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# ON INTERACTION WITH EXTERNAL FIELDS. I.

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О взаимодействии с внешними полями. I

Для взаимодействия квантованного заряженного поля с любыми зависящими от времени классическими внешними полями рассматривается S -матрица в N -упорядочной форме. Характеризуется распределение пар, раждающихся из вакуума.

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On Interaction with External Fields. I

For interaction of any charged field with arbitrary external fields S-matrix is considered in N-ordered form. Distribution of pairs, created from vacuum is characterized.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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## I. INTRODUCTION

The quantum electrodynamics with an external classical electromagnetic field served as an important model, when contemporary covariant quantum field theory was born[1-8]. The theory of interactions with external fields remains actual in connection with current investigations of phenomena in high intensity fields (see, e.g.[9-21]), and, in particular, of pair creation by black holes[22].

In the framework of quantum electrodynamics with an external electromagnetic field Feynman discussed repeatedly the problem of connection between spin and statistics[1,2]. He stressed the importance to make clear the intimate connection between the Dirac equation and the Pauli principle. Nikishov treated this problem as the one of the unitarity of S-matrix, and the unitarity condition was checked both for particular electromagnetic fields and for general ones[10-12]. Recently in ref.[18] the problem of unitarity has been investigated using the Bogolubov transformation. (Note that the S-matrix unitarity condition in spinor electrodynamics with an arbitrary external electromagnetic field was directly checked by Schwinger long ago [5], using the same covariant terms, which were implied by Feynman (in refs.[10-12,18] other terms were used). However, a more detailed information concerning mechanism of satisfying this condition is needed to inspect the connection between spin and statistics (cf. also [23])).

In Sec. 2 the normal form of S-matrix for interaction of any quantized charged field with arbitrary classical external fields is represented. We employ the usual second quantization formalism (Feynman and Schwinger avoided it). In Sec. 3 a distribution, which is generated by external fields from vacuum, is characterized. Appendix A contains a formalism of the particle number operator. In Appendix B a combinatorial treatment of N-ordering of S-matrix and unitarity condition is outlined.

## 2. NORMAL FORM OF S-MATRIX

Let $\mathrm{L}$ be an interaction Lagrangian of a quantized charged field, interacting with any external classical fields (electromagnetic, gravitational, etc.). We suppose only that this interaction Lagrangian $\mathrm{L}$ is, as usual, bilinear in the charged field, which may be scalar, spinor, vector, Duffin-Kemmer spin 0 or 1 field or any other. Counterterms of mass and wave function renormalizations can be added. In any case the S-matrix N-ordered with respect to charged field may be written as follows

$$S = T e^{i\mathrm{L}} =$$

$$= e^{i\overline{\mathrm{L}} + \frac{i^2}{2}\overline{\mathrm{L}\mathrm{L}} + \frac{i^3}{3}\overline{\mathrm{L}\mathrm{L}\mathrm{L}} + \frac{i^4}{4}\overline{\mathrm{L}\mathrm{L}\mathrm{L}\mathrm{L}} + \cdots} :e^{i\mathrm{L} + i^2\overline{\mathrm{L}\mathrm{L}} + i^3\overline{\mathrm{L}\mathrm{L}\mathrm{L}} + i^4\overline{\mathrm{L}\mathrm{L}\mathrm{L}\mathrm{L}} + \cdots}: \tag{1.a}$$

$$= e^{B} :e^{C}: = \langle 0|S|0\rangle :e^{C}: = e^{B} :e^{G-\mathcal{N}}: \quad , \tag{1.b}$$

where the Wick chronological pairings are implied. We shall illustrate general formulas of such a kind using spinor quantum electrodynamics as an example (in the $\beta$-formalism for spins 0, 1, $\frac{3}{2}$, ... all looks analogously). So, in spinor electrodynamics

$$\mathrm{L} = ie\int d^4x\, \hat{\bar{\psi}}(x)\gamma_\mu \hat{\psi}(x) A_\mu(x) \tag{2}$$

