of the operator A in state we get

$$\rho(A) = \int A(x, x') \rho(x, x') \, dx \, dx' = \mathrm{tr} A\rho ,$$

where we take the following representation for the operators A and ρ

$$A\psi(x) = \int A(x, x') \psi(x') dx'$$

and

$$\rho(x, x') = \Sigma \omega_k \psi_k^*(x') \psi_k(x) .$$

Here the functions $A(x,x')$ and $\rho(x,x')$ are the kernels of the operators A and ρ, respectively. These kernels can be considered as a kind of symbols of operators. In section III we will deal with kernels and symbols of operators more explicitly.

We can see that $<\psi, \rho\psi> \geq 0$ for any ψ, i.e., $\rho \geq 0$ and we get $\mathrm{tr}\rho = 1$ because $\mathrm{tr}\rho = \int \rho(x,x) dx = \Sigma \omega_k <\psi_k, \psi_k> = 1$ the typical properties of density operators. The treatment of physical problems with the help of density operators plays an important role in quantum statistics and algebraic quantum field theory. This above described approach to the description of quantum processes is contained in the so-called algebraical approach, where the observables form an $*$ - algebra $\mathfrak{A}$ and the states ρ are defined as positive linear functionals on the algebra $\mathfrak{A}$ of observables /4,13/. So the operators A, which stay for the physical observables form an algebra and the states $\rho(A)$ are positive linear functionals over this algebra of observables. Also in the quantum mechanical case the expectation value $<\psi, A\psi> = \mathrm{tr} P_\psi A$ (P_ψ is the projection operator on ψ) is of this form and a positive linear functional on the algebra of operators A. Our N-particle system is a special case of an observable-state system. The algebra of observables is generated by the position and momentum operators $Q_j = x_j$ and $P_j = \frac{1}{i} \frac{\partial}{\partial x_j}$. A natural domain of definition for polynoms in Q and P, $A = \sum_{n,m} a_{nm} Q^n P^m (Q^n = Q_1^{n_1} \dots Q_{3N}^{n_{3N}})$ is the Schwartz space $\mathcal{S} = \mathcal{S}(R^{3N}) \subset L_2(R^{3N})$ of rapidly decreasing functions. Since one is also interested in more complicated observables than polynoms in position and momentum operators, we take for the observable algebra $\mathfrak{A} = \mathcal{L}^+(\mathcal{S})$, the $*$-algebra of all operators A, so that A and also its adjoint A^+ maps $\mathcal{S}$ into itself, i.e., $A, A^+ : \mathcal{S} \to \mathcal{S}$. Special elements of $\mathcal{L}^+(\mathcal{S})$ are the creation, decreation and the number operator

$$A_i^+ = \frac{1}{\sqrt{2}}(x_i - \frac{\partial}{\partial x_i}), \quad A_i = \frac{1}{\sqrt{2}}(x_i + \frac{\partial}{\partial x_i}), \quad N = \Sigma A_i^+ A_i .$$

Let us introduce

$$\mathfrak{S}_1(\mathcal{S}) = \{ \rho \in \mathcal{L}^+(\mathcal{S}), \quad A\rho B \ \text{nuclear for all } A, B \in \mathcal{L}^+(\mathcal{S}) \} .$$

Now we can define the density operator more rigorously.

A density operator ρ is an element of $\mathfrak{S}_1(\mathcal{S})$, which is positive, $\rho \geq 0$ and normed, $\mathrm{tr}\rho = 1$. Any density operator defines a state on the observable algebra $\mathcal{L}^+(\mathcal{S})$ by $\rho(A) = \mathrm{tr}\rho A$. It is a deep mathematical result that these are all possible states on $\mathcal{L}^+(\mathcal{S})$. This question is investigated in /7,12,14/. A complete characterization of operator algebras on which the states are density operators was given by Schmüdgen /11/.

Theorem 1: Every state on $\mathcal{L}^+(\mathcal{S})$, i.e., a linear functional $\rho(A)$, which is positive for $A \geq 0$ and $\rho(I) = 1$, is of the form $\rho(A) = \mathrm{tr}\rho A$, where ρ is a density operator.

$\mathfrak{S}_1(\mathcal{S})$ is the complex linear space generated by all density operators. Let us give yet another equivalent definition of $\mathfrak{S}_1(\mathcal{S})$ which we need in what follows.

