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ON THE QUANTIZATION AND NONLOCALITY OF THE QUANTIZED GRAVITATIONAL FIELD

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О квантовании и нелокальности квантованного гравитационного поля

В работе сделана попытка квантовать гравитационное поле в конформно-плоской метрике $ds^2 = \psi(x)(c^2 dt^2 - dx^2 - dy^2 - dz^2)$ используя интегральную формулировку свернутых уравнений Эйнштейна /1,2/ $\psi(x) = \kappa/6 \int S(x,y) T(y) \psi(y) \sqrt{-\eta}\, d^4y + \Lambda'(x)$. Эта формулировка аналогична известному янг-фельдмановскому подходу в электродинамике.

Высказывается гипотеза, что квантовый оператор метрических коэффициентов $\psi(x)$ имеет вид $\hat{\psi} = (\kappa/6) \int S(x,y) \hat{T} \psi \sqrt{-\eta}\, d^4y + \Lambda'(x)$; это соотношение является определением оператора $\hat{\psi}(x)$; и под знаком объемного интегрирования в его правой части только $\hat{T}(y)$ (след тензора энергии-импульса материи) является квантовым оператором.

Показаны самосогласованность формализма и его полная эквивалентность формулизму традиционного квантования.

Показано также, что в квантовом случае это гравитационное поле нелокально и что возникающая нелокальность имеет характер нелокальности, предложенной М.А.Марковым /5,6/.

Работа выполнена в Лаборатории теоретической физики ОИЯИ.

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On the Quantization and Nonlocality of the Quantized Gravitational Field

An attempt is made to quantize a gravitational field in conformally flat metric $ds^2 = \psi(x)(c^2 dt^2 - dx^2 - dy^2 - dz^2)$ using the integral formulation of contracted Einstein equations /1,2/ $\psi = (\kappa/6) \int S(x,y) T \psi \sqrt{-\eta}\, d^4y + \Lambda'$. This formulation is an analog of the well-known Yang-Feldman approach in the electrodynamics.

A hypothesis is made that the quantum operator of the metric coefficients $\psi(x)$ has the form $\hat{\psi} = (\kappa/6) \int S(x,y) \hat{T} \psi \sqrt{-\eta}\, d^4y + \Lambda'(x)$; this relation is just the definition of the operator $\hat{\psi}(x)$; under the sign of the volume integration in the r.h.s. of it only $\hat{T}(y)$ (the trace of energy momentum tensor of the matter) is a quantum operator. The selfconsistency of the formalism and its equivalence with the formalism of traditional quantization are shown. It is shown also that in the quantum case this gravitational field is nonlocal and the appeared nonlocality has a character of that proposed by M.A.Markov /5,6/.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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1. **Besides the usual formulation of Einstein equations**

$$R_{ik} - \tfrac{1}{2} g_{ik} R = \kappa T_{ik} \tag{1}$$

($\kappa = 8\pi k/c^4$, k is the Newtonian gravitational constant) an integral formulation of them is known /1,2/:

$$g_{ik} = \kappa \int S^{\alpha\beta}_{ik} T_{\alpha\beta} \sqrt{-g}\, d^4y + \Lambda_{ik} ; \tag{2}$$

here the bitensor Green function $S^{\alpha\beta}_{ik}(x,y)$ and the free term $\Lambda_{ik}(x)$ obey the following equations:

$$D^{mn}_{ik} S^{\alpha\beta}_{mn} = \delta^{(\alpha}_{(i} \delta^{\beta)}_{k)} \delta^4(x-y), \quad D^{mn}_{ik} \Lambda_{mn} = 0 . \tag{3}$$

The differential operator of eqs. (3) D^{mn}_{ik} is chosen in such a way that eqs. (2) are equivalent with the Einstein eqs. (1); in particular we put

$$D^{mn}_{ik} g_{mn} \equiv G_{ik} \equiv R_{ik} - \tfrac{1}{2} g_{ik} R , \tag{4}$$

so that acting on eqs. (2) by the operator D^{mn}_{ik}, one obtains Einstein equations.

Both Einstein equations (1) and the integral formulation (2) are established only for the observable quantities g_{ik} and T_{ik}. But from the point of view of quantum theory the quantities T_{ik} are the mean values (m.v.) of a quantum operator $\hat{T}_{ik}$:

$$T_{ik} = \langle | \hat{T}_{ik} | \rangle \tag{5}$$

(our assumptions on q-numbers and state vectors

a priori repeat those of ref.[3]). Taking (5) into account one can rewrite (2) as follows

$$g_{ik} = \kappa \int S^{\alpha\beta}_{ik} <|\hat{T}_{\alpha\beta}|> \sqrt{-g}\, d^4 y + \Lambda_{ik} = \tag{2'}$$

$$= <|\{\kappa \int S^{\alpha\beta}_{ik} \hat{T}_{\alpha\beta} \sqrt{-g}\, d^4 y + \Lambda_{ik}\}|>,$$

and, denoting

$$\hat{g}_{ik} \equiv \kappa \int S^{\alpha\beta}_{ik} \hat{T}_{\alpha\beta} \sqrt{-g}\, d^4 y + \Lambda_{ik}, \tag{6}$$

one has eqs. (2) in the form

$$g_{ik} = <|\hat{g}_{ik}|>; \tag{7}$$

hence the m.v. of the q-number $\hat{g}_{ik}$ in the given state is the metric in the given physical situation.