$$\overline{\mathrm{L}\mathrm{L}} = (ie)^2\int d^4x_1 d^4x_2\, \hat{\bar{\psi}}(x_1)\gamma_{\mu_1}\overline{\hat{\psi}(x_1)\,\hat{\bar{\psi}}(x_2)}\gamma_{\mu_2}\hat{\psi}(x_2)A_{\mu_1}(x_1)A_{\mu_2}(x_2)$$

$$\overline{\mathrm{L}\mathrm{L}\mathrm{L}} = (ie)^3\int d^4x_1 d^4x_2 d^4x_3\, \hat{\bar{\psi}}(x_1)\gamma_{\mu_1}\overline{\hat{\psi}(x_1)\hat{\bar{\psi}}(x_2)}\gamma_{\mu_2}\overline{\hat{\psi}(x_2)\hat{\bar{\psi}}(x_3)}\gamma_{\mu_3}\hat{\psi}(x_3)A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)$$

$$\ldots \tag{3}$$

$$\overline{LL} = (ie)^2 \int d^4x_1 d^4x_2\, \hat{\bar\psi}(x_1)\gamma_{\mu_1}\hat\psi(x_1)\hat{\bar\psi}(x_2)\gamma_{\mu_2}\hat\psi(x_2)\, A_{\mu_1}(x_1)A_{\mu_2}(x_2)$$

$$\overline{LLL} = (ie)^3 \int d^4x_1 d^4x_2 d^4x_3\, \hat{\bar\psi}(x_1)\gamma_{\mu_1}\hat\psi(x_1)\hat{\bar\psi}(x_2)\gamma_{\mu_2}\hat\psi(x_2)\hat{\bar\psi}(x_3)\gamma_{\mu_3}\hat\psi(x_3)\, A_{\mu_1}(x_1)A_{\mu_2}(x_2)A_{\mu_3}(x_3)$$

$$\ldots \tag{4}$$

$$\hat\psi(x)\hat{\bar\psi}(y) = -\hat{\bar\psi}(y)\hat\psi(x) = i\,S_+(x-y) \tag{5}$$

The exponent C in spinor electrodynamics may be written also as follows

$$C = iL + ie^2 \int d^4x\, d^4y\, \hat{\bar\psi}(x)\gamma_\mu A_\mu(x) S^A_+(x,y)\gamma_\nu A_\nu(y)\hat\psi(y) = \tag{6.a}$$

$$= -i\int d^4x\, d^4y\, \hat{\bar\psi}(x)\left(\gamma\frac{\partial}{\partial x}+m\right) S^A_+(x,y)\left(-\gamma\frac{\overleftarrow{\partial}}{\partial y}+m\right)\hat\psi(y) = \tag{6.b}$$

$$= i\,\hat{\bar\psi}\, I\, \hat\psi = \tag{6.c}$$

$$= i\oint d\sigma_\mu \oint d\sigma'_\nu\, \hat{\bar\psi}(x)\gamma_\mu\left(S^A_+(x,x') - S_+(x-x')\right)\gamma_\nu\hat\psi(x') = \tag{6.d}$$

$$= i\left(\int\limits_{\substack{x_0=t''\\ y_0=t'}} - \int\limits_{\substack{x_0=t''\\ y_0=t''}} - \int\limits_{\substack{x_0=t'\\ y_0=t'}} + \int\limits_{\substack{x_0=t'\\ y_0=t''}}\right) d^3x\, d^3y\, \hat{\bar\psi}(x)\gamma_4\left(S^A_+(x,y) - S_+(x-y)\right)\gamma_4\hat\psi(y) = \tag{6.e}$$

$$= G - N. \tag{6.f}$$

Presentations like (6.c) and (6.d) and notations I and $B = iw$ were used by Schwinger in his non-operator terms[5] (unlike our $\hat\psi$ and $\hat{\bar\psi}$, which are operators). In eq. (6.f)