Lemma 1 [5]. A bounded operator ρ in L_2 is in $\mathfrak{S}_1(\mathcal{S})$, if and only if ρ and ρ^* maps L_2 into $\mathcal{S}$, i.e., $\mathfrak{S}_1(\mathcal{S}) = \{\rho ; \rho, \rho^* : L_2 \to \mathcal{S}\}$.

II. PHYSICAL TOPOLOGY

It is a consequence of the closed graph theorem that any operator $A \in \mathfrak{L}^+(\mathcal{S})$ is a continuous operator of the Schwartz space $\mathcal{S}[t]$ into itself. We have the canonical imbedding

$\mathcal{S}[t] \subset L_2 \subset \mathcal{S}'[t'],$

where $\mathcal{S}'$ is the space of distributions, the dual space to $\mathcal{S}$ and t' the dual topology. If $F \in \mathcal{S}'$ is a distribution and $\phi \in \mathcal{S}$ so $<F, \phi>$ is well-defined and coincides with the usual scalar product if $F \in L_2$. For $A \in \mathfrak{L}^+(\mathcal{S})$ we can define AF by $<AF, \phi> = <F, A^+\phi>$. In this sense any operator $A \in \mathfrak{L}^+(\mathcal{S})$ is extended to an operator of $\mathcal{S}'[t']$ into itself.

The topology t of the Schwartz space $\mathcal{S}$ is given by the following system of seminorms [9]

$$t: \ ||\phi(x)||'_k = \sup_{\substack{x \\ \ell, |\alpha| \le k}} |(1+x^2)^\ell \frac{\partial^\alpha}{\partial x^\alpha}\phi(x)|, \quad \phi(x) \in \mathcal{S}, \quad k = 0,1,2,\ldots$$

where as usual $\frac{\partial^\alpha}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdot \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \ldots, \ |\alpha| = \max|\alpha_i|$.

This system of seminorms is equivalent to the following one.

$$||\phi(x)||_n = ||T^n \phi(x)||_{L_2},$$

where

$$T = \sum_{i=1}^{3N} \left(-\frac{\partial^2}{\partial x_i^2} + x_i^2\right) = -\Delta + x^2 \ge I.$$

In what follows we shall need also functions $\phi(x,y)$ of two variables $(x,y) \in R^{3N} \times R^{3N}$, which are elements of $\mathcal{S}_2 = \mathcal{S} \otimes \mathcal{S} = \mathcal{S}(R^{6N})$. The above system of seminorms can be written as

$$||\phi(x,y)||_k = ||(T_x + T_y)^k \phi(x,y)||_{L_2},$$

where T_x and T_y are the operators T acting on x, y, respectively.

It is easy to see that this system of seminorms is equivalent to the following one:

$$||\phi(x,y)||_{(k)} = ||T_x^k T_y^k \phi(x,y)||_{L_2}.$$

Further we shall use this system of seminorms if we consider the topology in $\mathcal{S}_2$.

Now we remember the definition of the physical topology β^* on the set of states, which we had introduced in [5,6]. It is the topology of uniform convergence on every bounded set of observables. We define the topology β^* not only on the states, i.e., on the density operators, but on the whole linear space $\mathfrak{S}_1(\mathcal{S})$. $\mathfrak{L}^+(\mathcal{S})$ and $\mathfrak{S}_1(\mathcal{S})$ form a dual pair with respect to the expectation value $\rho(A) = \mathrm{tr}\rho A$. The physical topology β^* in $\mathfrak{S}_1(\mathcal{S})$ is then given by the system of seminorms [5,6]

$$\beta^*: \ q_{\mathfrak{A}}(\rho) = \sup_{A \in} |\rho(A)| < \infty,$$

where $\mathfrak{A}$ runs over all weakly bounded sets in $\mathfrak{L}^+(\mathcal{S})$. With the help of the operator T we can describe the physical topology more explicitely [5].

Theorem 2: The physical topology β^* is given by the system of seminorms $||\rho||_k = ||T^k \rho T^k||$, $k = 0,1,2,\ldots$ where $||\cdot||$ is the usual operator norm.

It is possible to show that the entropy $S = -\mathrm{tr}\rho\ln\rho$, which is uncontinuous with respect to the trace norm $||\rho||_1 = \mathrm{tr}\rho$, is continuous with respect to the physical topology β^* [5].

III. KERNELS AND SYMBOLS

Having investigated the set of linear operators, which are defined on $\mathcal{S}$, L_2 and $\mathcal{S}'$ and regarding their topological properties, we will turn to the symbols of these operators and like-wise investigate the symbols with respect to certain topologies.