Acting on both sides of eq. (6) by the differential operator D^{mn}_{ik}, we obtain equations for $\hat{g}_{ik}$

$$D^{mn}_{ik} \hat{g}_{mn} \equiv \hat{G}_{ik} = \kappa \hat{T}_{ik}, \tag{8}$$

where $\hat{G}_{ik}$ denotes some differential form of $\hat{g}_{ik}$ whose m.v. is the Einstein tensor G_{ik}:

$$<|\hat{G}_{ik}|> = G_{ik}, \tag{9}$$

so that eqs. (8) reduce to the classical Einstein equations when the m.v. is taken.

Generally speaking, some difficulty in q-numbers formulation of Einstein tensor lies in the fact that quantities g_{ik} are considered as the potentials of the gravitational field and, on the other hand, they are the pure geometrical objects which act on the tensor indices of variables of the given theory (i.e., the metric itself). In the classical theory this difference is not essential - e.g., in the Hamilton principle all components

g_{ik} are varied. On the other hand, the expressions A^i and $A_i = g_{ik} A^k$ are only different representations of the same quantity (the vector A). But in quantization the operators can be connected only with the field variables, and some quantities which determine the metric properties of the space should remain c-numbers (due to their meaning).

This difficulty is radically overcome in the theory based on the linear approximation to the Einstein equations: putting $g_{ik} = \eta_{ik} + h_{ik}$, where η_{ik} is the metric of the flat space and $|h_{ik}| \ll 1$, we use (due to linearity of the approximation) only η_{ik} as the metric, while the components h_{ik} become the dynamical variables which can be replaced by the operator $\hat{h}_{ik}$. In general (not in this linear approximation) to overcome this difficulty the "coordinate" representation is proposed, so that the acting of the operator $\hat{g}_{ik}$ is reduced to the multiplication by the components g_{ik}; nevertheless in this case the difficulty is preserved as well (except the loss of generality); e.g., A_i and A^k do not appear as co- and contravariant components of the same quantity, but rather as physically different quantities $\hat{A}_i \equiv g_{ik} A^k$ and A^k (e.g., A_i and A^i are commuted with $\pi^{ik} \sim \partial/\partial g_{ik}$ in different ways).

In the general case such a distinction between the potentials and geometrical quantities in the Einstein tensor means that we introduce the Riemannian space a differential operator D^{mn}_{ik} which acts on the symmetric tensors ϕ_{ik} defined in this space so that this operator (it depends on the metric of the given space, of course) in acting on the metric tensor g_{ik} gives the Eisntein tensor:

$$(D^{mn}_{ik} \phi_{mn})_{\phi_{mn} = g_{mn}} = G_{ik}. \tag{10}$$

In particular, finding of such a differential operator is the main problem of integral formulation of the Einstein equations[1,2]. As soon as such an operator is found, i.e., Einstein equations are written in the form which is analogous to that of linear theories

$$D^{mn}_{ik}\, g_{mn} = \kappa T_{ik} \,, \tag{11}$$

we can write the quantum variant of these equations as well in the form of (8):

$$D^{mn}_{ik}\, \hat{g}_{mn} = \kappa \hat{T}_{ik} \,. \tag{8'}$$

In so doing the components g_{ik} which are contained in D^{mn}_{ik} remain c-numbers, of course, and the operator $\hat{g}_{ik}$ enters into (8') in a linear manner (but the superposition principle has no place since the components g_{ik} are the m.v.'s of $\hat{g}_{ik}$). The solution of eq. (8) is $\hat{g}_{ik} = \hat{g}_{ik}(g_{mn})$ (since $D^{mn}_{ik} = D^{mn}_{ik}(g_{ab})$), where g_{ik} is to be considered as a parameter (not as a given field, of course); to know the value of this parameter for the given system, one has to solve Einstein equations (1) or (2); in this formalism it means that, after eq. (8) is solved, one has to solve eq. (7), where the m.v. is taken on the state vector of the given system (or one can take a m.v. of (8) in the given state). This formal situation is just analogous to that in the case of any field in general relativity: one has to solve both field equations and Einstein equations for the given physical system (however one can consider a simplified problem with given g_{ik}).

Thus we have a q-number $\hat{g}_{ik}$, whose m.v. is g_{ik} and whose motion eqs. are (8). Certainly, for consequent quantization we have to be sure of the fact that eqs. (8) for $\hat{g}_{ik}$ can be formulated in a canonical way; in this case we may consider operator $\hat{g}_{ik}$ as the quantum operator associated with the metric g_{ik}.

To check up the possibility of canonical quantization in such a way let us consider the formally simplified case of conformally flat spaces. Such a consideration is useful not only in the aspect of principle, but also due to the fact that such models as the Einstein, de Sitter and Friedman ones belong to the class of conformally flat spaces.