$$G = i\oint d\sigma_\mu \oint d\sigma'_\nu\, \hat{\bar\psi}(x)\gamma_\mu S^A_+(x,x')\gamma_\nu\hat\psi(x') = \tag{7.a}$$

$$= i\left(\int\limits_{\substack{x_0=t''\\ y_0=t'}} - \int\limits_{\substack{x_0=t''\\ y_0=t''}} - \int\limits_{\substack{x_0=t'\\ y_0=t'}} + \int\limits_{\substack{x_0=t'\\ y_0=t''}}\right) d^3x\, d^3y\, \hat{\bar\psi}(x)\gamma_4 S^A_+(x,y)\gamma_4\hat\psi(y) \tag{7.b}$$

$$N = i\oint d\sigma_\mu \oint d\sigma'_\nu\, \hat{\bar\psi}(x)\gamma_\mu S_+(x-x')\gamma_\nu\hat\psi(x') = \tag{8.a}$$

$$= i\left(\int\limits_{\substack{x_0=t''\\ y_0=t'}} - \int\limits_{\substack{x_0=t''\\ y_0=t''}} - \int\limits_{\substack{x_0=t'\\ y_0=t'}} + \int\limits_{\substack{x_0=t'\\ y_0=t''}}\right) d^3x\, d^3y\, \hat{\bar\psi}(x)\gamma_4 S_+(x-y)\gamma_4\hat\psi(y) = \tag{8.b}$$

$$= i\int d^3x\, \hat{\bar\psi}^{(+)}(x)\gamma_4\hat\psi(x)\,. \tag{8.c}$$

The construction $C = G - \mathcal{N}$ is common for all the cases. The operator $:\mathcal{N}:$ is the particle number operator (number of particles plus number of antiparticles). For more detailed information concerning properties of $\mathcal{N}$ see Appendix A. Operator $G$ is merely an operator representative for the Feynman one-particle Green function $S_+^A$, which is determined by the following integral equation/1/

$$S_+^A(x,z) = S_+(x-z) - ie\int_{t'}^{t''} d^4y\, S_+(x-y)\gamma_\mu A_\mu(y) S_+^A(y,z) . \qquad (9)$$

The expression for C in (1.a) corresponds to the solution of this equation in the form of the Neumann series. In/5,7/ the Fredholm form of the solution was considered.

The non-operator quantity B is not expressed via $S_+^A$ in a closed form, although its variation due to variation of $A_\mu$ is representable as/1/:

$$\delta B = ie\int d^4x\, Sp(\gamma_\mu S_+^A(x,x))\,\delta A_\mu(x) \quad . \qquad (10)$$

The coefficients in B are such that B and $e^B$ have the following forms

$$B = \pm Sp\ln(1\pm\lambda K),\ e^B = \mathrm{Det}(1+\lambda K) \text{ and } \mathrm{Det}^{-1}(1-\lambda K) \qquad (11)$$

(infinite determinants)

for the Fermi-Dirac and Bose-Einstein/5/ statistics (Schwinger obtained this by strightforward solving eq. (10) ).

The N-ordered form of S-matrix (1.a) may be constructed, using Feynman indications/1/. An expression, equivalent to eq. (1.a) was obtained by Schwinger/5/ (eq. (117)) in the coherent state representation (as one may call it to-day). Eq. (1.a) may be obtained also by means of the continual integration method or in a direct combinatorial way, using the standard Wick N-ordering technique (for the latter see Appendix B).

Note, that in the case of a closed system of interacting charged and electromagnetic fields one can represent the S-matrix N-ordered with respect to charged field as follows

$$S = T\, e^{iL} = T_A\left(e^B \mathcal{N}_\psi e^C\right) , \qquad (12)$$

where B and C are the same as above, but with the operators $\hat{A}_\mu(x)$ instead of $A_\mu(x)$, and $T_A$ means time ordering of $\hat{A}_\mu(x)$, entering into B and C. This is true for any other fields instead of $A_\mu(x)$.