A symbol of an operator is a function which is coordinated uniquely to the operator /1/. We will start with the integral representation of the operators and for the present consider the kernels of the operators as one of the possible kinds of symbols of operators and after this investigate general representations of symbols of operators on $\mathcal{S}$ according to their topological properties.

Let $A \in \mathcal{L}^+(\mathcal{S})$ and $\phi(x) \in \mathcal{S}$ then we can express the operator A with the help of a function of two variables $K(x,y)$

$$A\phi(x) = \int K(x,y)\phi(y)\,dy.$$

The kernels $K(x,y)$ can also be distributions, for example

$$I\phi(x) = \int \delta(x-y)\phi(y)dy.$$

Outgoing from this and remembering the fact that for every $A \in \mathcal{L}^+(\mathcal{S})$ $A\mathcal{S} \to \mathcal{S}'$ and because such a map A can be represented as bilinear form on $\mathcal{S}$ by application of the theorem of kernel /9/ we are able to prove

Lemma 2: If the operator A maps the Schwartz space $\mathcal{S}$ into the dual space $\mathcal{S}'$, then the kernel $K(x,y)$ of the operator A is an element of the space of distributions $\mathcal{S}'_2 = \mathcal{S}' \otimes \mathcal{S}'$, i.e., $A \in \mathcal{L}(\mathcal{S},\mathcal{S}') \longleftrightarrow K(x,y) \in \mathcal{S}'_2$.

For any real n, $-\infty < n < +\infty$, we define on $\mathcal{S}$ the norm $\|\phi\|_n = \|T^n\phi\|$ and denote by $\mathcal{H}_n$ the completion of $\mathcal{S}$ with respect to $\|\cdot\|_n$. Then $\{\mathcal{H}_n\}$ is a scale of Hilbert spaces with $\mathcal{S} = \bigcap_{-\infty<n} \mathcal{H}_n$ and $\mathcal{S}' = \bigcup_{n=+\infty}^{-\infty} \mathcal{H}_n$, $\mathcal{H}_0 = L_2$.

Lemma 3: Let $\rho \in \mathfrak{S}_1(\mathcal{S})$ then we can continue ρ to a continual mapping from $\mathcal{S}'$ into $\mathcal{S}$, i.e., $\rho F \subset \mathcal{S}$ for every distribution $F \in \mathcal{S}'$.

Proof: From Lemma 1 we know that $\rho L_2 \to \mathcal{S}$, i.e., for every k there is a constant c so that

$$\|\rho\phi\|_k \le c\|\phi\|_{L_2}.$$

Using this we have for any ρ' the following estimation

$$\|\rho'\phi\|_{k'} = \|\rho' T^n T^{-n}\phi\|_k \le c'\|T^{-n}\phi\|_{L_2} = c'\|\phi\|_{-n}$$

and therefore $\rho : \mathcal{H}_{-n} \to \mathcal{S}$ for every n.
That means $\rho : \mathcal{S}' \to \mathcal{S}$.
From Lemma 3 and the theorem of kernel /9/ we get

Lemma 4: If the operator $\rho \in \mathcal{L}(\mathcal{S}',\mathcal{S})$ then the kernel $\rho(x,y)$ of the operator ρ is an element of the space $\mathcal{S}_2$, i.e., $\rho \in \mathcal{L}(\mathcal{S}',\mathcal{S}) \longleftrightarrow \rho(x,y) \in \mathcal{S} \otimes \mathcal{S}$.

From Lemma 3 and Lemma 4 we see, that the kernels $\rho(x,y)$ of the operators $\rho \in \mathfrak{S}_1(\mathcal{S})$ are functions of $\mathcal{S}_2$. With respect to this correspondence, we have the following theorem

Theorem 3: The physical topology β^* on $\mathfrak{S}_1(\mathcal{S})$ is equal to the topology in $\mathcal{S}_2$.

Proof: The norm in the Schwartz space $\mathcal{S}$ with respect to two variables is, as is pointed out above, given by

$$\|\rho(x,y)\|_{(k)} = \|T_x^k T_y^k \rho(x,y)\|_{L_2}$$

with $T_x = (-\Delta_x + x^2)$ and $T_y = (-\Delta_y + y^2)$.

