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V.K.Mal'tsev

ON CANONICAL FORMULATION OF FIELD THEORIES WITH SINGULAR LAGRANGIANS

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* *Institute for Nuclear Research of the Academy of Sciences of the USSR, Moscow*

Мальцев В.К. E2 - 11265

О канонической формулировке теорий поля с сингулярными лагранжианами

В работе сделана попытка ввести формализм функций Рауса в теорию поля: только "невырожденные" компоненты рассматриваются как канонические переменные. Рассмотрены электродинамика и общая теория относительности. Формализмы оказывается достаточно простым и не зависящим от калибровки.

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Mal'tsev V.K. E2 - 11265

On Canonical Formulation of Field Theories with Singular Lagrangians

An attempt is made to introduce the Routh function formalism into the field theory: only "nondegenerated" field components are considered as canonical variables. Electrodynamics and general relativity are considered. The formalism appears to be quite simple and gauge-independent.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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It is well known that the direct Hamiltonian formulation of the field theories with singular Lagrangians is not easy. The Hessian degeneracy in such cases is usually attributed to some restrictions imposed on some momenta. In other words it is supposed that these momenta do exist but for some reasons, they are "degenerated". Nevertheless, difrefent ways of developing in this case of the Hamiltonian formalism are known.

However, there is one more possibility, namely, not to consider the "degenerated" coordinates as canonical ones, so that the conjugated momenta do not appear at all (e.g., a certain momentum is not equal to zero identically as is usually considered, but it does not exist). Certainly, the Hamiltonian formulation is now impossible. However, the so-called Routh function is known in mechanics /1/. This function is defined to be a Hamilton function with respect to some coordinates and a Lagrange one with respect to the others, while the choice of canonical and noncanonical coordinates is determined by the properties of the problem to be solved. It is natural in the case of singular Lagrangian to consider only "nondegenerated" coordinates to be canonical. Thus the canonical character is not forced upon the coordinates that have not it explicitly.

In mechanics the Routh function formalism is known well enough. In what follows at attempt is made to introduce this formalism in the field theory (by analogy with the Hamilton function formalism) applying it to some concrete fields.

ELECTRODYNAMICS

Classical case. The Lagrangian density $\mathfrak{L}$ of the electromagnetic field A_i ($F_{ik} = A_{k,i} - A_{i,k}$) with a source j^k is

$$\mathfrak{L} = -\frac{1}{4} F_{ik} F^{ik} + A_k j^k \tag{1}$$

(Latin indices run from 0 to 3, Greek ones run from 1 to 3, c=1, metric tensor $\eta_{ik} = \mathrm{diag}(-1,1,1,1,)$). Euler-Lagrange equations yield the Maxwell ones

$$F^{sk}_{,s} = - j^k , \tag{2}$$

obeying the identity

$$-j^k_{,k} = F^{sk}_{,sk} \equiv 0 ; \tag{3}$$

one equation appears to be a consequence of the others and to describe the given field one component A_i (any) appears to be superfluous.

In attempting of the canonical formulation one discovers immediately that $A_{0,0}$ ($\equiv \dot{A}_0$) is not contained in the Lagrangian density at all. In spite of this fact many ways of developing of the Hamiltonian formalism exist /2/. However let us introduce instead of the Hamilton function the Routh one considering only A_α as canonical coordinates. In so doing only three coordinates will be determined canonically, but just this number of components is enough to describe the field.

Conjugated to A_α momenta π^α are

$$\pi_\alpha = \partial \mathfrak{L} / \partial \dot{A}_\alpha = - F^{\circ\alpha} ; \tag{4}$$

Using these one obtains the Routh function density (by analogy with mechanics definition of Routh function):

$$\mathfrak{R} = \frac{1}{2}\pi^2_\alpha + \pi_\alpha A_{0,\alpha} + \frac{1}{4} F^2_{\alpha\beta} + A^\circ j^\circ - A_\alpha j_\alpha . \tag{5}$$

Considering Routh function $R = \int \mathfrak{R}\, d^3x$, one obtains

$$(\delta/\delta A_0) \int R\, dt = -j^\circ - \pi_{\alpha,\alpha} = -j^\circ - F^{\alpha\circ}_{,\alpha} , \tag{6}$$

so that the equation (6) = 0 is the zeroth component of (2). On the other hand

$$\delta R / \delta \pi_\alpha = \pi_\alpha + A_{0,\alpha} , \quad \delta R / \delta A_\alpha = - j_\alpha + F_{\alpha\beta,\beta} ; \tag{7}$$

the equation $\dot{A}_\alpha = \delta R / \delta \pi_\alpha$ coincides with (4); eliminating via it the momenta π_α from $\dot{\pi}_\alpha + \delta R / \delta A_\alpha$, one obtains

$$\dot{\pi}_\alpha + \delta R / \delta A_\alpha = - j^\alpha - F^{i\alpha}_{,i} . \tag{8}$$

Thus the Maxwell equations (2) are represented by the system

$$(\delta/\delta A_0) \int R\, dt = 0 , \tag{9}$$

$$\dot{A}_\alpha = \delta R / \delta \pi_\alpha , \quad \dot{\pi}_\alpha = - \delta R / \delta A_\alpha , \tag{9'}$$

and (9') has the explicit canonical form.