Recall also other decompositions of the S-matrix, where the operators $\hat{\psi}$ are kept under the sign of T-ordering, but the operators $\hat{A}_\mu$ are N-ordered /24-29/, or are taken as simple (in fact, symmetrized) products

$$S = T\, e^{iL} = T_\psi \left( e^{-\frac{1}{2}\int d^4y d^4z\, j_\nu(y) \overbrace{\hat{A}_\nu(y)\hat{A}_\lambda(z)} j_\lambda(z)}{}_{N_A} e^{iL} \right) , \quad (13.a)$$

$$= T_\psi \left( e^{\frac{i}{2}\int d^4y\, d^4z\, j_\nu(y)\Delta_{sym}(y-z) j_\nu(z)}\, e^{iL} \right) , \quad (13.b)$$

where $j_\mu(x) = ie{:}\hat{\bar{\psi}}(x)\gamma_\mu \hat{\psi}(x){:}$ , $\overbrace{\hat{A}_\nu(y)\hat{A}_\lambda(z)} = -i\delta_{\nu\lambda}\Delta_+(y-z)$. If one takes $j_\mu(x)$ to be a classical external current, then symbol $T_\psi$ is superfluous and eqs. (13) are the final normal product or symmetrized product decompositions of the S-matrix for this particular model.

Note also that one can decompose the discussed S-matrix in terms of antisymmetrized products (instead of the N-products in eq. (1) )of the charged field (see eq. (12.a) in ref. /28/ or eq. (145.b)in ref. /29/*) ).The only distinction from eq. (1) is the use of $S_{sym}$ – functions instead of $S_+$.

The construction of S-matrix

$$S \sim {:}\, e^{C} {:} = {:}\, e^{G-N} {:} = {:}\left(1 + G + \frac{1}{2!}G^2 + \cdots\right) e^{-N}{:} \qquad (14)$$

means**) that it is just the operator G (but not C) that corresponds to each charged line(e.g., electron-positron one), while the term $\frac{1}{\ell!}G^\ell$ corresponds to $\ell$ such lines. Hence, in non-operator terms, one charged line is presented by the propagator $S_+^A(x,y)$, while $\ell$ such lines by a determinant (permanent), constructed of $S_+^A$ for the Fermi-Dirac (Bose-Einstein) statistics. An S-matrix element between states with definite numbers of quanta ($m+n$ and $k+\ell$ ) equals

$$\langle f | S(t'',t') | i \rangle = e^{B} \langle f_1 \cdots f_k \bar{f}_1 \cdots \bar{f}_\ell | {:}\, e^{G-N} {:} | i_1 \cdots i_m \bar{i}_1 \cdots \bar{i}_n \rangle =$$

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*) Equations (12.a)-(12.b)in ref. /28/ and(145.a,b)-(147) in ref. /29/ must be corrected by inserting the factor $1/m!$ under $\sum_{m=1}^{\infty}$.

**) See Appendix A.

$$=\delta_{k-\ell-m+n}\frac{1}{(k+n)!}\langle f_1\cdots f_k\bar f_1\cdots\bar f_\ell|:G^{k+n}e^{-\mathcal{N}}:|i_1\cdots i_m\bar i_1\cdots\bar i_n\rangle=$$

$$=\delta_{k-\ell-m+n}\frac{1}{(k+n)!}\{a_{f_k}\cdots[a_{\bar f_1}[\cdots\{[:G^{k+n}:,a^+_{i_1}]a^+_{i_2}\}\cdots a^+_{\bar i_n}]_\pm]_\mp\cdots\}= \qquad (15.a)$$

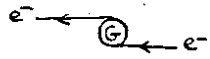
$$=\pm\delta_{k-\ell-m+n}\frac{1}{(k+n)!}\{\cdots[[a_{f_k}\cdots[a_{\bar f_1}[\cdots\{[:G^{k+n}:,a^+_{i_1}]a^+_{i_2}\}\cdots a^+_{i_m}]_\pm]_\mp\cdots]_\pm a^+_{\bar i_1}]_\mp\cdots a^+_{\bar i_n}\} \qquad (15.b)$$

$$=\pm\delta_{k-\ell-m+n}\sum_{\substack{\text{over all}\\ \text{permutations}\\ \text{of } a_{f_1}\cdots a_{f_k}a^+_{\bar i_1}\cdots a^+_{\bar i_n}}} p\{a_{f_1}[G,a^+_{i_1}]\}\{a_{f_2}[G,a^+_{i_2}]\}\cdots\{[a_{\bar f_\ell},G]a^+_{\bar i_n}\} \qquad (15.c)$$