Now we will consider the operator

$$(T^k \rho T^k \phi)(x) = T^k(\rho T^k\phi)(x) = T^k \int \rho(x,y) T^k \phi(y)dy =$$

$$= T_x^k \int \rho(x,y) T_y^k \phi(y)dy$$

because the operator T^k under the integral acts only on the variable y and the operator T^k in front of the integral only on the variable x. After integration by parts we have

$$(T^k \rho T^k \phi)(x) = \int (T_x^k T_y^k \rho(x,y))\phi(y)dy$$

that means that the kernel to the operator $T^k \rho T^k$ is $T_x^k T_y^k \rho(x,y)$. Since the Hilbert-Schmidt-norm of an operator is equal to the norm of the kernel of this operator, we get

$$||T^k \rho T^k||_{H.S.} = ||T_x^k T_y^k \rho(x,y)||_{L_2} .$$

We must yet estimate the Hilbert-Schmidt-norm and the operator norm, mutually.

On the one hand it is obvious

$$||T^k \rho T^k|| < ||T^k \rho T^k||_{H.S.}$$

and on the other hand we have

$$||T^k \rho T^k||_{H.S.} = ||T^{-1} T^{k+1} \rho T^{k+1} T^{-1}||_{H.S.} \leq$$

$$\leq ||T^{-1}||_{H.S.} \; ||T^{k+1} \rho T^{k+1} T^{-1}||$$

by using the fact that

$$|| A \cdot B||_{H.S.} \leq || A||_{H.S.} \; ||B||$$

and that $T^{-1} \leq I$ is a Hilbert-Schmidt operator. Therefore, we get

$$||T^k \rho T^k||_{H.S.} \leq ||T^{-1}||_{H.S.} \; ||T^{k+1} \rho T^{k+1}||.$$

From Lemma 4 we know that the kernel $\rho(x,y)$ of an operator $\rho \in \mathfrak{S}_1(\mathcal{S})$ is a function of $\mathcal{S}_2$. If we take an operator $A \in \mathcal{L}(\mathcal{S},\mathcal{S}')$ then the kernel $K(x,y)$ is in $\mathcal{S}_2'$ (Lemma 2) and A_ρ has a finite trace

$$\mathrm{tr}\, A\rho = \int A(x,y)\rho(y,x)\,dy\,dx .$$

So we can regard all operators $A \in \mathcal{L}(\mathcal{S},\mathcal{S}')$ as generalized observables, which give us finite expectation values for any state ρ. But $\mathcal{L}(\mathcal{S},\mathcal{S}')$ is not an algebra, since the multiplication of two observables is not always defined. Since $\mathcal{S}_2'$ is the dual space of $\mathcal{S}_2$ we have got the result that the space $\mathcal{L}(\mathcal{S},\mathcal{S}')$ of all generalized observables is the dual space of $\mathfrak{S}_1(\mathcal{S})$.

Now we will turn to the Weyl symbols of operators. Roughly speaking symbols of operators are functions $A(p,q)$ which are related to operators $\hat{A}$ in a linear one-to-one manner $\hat{A} \longleftrightarrow A(p,q)$, so that algebraic operations in the set of operators correspond to appropriate functional operations in the set of symbols.

The mapping $A(p,q) \longleftrightarrow \hat{A}$ is called quantization. It yields a possibility to find for classical functions of the position $q = (q_1,\dots,q_{3N})$ and momentum $p = (p_1,\dots,p_{3N})$ the corresponding quantum mechanical observable $\hat{A}$. It is well-known that such a correspondence is not uniquely determined by physical conditions.

An appropriate quantization is the so-called Weyl quantization, which started from the following correspondence

$$q_i^n \to \hat{q}_i^n = Q_i^n, \; p_i^n \to \hat{p}_i^n = P_i^n$$

$$q_i p_j \to \widehat{q_i p_j} = \frac{1}{2}(Q_i P_j + P_j Q_i) .$$

For a general monom $p_i^n q_j^m$ the corresponding operator $\widehat{p_i^n q_j^m}$ is defined as the (operator valued) coefficient /3/ $\hat{A}_{nm}$ of $\lambda^n \mu^m$ by

$$(\lambda p_i + \mu q_j)^{n+m} = \Sigma \frac{(n+m)!}{\ell!\; k!} \hat{A}_{\ell k} \lambda^\ell \mu^k .$$

By linearity it is now related to any polynom

$$A(p,q) = \Sigma\, a_{nm}\, p^n q^m, \; p^n = p_1^n p_2^n \dots \quad \text{and analog } q^n,$$

a linear operator $\hat{A} = \Sigma\, a_{nm} \widehat{p^n q^m}$. $A(p,q)$ is called the Weyl symbol of the operator $\hat{A}$, and $\hat{A}$ the Weyl quantization of $A(p,q)$. It is straightforward to prove the following