2. First of all we should note that the formalism developed in ref. /2/ is to be transformed to a more suitable (for our purpose) form. Namely, in ref. /2/ the spaces with metric $g_{ik} = \phi^2(x)\,\eta_{ik}$ were considered; here η_{ik} is the metric of the flat space, and $\phi(x)$ obeys the equation

$$\Box\phi = \frac{\kappa}{6} T \phi^3 \tag{12}$$

($\Box$ is the covariant with respect to η_{ik} d'Alembertian, and $T = T^i_i$ is the trace of the matter energy-momentum tensor); this equation is the contraction of Einstein equations

$$-R = \kappa T \,. \tag{13}$$

The integral equation in this case has the form

$$\phi(x) = \frac{\kappa}{6}\int G T \phi^3 \sqrt{-\eta}\, d^4y + \Lambda(x) \,, \quad \eta = \det \eta_{ik} \,, \tag{14}$$

where the Green function $G(x, y)$ and the free term $\Lambda(x)$ obey the equations

$$\Box G(x,y) = \delta^4(x-y) \,, \quad \Box\Lambda(x) = 0 \,. \tag{15}$$

In this case the operator D^{mn}_{ik} is reduced to d'Alembertian $\Box$. Acting by it on eq. (14) we obtain eq. (12) due to eq. (15).

Let us put now $\phi^2(x) = \psi(x)$, so that the metric has the form $g_{ik} = \psi(x)\eta_{ik}$; $\psi(x)$ coincides (with an accuracy to a sign) with metric tensor components in Cartesian coordinate frame. The transformation of the formalism to this new potential is easy. Indeed, from (14) we have

$$\psi(x) = \frac{\kappa}{6}\int S T \psi \sqrt{-\eta}\, d^4y + \Lambda'(x) \,, \tag{16}$$

where the new Green function has the form $S(x, y) = \psi^{1/2}(x) G(x, y) \psi^{1/2}(y)$ and obeys the equation

$$DS \equiv \frac{1}{\psi}\Box S - \frac{1}{\psi^2}\eta^{ik}\psi_{,i}S_{,k} -$$

$$-\frac{1}{2\psi^2}(\Box\psi - \frac{3}{2\psi}\eta^{ik}\psi_{,i}\psi_{,k})S = \delta^4(x-y) \qquad (17)$$

(comma denotes a partial derivative). The free term $\Lambda'(x)$ has the form

$$\Lambda' = \int \frac{1}{\psi}(S_{,i}\psi - S\psi_{,i})\sqrt{-\eta}\, d\Sigma^i \qquad (18)$$

(here the integration is made over the surface Σ covering the region of volume integration in (16)) and obeys the equation

$$D\Lambda' = 0. \qquad (19)$$

In acting on (16) by the operator D, we obtain the analog of eqs. (11)

$$D\psi \equiv [\frac{1}{\psi}\Box - \frac{1}{\psi^2}\eta^{ik}\psi_{,i}\partial_k -$$

$$-\frac{1}{2\psi^2}(\Box\psi - \frac{3}{2\psi}\eta^{ik}\psi_{,i}\psi_{,k})]\psi = \frac{\kappa}{6}T\psi, \qquad (20)$$

i.e., we obtain the Einstein equation contraction (13)

$$\frac{1}{2\psi}(\Box\psi - \frac{1}{2\psi}\eta^{ik}\psi_{,i}\psi_{,k}) = \frac{\kappa}{6}T\psi \qquad (21)$$

(the scalar curvature in this case is $R = -3\psi^{-2}[\Box\psi - (2\psi)^{-1}\eta^{ik}\psi_{,i}\psi_{,k}]$).

Substitute now into (16) a quantum operator $\hat{T}$ instead of the c-number T; in so doing we obtain some operator $\hat{\psi}$ which is equal to

$$\hat{\psi}(x) = \frac{\kappa}{6}\int S\hat{T}\psi\sqrt{-\eta}\, d^4y + \Lambda'(x). \qquad (22)$$

Multiplying (17) by $\hat{\psi}\sqrt{-\eta}$ and integrating over d^4x, we obtain after simple transformations that in this case

$$\Lambda' = \int \psi^{-1}(S_{,i}\hat{\psi} - S\hat{\psi}_{,i})\sqrt{-\eta}\, d\Sigma^i. \qquad (23)$$

The operator $\hat{\psi}(x)$ obeys an equation which is derived from (22) like (20) is derived from (16)

$$D\hat{\psi} \equiv \frac{1}{\psi}\Box\hat{\psi} - \frac{1}{\psi^2}\eta^{ik}\psi_{,i}\hat{\psi}_{,k} -$$

$$-\frac{1}{2\psi^2}(\Box\psi - \frac{3}{2\psi}\eta^{ik}\psi_{,i}\psi_{,k})\hat{\psi} = \frac{\kappa}{6}\hat{T}\psi. \qquad (24)$$

The m.v. of the r.h.s. of eq. (22) coincides with r.h.s. of eq. (16). Hence,

$$\langle|\hat{\psi}|\rangle = \psi; \qquad (25)$$

m.v. of (24) gives (20). Therefore eq. (24) is the q-number variant of eq. (21).

As for the nonuniqueness of the choice of the operator D (mentioned in ref. /2/) we would like to note that the given choice (i.e., $\Box$ for ϕ and the related one of eq. (24)) results in a conformally invariant equation for $\hat{\psi}(x)$; indeed, if one introduces a tensor $\hat{\psi}_{ik} = \hat{\psi}\eta_{ik}$ (like $g_{ik} = \psi\eta_{ik}$), eq. (24) can be rewritten identically as

$$(\nabla^m\nabla_m - R/6)\hat{\psi}^i_i = \frac{2\kappa}{3}\hat{T}^i_i, \qquad (26)$$

where $\nabla^m\nabla_m$ is a d' Alembertian covariant with respect to metric g_{ik}, and $\hat{\psi}^i_i \equiv g^{ik}\hat{\psi}_{ik}$. The property of conformal invariance is natural in this case since all conformally flat spaces are conformal to each other; at the same time other possible choices of D (noted in /2/) do not imply conformally invariant equations.