(p – parity of permutation)

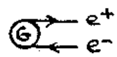
(see Appendix A, eq. (A.11)). Here $i$ and $f$ label initial and final particle states, while $\bar i$ and $\bar f$, initial and final antiparticle ones. (e.g., $i_j$ is a complete set of quantum numbers of a $j$-th initial particle). This matrix element is zero, if quantum numbers are such that it is not realizable by the diagram with $k+n=m+\ell$ charged lines (according to construction of G). Being realizable it is just the mentioned determinant (permanent) constructed of "one-particle" matrix elements $\langle e^-|:G:|e^-\rangle$, $\langle e^+|:G:|e^+\rangle$, $\langle 0|:G:|e^+e^-\rangle$, $\langle e^+e^-|:G:|0\rangle$. The latter are written in the $x$-representation as follows

$$\langle 0|\hat\psi^{(-)}(x):G:\hat{\bar\psi}^{(+)}(y)|0\rangle = i\,S^A_+(x,y)_{x_0=t'',\,y_0=t'} \qquad (16)$$

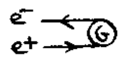



$$\langle 0|\hat{\bar\psi}^{(-)}(y):G:\hat\psi^{(+)}(x)|0\rangle = -i\,S^A_+(x,y)_{x_0=t',\,y_0=t''} \qquad (17)$$


$$\langle 0|:G:\hat{\bar\psi}^{(+)}(y)\psi^{(+)}(x)|0\rangle = -i\,S^A_+(x,y)_{x_0=y_0=t'} \qquad (18)$$




$$\langle 0|\hat\psi^{(-)}(x)\hat{\bar\psi}^{(-)}(y):G:|0\rangle = i\,S^A_+(x,y)_{x_0=y_0=t''} \qquad (19)$$




For the sake of completeness let us give the S-matrix element in non-operator terms of the mentioned determinant (permanent). Let us put

$$a^+_i=\int d^3x_i\,\hat{\bar\psi}(x_i)\gamma_4 u^{(i)}(x_i)\,,\quad a^+_{\bar i}=\int d^3x_{\bar i}\,\bar u^{(\bar i)}(x_{\bar i})\gamma_4\hat\psi(x_{\bar i})$$

$$a_f=\int d^3x_f\,\bar u^{(f)}(x_f)\gamma_4\hat\psi(x_f)\,,\quad a_{\bar f}=\int d^3x_{\bar f}\,\hat{\bar\psi}(x_{\bar f})\gamma_4 u^{(\bar f)}(x_{\bar f}) \qquad (20)$$

in eq. (15) . Then

$$\langle f_1\cdots f_k\bar f_1\cdots\bar f_\ell|\,S(t'',t')\,|i_1\cdots i_m\bar i_1\cdots\bar i_n\rangle=$$

$$=e^{B}\langle f_1\cdots f_k\bar f_1\cdots\bar f_\ell|:e^{G-\mathcal{N}}:|i_1\cdots i_m\bar i_1\cdots\bar i_n\rangle=$$

$$=\pm i^{k+n}\delta_{k-l-m+n}e^{B}\int d^3x_{f_1}\cdots d^3x_{f_k}\, d^3x_{\bar f_1}\cdots d^3x_{\bar f_l}\, d^3x_{i_1}\cdots d^3x_{i_m}\, d^3x_{\bar i_1}\cdots d^3x_{\bar i_n}\, \bar u_{f_1}\gamma_4\cdots\bar u_{f_k}\gamma_4$$