Lemma 5 /3/

i) The Weyl quantization $A(p,q) \to \hat{A}$ defines a one-to-one linear mapping of the set $\mathcal{P}(p,q)$ of all polynoms in p_i, q_i onto the set $\mathcal{P}(P,Q)$ of all operator polynoms in P_i, Q_i.

ii) $\hat{A} \to A(p,q)$ then we get

$$\hat{p}_k \hat{A} \to (p_k - \frac{i}{2}\frac{\partial}{\partial q_k}) A \qquad \hat{A}\hat{p}_k \to (p_k + \frac{i}{2}\frac{\partial}{\partial q_k}) A$$

$$\hat{q}_k \hat{A} \to (q_k + \frac{i}{2}\frac{\partial}{\partial p_k}) A \qquad \hat{A}\hat{q}_k \to (q_k - \frac{i}{2}\frac{\partial}{\partial p_k}) A$$

The Weyl quantization can now extended to an arbitrary distribution $A(p,q) \in \mathcal{S}'_2$. But first we will remark

Theorem 4: Let $A(p,q)$ be a polynom, $\hat{A} \in \mathcal{L}^+(\mathcal{S})$ the corresponding operator by the Weyl quantization and $K(x,y)$ the kernel of $\hat{A}$, then

$$A(p,q) = \int e^{ip\xi} K(q+\xi/2,\ q-\xi/2)\, d\xi \equiv \mathfrak{F}K$$

$$K(x,y) = \frac{1}{(2\pi)^{3N}} \int A(p, \frac{x+y}{2})\, e^{-ip(x-y)} dp \equiv \mathfrak{G}A.$$

This theorem is proved in ref. /3/.

The relations between kernels and symbols of Theorem 4 suggest immidiately the following

Theorem 5: The integral transformations $\mathfrak{F}$, $\mathfrak{G}$ of theorem 4 define one-to-one continuous mappings of $\mathcal{S}'_2$ onto itself. For any $A(p,q) \in \mathcal{S}'_2$ the operator $A \in \mathcal{L}(\mathcal{S}, \mathcal{S}')$ with the corresponding kernel $K(x,y) = \mathfrak{G}A(p,q)$ is called the Weyl quantization of $A(p,q)$ and $A(p,q) = \mathfrak{F}K(x,y)$ is called the Weyl symbol of $\hat{A}$.

Proof: As usual, for a distribution $A(p,q) \in \mathcal{S}'_2$ the integral trandformation $\mathfrak{G}A$ is defined by

$$<\mathfrak{G}A, \phi> = \frac{1}{(2\pi)^{3N}} \iint A(p, \frac{x+y}{2})\, e^{-ip(x-y)}\, \overline{\phi(x,y)}\, dp\, dx\, dy,$$

$$= \frac{1}{(2\pi)^{3N}} \iint \overline{A(p,q)}\, e^{ipz}\, \phi(q+\frac{z}{2}, q-\frac{z}{2})\, dp,$$

$$= <A, \mathfrak{G}^*\phi>.$$

Quite analog is $\mathfrak{F}K$ defined by $<\mathfrak{F}K,\phi> = <K, \mathfrak{F}^*\psi>$.

We yet see that $\mathfrak{G}^* = \frac{1}{(2\pi)^{3N}}\mathfrak{F}$ and $\mathfrak{F}^* = (2\pi)^{3N}\mathfrak{G}$.

Therefore, Theorem 5 is a consequence of the following Lemma

Lemma 6: The integral transformations $\mathfrak{F}$, $\mathfrak{G}$ are continuous one-to-one transformations of the space $\mathcal{S}_2$ into itself. Further, $\mathfrak{F}\cdot\mathfrak{G} = \mathfrak{G}\cdot\mathfrak{F} = I$.