For the time derivative of any quantity f one has

$$df/dt = \partial f / \partial t + \{R, f\} , \tag{10}$$

where $\partial f/\partial t$ is taken at constant A_α and π_α, and the Poisson brackets are

$$\{g, f\} \equiv \int \left(\frac{\partial g}{\delta \pi_\alpha} \frac{\partial f}{\delta A_\alpha} - \frac{\partial g}{\delta A_\alpha} \frac{\partial f}{\delta \pi_\alpha} \right) d^3x ; \tag{11}$$

it is easy to check that

$$\{\pi_\alpha(\vec{x}), A_\beta(\vec{x}')\} = \delta_{\alpha\beta} \delta(\vec{x} - \vec{x}') , \tag{12}$$

$$\{\pi_\alpha, \pi_\beta\} = \{A_\alpha, A_\beta\} = \{A_0, \pi_\alpha\} = \{A_0, A_\alpha\} = 0 .$$

Let us condiser now the Fourier transformation of the field

$$A^n = \frac{1}{\sqrt{V}} \sum_{\vec{k}} q^n_{(\vec{k})} e^{i\vec{k}\vec{x}} , \quad \pi^\alpha = \frac{1}{\sqrt{V}} \sum_{\vec{k}} p^\alpha_{(\vec{k})} e^{-i\vec{k}\vec{x}} ,$$

$$j^n = \frac{1}{\sqrt{V}} \sum_{\vec{k}} J^n_{(\vec{k})} e^{-i\vec{k}\vec{x}} \qquad (13)$$

the Routh function in this representation is

$$R = \frac{1}{2} \sum_{\vec{k}} (\vec{p}_{(\vec{k})} \vec{p}^*_{(\vec{k})} + [\vec{k} \times \vec{q}_{(\vec{k})}][\vec{k} \times \vec{q}^*_{(\vec{k})}] - \sum_{\vec{k}} (i\vec{k}\vec{p}_{(\vec{k})} - J^\circ_{(\vec{k})}) q^\circ_{(\vec{k})} - \sum_{\vec{k}} \vec{q}_{(\vec{k})} \vec{J}_{(\vec{k})} , \qquad (14)$$

here $\vec{k} \times \vec{q}$ is the vector product (indices $(\vec{k})$ will be often omitted below). Varying (14) one has

$$\delta R/\delta q_0 = i\vec{p}\vec{k} - J^\circ, \quad \delta R/\delta \vec{p} = \vec{p}^* + i\vec{k} q_0 , \qquad (15)$$
$$-\delta R/\delta \vec{q} = \vec{J} + \vec{k} \times (\vec{k} \times \vec{q}^*),$$

so that the equations

$$\delta R/\delta q_0 = 0, \quad \dot{\vec{q}} = \delta R/\delta \vec{p} , \quad \dot{\vec{p}} = -\delta R/\delta \vec{q} \qquad (16)$$

are the Fourier components of (9), (9').

Considering the (canonical) energy-momentum tensor

$$T^k_i = A_{s,i} F^{sk} - \delta^k_i \mathfrak{L} , \qquad (17)$$

one obtains that the energy density T^0_0 coincides with the Routh function density (as one should expect from the obvious analogy with mechanics). If now all variables are considered as solutions of the field equations, then decomposing $\vec{p}$ and $\vec{q}$ into longitudinal (||) and transverse ($\perp$) components one has the energy

$$E = R^{(s)} = \frac{1}{2} \sum_{\vec{k}} (\vec{p}_\perp \vec{p}^*_\perp + \vec{k}^2 \vec{q}_\perp \vec{q}^*_\perp) + \frac{1}{2} \sum_{\vec{k}} \frac{1}{\vec{k}^2} \vec{J}_0 \vec{J}^*_0 - \sum_{\vec{k}} \vec{q}\vec{J} . \qquad (18)$$

Thus the free field energy is defined by transverse components only, as one should expect.