3. Let us make sure now of the fact that the quantity $\hat\psi(x)$ obeying eq. (24) is a canonical variable. A Lagrangian density related to (24) is *

$$\mathfrak{L} = -\frac{3}{2\kappa}\frac{1}{\psi}\eta^{ik}\left(\hat\psi_{,i}\hat\psi_{,k} - \frac{1}{\psi}\hat\psi\hat\psi_{,i}\psi_{,k} + \frac{1}{4\psi^2}\hat\psi^2\psi_{,i}\psi_{,k}\right) + \mathfrak{L}_m, \qquad (27)$$

where $\mathfrak{L}_m$ is a Lagrangian density of matter; below the material sources will be omitted, but no transformations (possible via $R=0$ when $T=0$) of the r.h.s. of (24) will be done, i.e., we shall consider a homogeneous equation

$$D\hat\psi = 0. \qquad (28)$$

The canonically conjugated with $\hat\psi$ field π is

$$c\pi = \frac{\partial\mathfrak{L}}{\partial\hat\psi_{,0}} = -\frac{3}{\kappa\psi}\left(\hat\psi_{,0} - \frac{1}{2\psi}\psi_{,0}\hat\psi\right). \qquad (29)$$

The Hamiltonian is

$$\mathfrak{H} = c\pi\hat\psi_{,0} - \mathfrak{L} = -\frac{\kappa c^2}{6}\psi\pi^2 + \frac{1}{2\psi}\dot\psi\pi\hat\psi - \frac{3}{2\kappa}\frac{1}{\psi}\left[(\vec\nabla\hat\psi)^2 - \frac{1}{\psi}\hat\psi\vec\nabla\hat\psi\vec\nabla\psi + \frac{1}{4\psi^2}\hat\psi^2(\vec\nabla\psi)^2\right] \qquad (30)$$

(here $\vec\nabla = \partial/\partial\vec r$) and it is (with an accuracy up to a factor $\sqrt{-g}$) a component t^0_0 of the canonical energy-momentum tensor of the field $\hat\psi$:

$$t^k_i\sqrt{-g} = -\frac{3}{2\kappa}\frac{1}{\psi}\eta^{kn}\left(2\hat\psi_{,n}\hat\psi_{,i} - \hat\psi_{,i}\hat\psi_{,n}\right) - \delta^k_i\mathfrak{L}, \qquad (31)$$

so that the Hamilton function $H = \int\mathfrak{H}d^3x$ is the energy of the field $\hat\psi$ **

$$H = \int t^0_0\sqrt{-g}\,d^3x. \qquad (32)$$

In varying the Hamilton function with respect to π and $\hat\psi$, one obtains that the system of canonical equations

$$c\pi_{,0} = -\delta H/\delta\hat\psi, \quad c\hat\psi_{,0} = \delta H/\delta\pi \qquad (33)$$

is equivalent to eq. (28). Now we can pass to quantization of the field $\hat\psi$ putting $\hat\psi$ and π to be operators with commutation relations

$$[\hat\psi(\vec x), \pi(\vec x')] = i h\delta(\vec x - \vec x'),$$
$$[\pi(\vec x), \pi(\vec x')] = [\hat\psi(\vec x), \hat\psi(\vec x')] = 0, \qquad (34)$$

as usual. Considering Hamiltonian (30) as an operator (and symmetrizing the product $\pi\hat\psi$)) we obtain that equations

$$\dot{\hat\psi} = \frac{i}{h}[H, \hat\psi], \quad \dot\pi = \frac{i}{h}[H, \pi] \qquad (35)$$

coincide formally with eqs. (33) and (28).

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*In what follows η_{ik} is chosen (due to its triviality) to be diag(1,–1,–1,–1).

**This energy is indefinite just like the (canonical) energy of gravitational field itself /4/; The (canonical) energy of classical gravitational field in conformally flat spaces is nonpositive.

Therefore, equation (28) is the canonically quantized equation (21) and the operator $\hat{\psi}$ is the quantum operator of metric coefficients ψ.

4. The next step in traditional schemes of quantization is a transition to the Fourier representation. It is clear, however, that in the case of any Riemannian space the expansion of a metric into plane waves $\sim \exp(ikx)$ is useless (generally speaking).