Proof: We show that $A \to \mathfrak{G}A = K$ is continuous. To do this we prove that for every semi-norm $\|\cdot\|'_m$ of $\mathcal{S}_2$ there exists a $s(\geq m)$ so that

$$\|K\|'_m = \|\mathfrak{G}A\|'_m \leq c\|A\|'_s.$$

For this let us first estimate $(1+x^2)^\ell (1+y^2)^k K(x,y)$, where

$$K(x,y) = \frac{1}{(2\pi)^{3N}} \int e^{-ip(x-y)} A(p, \frac{x+y}{2})\, dp,$$

$$= \frac{1}{(2\pi)^{3N}} \int \frac{e^{-ip(x-y)}}{[1+(\frac{x-y}{2})^2]^r} (1-\frac{1}{2}\Delta p)^r A(p, \frac{x+y}{2})\, dp.$$

Here Δp is the Laplace operator acting on p. For r we put later $\ell + k$. From the last relation we get

$$\sup_{x,y} |(1+x^2)^\ell (1+y^2)^k K(x,y)| \leq$$

$$\leq \frac{1}{(2\pi)^{3N}} \int \frac{dp}{(1+p^2)^t} \sup_{x,y} \frac{(1+x^2)^\ell (1+y^2)^k}{[1+(\frac{x-y}{2})^2]^r [1+(\frac{x+y}{2})^2]^r} \times$$

$$\times \sup_{p,q} |(1+p^2)^t (1+q^2)^r (1-\frac{1}{2}\Delta p)^r A(p,q)|.$$

We choose t so large that the integral is finite and $r = \ell + k$. The last supremum can be estimated by $c\|A\|'_s$, where $s \geq \max(t, 2r)$. To estimate the first supremum we apply the inequality

$$1 + (\xi \pm \eta)^2 \leq 2(1+\xi^2)(1+\eta^2) \quad \text{where } \xi, \eta \in R^n .$$

Then we get $(1+x^2)^\ell \leq 2^\ell [1 + (\frac{x-y}{2})^2]^\ell [1 + (\frac{x+y}{2})^2]^\ell$ and an analog for $(1+y^2)^k$. So the first supremum is less than $2^{\ell+k}$.

In a quite analogous way we also can estimate an arbitrary $\sup|(1+x^2)^\ell (1+y^2)^k \frac{\partial^\alpha}{\partial x^\alpha} \frac{\partial^\beta}{\partial y^\beta} K(x,y)|$. Therefore $\mathcal{G}$ is continuous. In the same way one proves the continuity of $\mathcal{F}$. $\mathcal{F} \cdot \mathcal{G} = \mathcal{G}\mathcal{F} = I$ can be shown by a straightforward calculation.

An immediate consequence of Lemma 1 together with Theorem 3 is

<u>Theorem 6:</u> If $\hat{\rho} \in \mathfrak{S}_1(\mathcal{S})$ then its symbol $\rho(p,q)$ is a functional of $\mathcal{S}_2$. The correspondence $\rho(p,q) \longleftrightarrow \hat{\rho}$ defines a one-to-one linear in both directions continuous mapping between the operators of $\mathfrak{S}_1(\mathcal{S})$ with the topology β^* and $\mathcal{S}_2$.

Summing up we can say that we have described the connection between the operators, their kernels and symbols of an N-particle system. We can see that for every density operator, for which the expectation values formed by all polynomial functions of momentum and position are finite, the expectation values formed by the generalized observables $A \in \mathcal{L}(\mathcal{S}, \mathcal{S}')$ are also finite. The correspondence between operators, kernels and symbols can be summarized by the following scheme:

operator $\hat{A}$	$\longleftrightarrow$	kernel $K(x,y)$	$\longleftrightarrow$	symbol $A(p,q)$
$\mathcal{L}(\mathcal{S}, \mathcal{S}')$	$\longleftrightarrow$	$\mathcal{S}'_2$	$\longleftrightarrow$	$\mathcal{S}'_2$
		$\cup$		$\cup$
$\mathfrak{S}_1(\mathcal{S})$	$\longleftrightarrow$	$\mathcal{S}_2$	$\longleftrightarrow$	$\mathcal{S}_2$

with the correspondence of the topologies

$$\beta^* \longleftrightarrow t \longleftrightarrow t$$

We have discussed here the topological relations between operators, symbols and kernels only for density operators, more precisely for $\mathfrak{S}_1(\mathcal{S})$. Of course, since $\mathcal{L}(\mathcal{S}, \mathcal{S}')$ is the dual space of $\mathfrak{S}_1(\mathcal{S})$, we have there the dual topology β in a canonical way, which is related to the topology in the space $\mathcal{S}'_2$ of distributions by the correspondence between the observables A and their kernels or symbols. Another set of observables, smaller than $\mathcal{L}(\mathcal{S}, \mathcal{S}')$ and their natural topology were studied in /10/.

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