<u>Quantum case.</u> The quantization is obvious. Considering the canonical variables as operators one obtains from (12) that the only nonvanishing commutator is

$$[A_\alpha(\vec{x}), \pi_\beta(\vec{x}')] = i h \delta_{\alpha\beta} \delta(\vec{x} - \vec{x}') \qquad (19)$$

(there is no reason to quantize A_0, hence). Interpreting now R as an operator one obtaines that the definition (in accordance with (10))

$$\dot{\pi}_\alpha = \frac{i}{h}[R, \pi_\alpha], \quad \dot{A}_\alpha = \frac{i}{h}[R, A_\alpha] \qquad (20)$$

leads to equations which coincide formally with (9'); at the same time eq. (9) formally remains unchanged. For Fourier components one has

$$[q_{\alpha(\vec{k})}, p_{\beta(\vec{n})}] = i h \delta_{\alpha\beta} \delta_{\vec{k}\vec{n}} , \qquad (21)$$

and the definition

$$\dot{\vec{p}} = \frac{i}{h}[R, \vec{p}], \quad \dot{\vec{q}} = \frac{i}{h}[R, \vec{q}] \qquad (22)$$

yields the Fourier components of (9'). Introducing the creation and annihilation operators one can easily be convinced that the free field energy and momentum are provided only by $\vec{a}_\perp$ and $\vec{a}^+_\perp$ while $\vec{a}_{||}$ and $\vec{a}^+_{||}$ appear only due to the interaction with the source (and scalar photons are absent from the very beginning). Thus the canonical quantization of the electrodynamics appears formally to be rather easy (almost like that of the scalar field) and it should be emphasized that the traditional results are obtained without any gauge condition: no restrictions were imposed on A_k and π_α (but the only convention that A_0 is not a canonical variable). Now one can introduce a gauge that is suitable for a concrete problem to be solved, since the general formalism of quantization appears to be gauge-invariant.

GENERAL RELATIVITY

The linear approximation. This case has no peculiarities of principle as compared with the electrodynamics, though the "gauge-unnecessity" of the proposed formalism makes the advantages of it to be more vivid when applying it to the set of ten components h_{ik}. Letting $g_{ik} = \eta_{ik} + h_{ik}$, one has the Lagrangian density

$$\mathfrak{L} = \sqrt{-g}\, g^{ik} \left(\Gamma^m_{in}\Gamma^n_{km} - \Gamma^m_{ik}\Gamma^n_{mn}\right) \tag{23}$$

(here $16\pi\kappa = c = 1$, κ is the Newtonian constant) in this approximation

$$\mathfrak{L} = \frac{1}{4}\big(2h_{kn,m}\, h^{km,n} - h_{mn,k}\, h^{mn,k} - 2h^{kn}_{,n}\, h^m_{m,k} + h^{n,k}_n\, h^m_{m,k}\big). \tag{24}$$

Now $\partial\mathfrak{L}/\partial\dot{h}_{on} \neq 0$, but these quantities do not contain any velocity, thus we would consider only $h_{\alpha\beta}$ as canonical variables; in so doing one has

$$\pi_{\alpha\beta} = \partial\mathfrak{L}/\partial\dot{h}_{\alpha\beta} = \frac{1}{2}\big(\dot{h}_{\alpha\beta} - \eta_{\alpha\beta}\dot{h}_{\nu\nu} - h_{o\alpha,\beta} - h_{o\beta,\alpha} + \eta_{\alpha\beta}\, h_{o\nu,\nu}\big), \tag{25}$$

and the Routh function density is

$$\begin{aligned}\mathfrak{R} = {} & \pi_{\alpha\beta}\dot{h}_{\alpha\beta} - \mathfrak{L} = \pi^2_{\alpha\beta} - \frac{1}{2}\pi^2_{\alpha\alpha} + 2\pi_{\alpha\beta}\, h_{o\alpha,\beta} - \\ & - \frac{1}{2}\pi_{\alpha\alpha} h_{o\beta,\beta} + \frac{1}{2} h_{o\alpha,\alpha}\dot{h}_{oo} - \frac{1}{2}(h_{oo,\beta} + h_{\alpha\alpha,\beta})\dot{h}_{o\beta} + \\ & + \frac{1}{4}\Big(-\frac{3}{2}h^2_{o\alpha,\alpha} + 2h^2_{o\alpha,\beta} + 2h_{o\alpha,\beta}\, h_{o\beta,\alpha} - \\ & - 2h_{k\alpha,\beta}\, h^k_{\beta,\alpha} + h^{nk}_{,\alpha}\, h_{nk,\alpha} + 2h_{\beta\alpha,\alpha} h^n_{n,\beta} - h^n_{n,\alpha}\, h^m_{m,\alpha}\Big).\end{aligned} \tag{26}$$