But in the case of conformally flat spaces one can use something like such an expansion. Let us consider the field $\hat{\psi}$ as usual in a cube (with colume V) and put

$$\hat{\psi}(\vec{x}) = [\psi(x)/3V]^{1/2} \sum_{\vec{k}} q_{\vec{k}} e^{i\vec{k}\vec{x}}, \qquad \pi(\vec{x}) = [3/V\psi(x)]^{1/2} \sum_{\vec{k}} p_{\vec{k}} e^{-i\vec{k}\vec{x}}. \tag{36}$$

From (36) and (34) one has

$$q_{\vec{k}} = (3/V)^{1/2} \int d\vec{x}\, \hat{\psi}\, \psi^{-1/2} e^{-i\vec{k}\vec{x}}, \quad p_{\vec{k}} = (3V)^{-1/2} \int d\vec{x}\, \pi \psi^{1/2} e^{i\vec{k}\vec{x}}, \tag{37}$$

and

$$[q_{\vec{n}}, p_{\vec{k}}] = i h \delta_{\vec{k}\vec{n}}, \quad [p_{\vec{k}}, p_{\vec{n}}] = [q_{\vec{k}}, q_{\vec{n}}] = 0. \tag{38}$$

Substituting (36) into (30) we obtain

$$H = -(1/2\kappa) \sum_{\vec{k}} (\kappa^2 c^2 p_{\vec{k}} p^*_{\vec{k}} + \vec{k}^2 q_{\vec{k}} q^*_{\vec{k}}) + + (1/4V) \sum_{\vec{k},\vec{n}} (p_{\vec{k}} q_{\vec{n}} + q_{\vec{n}} p_{\vec{k}}) \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}}\, \dot{\psi}/\psi. \tag{39}$$

The momentum of the field in volume V is

$$\vec{P} = (i/2) \sum_{\vec{k}} \vec{k} (q_{\vec{k}} p_{\vec{k}} + p_{\vec{k}} q_{\vec{k}}) - - (1/4V) \sum_{\vec{k},\vec{n}} (p_{\vec{k}} q_{\vec{n}} + q_{\vec{n}} p_{\vec{k}}) \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}} (\vec{\nabla}\psi)/\psi. \tag{40}$$

The second sums in (39) and (40) disappear in flat space (when $\psi \equiv 1$) and are, therefore, specific properties of a curved space.

We note that operators $p_{\vec{k}}$ and $q_{\vec{k}}$ depend on the time explicitly in accordance with (37):

$$\partial p_{\vec{k}}/\partial t = (2V)^{-1} \sum_{\vec{n}} p_{\vec{n}} \int d\vec{x}\, e^{i(\vec{k}-\vec{n})\vec{x}}\, \dot{\psi}/\psi, \qquad \partial q_{\vec{k}}/\partial t = -(2V)^{-1} \sum_{\vec{n}} q_{\vec{n}} \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}}\, \dot{\psi}/\psi. \tag{41}$$

If one puts $H = H_0 + \Delta H$, where

$$H_0 \equiv -(2\kappa)^{-1} \sum_{\vec{k}} (\kappa^2 c^2 p_{\vec{k}} p^*_{\vec{k}} + \vec{k}^2 q_{\vec{k}} q^*_{\vec{k}}) \tag{42}$$

corresponds to a set of oscillators with nonpositive energies and

$$\Delta H \equiv (4V)^{-1} \sum_{\vec{k},\vec{n}} (p_{\vec{k}} q_{\vec{n}} + q_{\vec{n}} p_{\vec{k}}) \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}}\, \dot{\psi}/\psi, \tag{43}$$

then a direct computation via (41) and (43) gives

$$\partial p_{\vec{k}}/\partial t = (i/h)[p_{\vec{k}}, \Delta H], \quad \partial q_{\vec{k}}/\partial t = (i/h)[q_{\vec{k}}, \Delta H], \tag{44}$$

and we obtain

$$\dot{p}_{\vec{k}} \equiv \partial p_{\vec{k}}/\partial t + (i/h)[H, p_{\vec{k}}] = (i/h)[H_0, p_{\vec{k}}] = (\vec{k}^2/\kappa) q^*_{\vec{k}}, \qquad \dot{q}_{\vec{k}} \equiv \partial q_{\vec{k}}/\partial t + (i/h)[H, q_{\vec{k}}] = (i/h)[H_0, q_{\vec{k}}] = -\kappa c^2 p^*_{\vec{k}}. \tag{45}$$

Eqs. (35) are reduced now to equations of a formally simple type, but it should be noted that here $\dot{p}_{\vec{k}}$ and $\dot{q}_{\vec{k}}$ differ essentially from $dp_{\vec{k}}/dt$ and $dq_{\vec{k}}/dt$, since both H and H_0 explicitly depend on time and the transition to Heisenberg or Dirac (with $H_{int} = \Lambda H$) picture in a trivial way is impossible. However $\dot{H}_0 = 0$ and this confirms its interpretation as the Hamilton function of the set of oscillators with definite (nonpositive) energies; thus operator H_0 adds a negative energy to that of material system in the given gravitational field * (or it decreases the positive energy of this system).

Furthermore one can use another expansion instead of (36); letting

$$\hat{\psi}(\vec{x}) = [\kappa c^2 \psi(x)/3V]^{1/2} \sum_{\vec{k}} q_{\vec{k}} e^{i\vec{k}\vec{x}},$$

$$\pi(\vec{x}) = [3/\kappa c^2 V \psi(x)]^{1/2} [\sum_{\vec{k}} p_{\vec{k}} e^{-i\vec{k}\vec{x}} + (\dot{\psi}/2\psi) \sum_{\vec{k}} q_{\vec{k}} e^{i\vec{k}\vec{x}}], \tag{36'}$$

so that

$$q_{\vec{n}} = (3/\kappa c^2 V)^{1/2} \int d\vec{x}\, \hat{\psi} \psi^{-1/2} e^{-i\vec{n}\vec{x}},$$

$$p_{\vec{n}} = (\kappa c^2/3V)^{1/2} \int d\vec{x}\, \pi \psi^{1/2} e^{i\vec{n}\vec{x}} - (3/4\kappa c^2 V)^{1/2} \int d\vec{x}\, \hat{\psi} \dot{\psi} \psi^{-3/2} e^{i\vec{n}\vec{x}}, \tag{37'}$$

one has

$$H = -2^{-1} \sum_{\vec{k}} (p_{\vec{k}} p^*_{\vec{k}} + c^2 \vec{k}^2 q_{\vec{k}} q^*_{\vec{k}}) + (8V)^{-1} \sum_{\vec{k},\vec{n}} q_{\vec{k}} q_{\vec{n}} \int d\vec{x} (\dot{\psi}/\psi)^2 e^{i(\vec{k}+\vec{n})\vec{x}}; \tag{39}$$

The commutators (38) are preserved, and, though

$$\partial q_{\vec{n}}/\partial t = -(2V)^{-1} \sum_{\vec{k}} q_{\vec{k}} \int d\vec{x}\, e^{i(\vec{k}-\vec{n})\vec{x}} \dot{\psi}/\psi,$$

$$\partial p_{\vec{n}}/\partial t = (2V)^{-1} \sum_{\vec{k}} p_{\vec{k}} \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}} \dot{\psi}/\psi + (2V)^{-1} \sum_{\vec{k}} q_{\vec{k}} \int d\vec{x} (2\psi^{-2} \dot{\psi}^2 - \psi^{-1} \ddot{\psi}) e^{i(\vec{k}+\vec{n})\vec{x}}, \tag{41'}$$

they do not depend on time:

$$\frac{\partial}{\partial t}[p_{\vec{k}}, q_{\vec{n}}] = \frac{\partial}{\partial t}[q_{\vec{k}}, q_{\vec{n}}] = \frac{\partial}{\partial t}[p_{\vec{k}}, p_{\vec{n}}] = 0, \tag{46}$$

(the same for (38), of course). For $\dot{p}_{\vec{n}}$ and $\dot{q}_{\vec{n}}$ one has

$$\dot{q}_{\vec{n}} = -p^*_{\vec{n}} - (2V)^{-1} \sum_{\vec{k}} q_{\vec{k}} \int d\vec{x}\, e^{i(\vec{k}-\vec{n})\vec{x}} \dot{\psi}/\psi,$$

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*It can be seen from the structure of Weyl tensor that conformally flat models always are the solutions of inhomogeneous Einstein equations (but the trivial case of flat space), i.e., they are the solutions of Einstein eqs. with a source T_{ik} or, at least, with a source λg_{ik} (e.g., de Sitter model; λ-term is to be considered here as a material source as well). Thereby in the framework of Einstein theory, we always have to assume the existence of material source with definite positive energy.

$$\dot p_{\vec n} = c^2\vec n^2 q^*_{\vec n} + (2V)^{-1}\sum_{\vec k} p_{\vec k}\int d\vec x\, e^{i(\vec n-\vec k)\vec x}\,\dot\psi/\psi + \qquad (45')$$

$$+ (2V)^{-1}\sum_{\vec k} q_{\vec k}\int d\vec x\left(\frac{3}{2}\psi^{-2}\dot\psi^{2}-\psi^{-1}\ddot\psi\right)e^{i(\vec n+\vec k)\vec x} .$$

The field $\hat\psi$ is represented as a set of oscillators with variable frequencies when $\dot\psi/\psi \equiv \chi = \chi(t)$ (e.g., in the flat Fridman model and, in some approximation, in its other types), namely

$$H = -2^{-1}\sum_{\vec k}(p_{\vec k}p^*_{\vec k} + \Omega^2 q_{\vec k}q^*_{\vec k}),\quad \Omega^2 = c^2\vec k^2 - \chi^2/4; \qquad (47)$$

Ω^2 is indefinite, and it is of interest that the condition $\Omega^2 \geq 0$ results in $\lambda \equiv 2\pi/|\vec k| \leq 4\pi c/|\chi| \sim$ radius of our Universe (since $|\chi| \sim t^{-1} \sim 10^{-17}$sec).

5. In Sec. 3 quantization has been performed for a given time moment. Let now all operators be time dependent in such a way that on every surface $t = \mathrm{const}$ they coincide with the previous operators and $\hat\psi_0(x) = c\hat{\dot\psi}(\vec x)$, and let us compute the commutator $[\hat\psi(x), \hat\psi(x')]$ between $\hat\psi$'s taken at different space-time points x and x'. This commutator obeys the equation of type (28)

$$D[\hat\psi(x), \hat\psi(x')] = 0 ; \qquad (48)$$

furthermore one has

$$[\hat\psi(x), \hat\psi(x')]_{t=t'} = 0,$$

$$(\partial/\partial t)[\hat\psi(x), \hat\psi(x')]_{t=t'} = [\hat{\dot\psi}(\vec x), \hat\psi(\vec x')] =$$

$$= (ih\kappa c^2/3)\psi\,\delta(\vec x - \vec x'), \qquad (49)$$

(these are obtained from (29) and (34)). If one denotes

$$[\hat\psi(x), \hat\psi(x')] = (ih\kappa c/3)\psi^{1/2}(x)\Delta(x,x')\psi^{1/2}(x'), \qquad (50)$$