Varying the Routh function R one can see that the equation $\dot{h}_{\alpha\beta} = \delta R/\delta\pi_{\alpha\beta}$ coincides with (25); hence, one has

$$\pi_{\alpha\beta} + \delta R/\delta h_{\alpha\beta} = G^{\alpha\beta}, \tag{27}$$

$$(\delta/\delta h_{oo})\int R\, dt = G^{oo}, \qquad (\delta/\delta h_{o\alpha})\int R\, dt = 2G^{o\alpha},$$

where G^{ab} are components of the Einstein tensor in this approximation

$$G^{ab} = \frac{1}{2}\big(-h^{ab,n}_{,n} - h^{n,ab}_n + h^{na,b}_{,n} + h^{nb,a}_{,n} + \eta^{ab} h^{n,m}_{n,m} - \eta^{ab} h^{mn}_{,mn}\big). \tag{28}$$

Thus the system

$$(\delta/\delta h_{on})\int R\, dt = 0, \tag{29}$$

$$\dot{h}_{\alpha\beta} = \delta R/\delta\pi_{\alpha\beta}, \quad \dot{\pi}_{\alpha\beta} = -\delta R/\delta h_{\alpha\beta} \tag{30}$$

represents the equations $G_{ab} = 0$. It is worth recalling that the eqs. (29) (noncanonical ones) are not equations at all but only some initial conditions since G^{on} are integrals for the (canonical) motion equations (30).

Let us transform the density (24) into the following

$$\tilde{\mathfrak{L}} = \mathfrak{L} + \frac{1}{2}(h_{o\alpha,\alpha} h_{oo} - h_{\alpha\alpha,\beta} h_{o\beta})_{,o} + \qquad (31)$$

$$+ \frac{1}{2}(\dot h_{\alpha\alpha} h_{o\beta} - \dot h_{o\beta} h_{oo})_{,\beta} .$$

Now

$$\partial\tilde{\mathfrak{L}}/\partial\dot h_{on} = 0, \qquad (32)$$

$$\tilde\pi_{\alpha\beta} = \partial\tilde{\mathfrak{L}}/\partial\dot h_{\alpha\beta} = \frac{1}{2}(\dot h_{\alpha\beta} - \eta_{\alpha\beta}\dot h_{\nu\nu} - h_{o\alpha,\beta} - h_{o\beta,\alpha} + 2\eta_{\alpha\beta} h_{o\nu,\nu}), \qquad (33)$$

and the Routh function density is

$$\tilde{\mathfrak{R}} = \tilde\pi^2_{\alpha\beta} - \frac{1}{2}\tilde\pi^2_{\alpha\alpha} + 2\tilde\pi_{\alpha\beta} h_{o\alpha,\beta} + \frac{1}{4}(2h^2_{o\alpha,\beta} +$$

$$+ 2h_{o\alpha,\beta} h_{o\beta,\alpha} - 4h^2_{o\alpha,\alpha} - 2h_{k\alpha,\beta} h^{k}_{\beta,\alpha} + \qquad (34)$$

$$+ 2h_{\beta\alpha,\alpha} h^{m}_{m,\beta} + h^{nk}_{,\alpha} h_{nk,\alpha} - h^{n}_{n,\alpha} h^{m}_{m,\alpha}),$$

(eqs. (29), (30) remain unchanged, of course) and (34) coincides with the canonical energy density due to (32). Considering now the Fourier transformation in a cube V

$$h^{mn} = \frac{1}{\sqrt{V}}\sum_{\vec k} q^{mn}_{(\vec k)} e^{i\vec k\vec x}, \quad \tilde\pi^{\alpha\beta} = \frac{1}{\sqrt{V}}\sum_{\vec k} p^{\alpha\beta}_{(\vec k)} e^{-i\vec k\vec x}, \qquad (35)$$

one has

$$\tilde R = \frac{1}{2}\sum_{\vec k}[2p_{\alpha\beta}p^*_{\alpha\beta} - p_{\alpha\alpha}p^*_{\beta\beta} + k_\mu k_\nu(q_{\mu\nu}q^*_{\lambda\lambda} - q_{\lambda\mu}q^*_{\lambda\nu}) + \qquad (36)$$

$$+ \frac{1}{2}k^2_\lambda(q_{\mu\nu}q^*_{\mu\nu} - q_{\mu\mu}q^*_{\nu\nu})] +$$

$$+ \frac{1}{2}\sum_{\vec k}(q^*_{\nu\nu}k_\lambda - q^*_{\lambda\nu}k_\lambda k_\nu)q_{oo} + 2i\sum_{\vec k} k_\alpha p_{\alpha\beta} q_{o\beta} .$$