then for $\Delta(x,x')$ one has via (48) and (49)

$$\Box\Delta(x,x') = 0 ,$$

$$\Delta(x,x')|_{t=t'} = 0,\ \partial\Delta(x,x')/\partial t|_{t=t'} = c\,\delta(\vec x - \vec x'); \qquad (51)$$

this implies the function $\Delta(x,x')$ to be the usual function $\Delta(x-x')$:

$$\Delta(a) = (a^0/2\pi|a^0|)\delta(\eta_{ik}a^i a^k) . \qquad (52)$$

The other commutarors can be derived from (50) e.g.,

$$[\pi(x), \hat\psi(x')] = -\frac{ih}{c}\psi^{-1/2}(x)\frac{\partial\Delta(x-x')}{\partial t}\psi^{1/2}(x'),$$

$$[\pi(x), \pi(x')] = -\frac{3ih}{\kappa c^3}\psi^{-1/2}(x)\frac{\partial^2\Delta(x-x')}{\partial t\,\partial t'}\psi^{-1/2}(x'). \qquad (50')$$

In transition $t \to t'$ commutators (50), (50') reduce to corresponding ones (34). Thus, the performed quantization is invariant with respect to the usual Lorentz transformations of the flat space. (It is reasonable to develop the general covariant quantization for fields $\hat g_{ik}$ of the general type, of course).

6. Let us now discuss some consequences of the developed quantization.

The relations (29) and (34) lead to the commutator between $\hat{\dot\psi}$ and $\hat\psi$

$$[\hat{\dot\psi}(x), \hat\psi(\vec x')] = (ih\kappa c^2/3)\psi\,\delta(\vec x - \vec x'). \qquad (53)$$

The fact that $\hat\psi$ and $\hat{\dot\psi}$ do not commute means that the simultaneous measurement of the related quantities is impossible, and here it leads to additional (as compared with traditional field theories) results due to factor ψ in the r.h.s. of (53).

Since the measuring of any quantity f takes place always in some space-time region V·T and the measured quantity f_M is

$$f_M = (VT)^{-1} \int_{VT} f\, dV\, dT \qquad (54)$$

(so that one means the value f at a given point to be the limit of f_M when $V, T \to 0$), we should average (53) in some volume VT:

$$[\hat{\dot{\psi}}_M, \hat{\psi}_M] = (8i\pi hk/3c^2V)\psi_M . \qquad (55)$$

Strictly speaking one has to consider $\hat{\dot{\psi}} = \hat{\dot{\psi}}(\vec{x}, t)$ and $\hat{\psi} = \hat{\psi}(\vec{x}, t)$ and use the commutator of the type (50'), but in the first order the latter has the form of (53). The commutator (55) yields

$$\Delta\dot{\psi}_M \Delta\psi_M \geq (4\pi hk/3c^2V)\psi_M . \qquad (56)$$

It is clear that

$$\Delta\dot{\psi}_M \sim \Delta\psi_M /T , \qquad (57)$$

hence (56) yields

$$(\Delta\psi_M)^2 \geq (4\pi hkT/3c^2V)\psi_M , \qquad (58)$$

that appears to be rather natural - the larger is volume V and the quicker is the measuring, the less is inaccuracy produced by the process of measuring itself. But in virtue of the causality principle the measuring corresponding to a finite volume cannot be performed during an arbitrary short period T. If the considered volume is a sphere whose radius is r, then at least $T \geq r/c$. Substituting this T and $V = 4\pi r^3/3$ into (58) we obtain

$$(\Delta\psi_M)^2 \geq (hk/c^3 r^2)\psi_M \qquad (59)$$

or, introducing the Planck length $\ell_0 = (hk/c^3)^{1/2}$,

$$\Delta\psi_M \geq (\ell_0/r)\sqrt{\psi_M} . \qquad (59')$$

Therefore, the notion of a metric as that measured at a point (i.e., the limit of ψ_M at $r \to 0$) losses its meaning; the latter can be preserved only for large enough domains. To define more precisely this conclusion we should note that the measurable length corresponding to the coordinate difference r is

$$\ell_r = \int_0^r dr\sqrt{-g_{11}} = \int_0^r dr\sqrt{\psi} = r\sqrt{\psi_r} , \qquad (60)$$

where ψ_r is the mean value of ψ in the interval (0, r). With sufficiently high accuracy we can put $\psi_M = \psi_r$, then (59') yields

$$\Delta\psi_M \geq (\ell_0/\ell_r)\psi_M ; \qquad (61)$$

i.e., the metric has its meaning only in the regions whose linear dimensions are larger than ℓ_0 , while already at $\ell_r = \ell_0$ the inaccuracy in measuring of ψ_M cannot be less than ψ_M itself, and this result does not depend on the manner of measuring in virtue of the fundamental principles of the quantum theory.

Due to the fundamental character of the metric its immeasurability with an accuracy higher than $(\ell_0/\ell_r)\psi_M$ leads to immeasurability of every quantity (distributed in the space) in small enough regions. Indeed, the measurable length related to coordinate difference 2r is equal to $\ell = 2r\sqrt{\psi_r}$, therefore according to (59') the error of this length measurement $\Delta\ell$ always is

$$\Delta\ell \geq \ell_0 \qquad (62)$$

without any dependence on the length and manner of its measuring. Thus the Planck length appears to be shortest measurable length. As can be easily seen, this result implies the principal nonlocality of the quantum field theory involving gravitation, and this nonlocality has the character of that proposed by M.A.Markov /5,6/ . Indeed, the appearance of a minimal measurable length ℓ_0 means that at lengthes shorter than ℓ_0 no experimental distinction of any characteristic of some object is possible: if such distinction was possible, the measuring of a length shorter than ℓ_0 would be possible too (namely as the distance between the points of the object with different values of this characteristics). Hence the notion of a field characteristic (e.g., the potential) as a coordinate function losses its direct meaning of the value of some field quantity at the given point (since the field has no determined characteristics not only at a point but also in a region smaller than ℓ_0^3).

It is worth noting that this nonlocality concerns just the observable quantities - the field functions and the distances between objects of the system with different values of this functions. These distances are determined via the metric. On the other hand, the coordinates of these objects are not observable but are set by the arbitrariness of the observer. Thus the relation (62) does not imply impossibility of representation of physical quantities as coordinate functions. It means only that the exact location of an object with definite characteristics and coordinates is impossible. To know the localization we are to know the metric, but knowledge of the latter is restricted by (61), hence the accuracy of location is restricted by (62). Thus the representation of fields as coordinate functions losses its traditional meaning as the value of the field at the given point /5,6/ and becomes an auxiliary notion just like the coordinates themselves do in general relativity.

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APPENDIX

One can introduce operators $a_{\vec{k}}$ and $a^+_{\vec{k}}$ instead of $p_{\vec{k}}$ and $q_{\vec{k}}$ of (36):

$$q_{\vec{k}} = (hc\kappa/2|\vec{k}|)^{1/2}\,(a_{-\vec{k}} + a^+_{\vec{k}}),$$
$$p_{\vec{k}} = i(h|\vec{k}|/2c\kappa)^{1/2}\,(a^+_{-\vec{k}} - a_{\vec{k}}), \qquad \text{(A1)}$$

(and analogous ones for (36")). Then (38) yields

$$[a_{\vec{n}}, a^+_{\vec{k}}] = \delta_{\vec{k}\vec{n}}, \quad [a_{\vec{k}}, a_{\vec{n}}] = [a^+_{\vec{k}}, a^+_{\vec{n}}] = 0, \qquad \text{(A2)}$$

and (39), (40), (41), (45) can be rewritten as follows:

$$P_m = (h/2)\sum_{\vec{k}} k_m (a^+_{\vec{k}} a_{\vec{k}} + a_{\vec{k}} a^+_{\vec{k}}) +$$
$$+(ih/4V)\sum_{\vec{k},\vec{n}} (|\vec{k}|/|\vec{n}|)^{1/2} (a^+_{-\vec{k}} a^+_{\vec{n}} - a_{\vec{k}} a_{-\vec{n}} +$$
$$+ a^+_{-\vec{k}} a_{-\vec{n}} - a^+_{\vec{n}} a_{\vec{k}}) \int d\vec{x}\, e^{i(\vec{k}-\vec{n})\vec{x}}\, \psi_{,m}/\psi,$$
$$P_m = (H/c, \vec{P}), \quad k_m = (-|\vec{k}|, \vec{k}), \qquad \text{(A3)}$$

and

$$\partial a_{\vec{k}}/\partial t = (4V)^{-1} \sum_{\vec{n}} [(|\vec{n}|/|\vec{k}|)^{1/2} - (|\vec{k}|/|\vec{n}|)^{1/2}]\, a_{\vec{n}} \int d\vec{x}\, e^{i(\vec{n}-\vec{k})\vec{x}}\, \dot{\psi}/\psi -$$
$$-(4V)^{-1} \sum_{\vec{n}} [(|\vec{n}|/|\vec{k}|)^{1/2} + (|\vec{k}|/|\vec{n}|)^{1/2}]\, a^+_{\vec{n}} \int d\vec{x}\, e^{-i(\vec{k}+\vec{n})\vec{x}}\, \dot{\psi}/\psi, \qquad \text{(A4)}$$

and

$$\dot{a}_{\vec{k}} = i|\vec{k}|c a_{\vec{k}}, \quad \dot{a}^{+}_{\vec{k}} = -i|\vec{k}|c a^{+}_{\vec{k}}. \tag{A5}$$

REFERENCES

1. *Altshuler B.L. Zh.Exp.Teor.Fiz., 1966, 51, p.1143, 1968, 55, p.1311. Linden-Bell D. Mon.Not. R.Astr. Soc., 1968, 135, p.413; Sciama D.W.,* Waylen P.C., *Gilman R.C. Phys. Rev., 1970, D2, p.1400.*
2. *Mal"tsev V.K., Markov M.A. JINR, E2-9722, Dubna, 1976; Proceedings (Trudy) of Lebedev Physical Institute, 1977, v.96, p.11, Moscow.*
3. *Dirac P.A.M. Lectures on Quantum Field Theory. Yeshiva University, New York, 1967.*
4. *Pauli W. Relativitatstheorie. Enc.math.Wiss., 1921, B5, no. 4.*
5. *Markov M.A. On the Nature of Matter, Moscow, 1976.*
6. *Markov M.A. JINR, E2-8838, Dubna, 1975; ICTP Preprint IC/75/168 (1975).*

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