If all the variables are solutions of (29), (30) then the energy is

$$E = \tilde R^{(s)} = \frac{1}{2}\sum_{\vec k_\perp}(2p^\perp_{\alpha\beta}p^{*\perp}_{\alpha\beta} - p^\perp_{\alpha\alpha}p^{*\perp}_{\beta\beta} + \frac{1}{2}k^2_\lambda q^\perp_{\mu\nu}q^{*\perp}_{\mu\nu}), \qquad (37)$$

where $p^\perp_{\alpha\beta}$, $q^\perp_{\alpha\beta}$ are the transverse components of $p_{\alpha\beta}$ and $q_{\alpha\beta}$ – all other components disappear just like in the case of electrodynamics (the same holds for the field momentum $\vec P$ as well). Now one can proceed to quantization without any care of a gauge.

<u>The general case</u>. The Lagrange density (23) does not contain squares of $\dot g_{on}$, hence only $g_{\alpha\beta}$ (the intrinsic components) are to be considered as canonical variables in correspondence with the above programme; however the density (23) is known to result in cumbersome expressions in canonical formalism. For this reason we would consider the density proposed by Dirac /3/

$$\tilde{\mathfrak{L}} = \frac{1}{4}\sqrt{-g}\{(e^{\alpha\beta}e^{\mu\nu} - e^{\alpha\mu}e^{\beta\nu}) \times \qquad (38)$$

$$\times (g^{oo}\dot g_{\mu\nu} - 4g^{oo}g_{o\mu,\nu} - 4g^{o\lambda}\Gamma_{\lambda,\mu\nu})\dot g_{\alpha\beta} +$$

$$+[(g^{ik}g^{mn} - g^{im}g^{kn})g^{\nu\lambda} + 2(g^{im}g^{n\nu} - g^{i\nu}g^{mn})g^{k\lambda}]g_{ik,\nu}g_{mn,\lambda}\}$$

(here $e^{\alpha\beta} = g^{\alpha\beta} - g^{o\alpha}g^{o\beta}/g^{oo}$); this one coincides with (23) with accuracy to some divergence-type terms. Now

$$\pi^{\alpha\beta} = \partial\tilde{\mathfrak{L}}/\partial\dot g_{\alpha\beta} = \sqrt{-g}(e^{\alpha\mu}e^{\beta\nu} - e^{\alpha\beta}e^{\mu\nu})\Gamma^{o}_{\mu\nu} . \qquad (39)$$

The Routh function density is

$$\mathfrak{R} = -\frac{1}{g^{oo}\sqrt{-g}}(g_{\alpha\mu}g_{\beta\nu} - \frac{1}{2}g_{\alpha\beta}g_{\mu\nu})\pi^{\alpha\beta}\pi^{\mu\nu} +$$

$$+ 2\pi^{\alpha\beta}(g_{o\alpha,\beta} + 2g^{o\nu}\Gamma_{\nu,\alpha\beta}/g^{oo}) -$$

$$-\frac{1}{4}\sqrt{-g}\{\frac{1}{g^{oo}}(e^{\alpha\mu}e^{\beta\nu}-e^{\alpha\beta}e^{\mu\nu})[g_{o\alpha,\beta}+g_{o\beta,\alpha})g^{oo}+2g^{o\lambda}\Gamma_{\lambda,\alpha\beta}]\times \qquad (40)$$

$$\times[(g_{o\mu,\nu}+g_{o\nu,\mu})g^{oo}+2g^{o\sigma}\Gamma_{\sigma,\mu\nu}]+$$

$$+[(g^{ik}g^{mn}-g^{im}g^{kn})g^{\lambda\nu}+2(g^{im}g^{n\nu}-g^{i\nu}g^{mn})g^{k\lambda}]g_{ik,\nu}g_{mn,\lambda}\}.$$

Then the equation $\dot{g}_{\alpha\beta}=\delta R/\delta\pi^{\alpha\beta}$ coincides with (39) and eliminating $\pi^{\alpha\beta}$ via it from the Routh equations one obtains that the system

$$(\delta/\delta g_{on})\int R\,dt = 0, \qquad (41)$$

$$\dot{g}_{\alpha\beta}=\delta R/\delta\pi^{\alpha\beta}, \quad \dot{\pi}^{\alpha\beta}=-\delta R/\delta g_{\alpha\beta}, \qquad (41')$$

represents the (homogeneous) Einstein equations.

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