$$\bar u_{\bar i_1}\gamma_4\cdots\bar u_{\bar i_n}\gamma_4 \begin{vmatrix} S^A_{+f_1 i_1}\cdots S^A_{+f_1 i_m} & S^A_{+f_1\bar f_1}\cdots S^A_{+f_1\bar f_l} \\ \cdots\cdots & \cdots\cdots \\ S^A_{+f_k i_1}\cdots S^A_{+f_k i_m} & S^A_{+f_k\bar f_1}\cdots S^A_{+f_k\bar f_l} \\ S^A_{+\bar i_1 i_1}\cdots S^A_{+\bar i_1 i_m} & S^A_{+\bar i_1\bar f_1}\cdots S^A_{+\bar i_1\bar f_l} \\ \cdots\cdots & \cdots\cdots \\ S^A_{+\bar i_n i_1}\cdots S^A_{+\bar i_n i_m} & S^A_{+\bar i_n\bar f_1}\cdots S^A_{+\bar i_n\bar f_l} \end{vmatrix}_{\mp} \gamma_4 u_{i_1}\cdots\gamma_4 u_{i_m}\gamma_4 u_{\bar f_1}\cdots\gamma_4 u_{\bar f_l} \tag{21}$$

where, e.g., $u_{i_1}=u^{(i_1)}_{\alpha_{i_1}}(x_{i_1})$, $S^A_{+f_1 i_1}=S^A_{+\alpha_{f_1}\alpha_{i_1}}(x_{f_1},x_{i_1})$, $(i_1)$ labelling a state and $\alpha$ being spinor indices; $x_{0i}=x_{0\bar i}=t'$ $(\to-\infty)$, $x_{0f}=x_{0\bar f}=t''$ $(\to+\infty)$. The $|\quad|_{\mp}$ denote the mentioned determinant or permanent.

If one switches off interactions, the determinant (permanent) turns out to be

$$(-i)^k i^n\delta_{km}\delta_{ln}\begin{vmatrix} S^{(-)}_{f_1 i_1}\cdots S^{(-)}_{f_1 i_m} & 0\cdots 0 \\ \cdots\cdots & \cdots\cdots \\ S^{(-)}_{f_k i_1}\cdots S^{(-)}_{f_k i_m} & 0\cdots 0 \\ 0\cdots 0 & S^{(+)}_{\bar i_1\bar f_1}\cdots S^{(+)}_{\bar i_1\bar f_l} \\ \cdots\cdots & \cdots\cdots \\ 0\cdots 0 & S^{(+)}_{\bar i_n\bar f_1}\cdots S^{(+)}_{\bar i_n\bar f_l} \end{vmatrix}_{\mp} \tag{22}$$

It represents the unit operator ( S =1 ).*)

## 3. STATISTICS OF PAIRS CREATED BY EXTERNAL FIELDS FROM VACUUM

For the N-ordered S-matrix

$$S=e^{B}:e^{C}:\,,\qquad S^{+}=e^{B^*}:e^{C^+}: \tag{23}$$

from the unitarity condition

$$S^{+}S=e^{B^*+B}\;:e^{C^+}::e^{C}:=1 \tag{24}$$

it follows that

$$:e^{C^+}::e^{C}:=e^{-B^*-B}. \tag{25}$$

Hence, all operator terms vanish after decomposition of $:e^{C^+}::e^{C}:$ into N-products. The probability of creation of any number of pairs in any states from vacuum is given by

*) Let us give the transition from our notations to Schwinger ones: $S^{(+)}\to S^{(-)}$, $S^{(-)}\to S^{(+)}$, $S^A_+\to -G_+$, $\gamma_\mu\to -i\gamma_\mu$.

$$1 = \langle 0|S^{+}S|0\rangle = e^{B^{*}+B}\langle 0|:e^{C^{+}}::e^{C}:|0\rangle =$$

$$= e^{B^{*}+B}\sum_{n=0}^{\infty}\frac{1}{(n!)^{2}}\langle 0|:C^{+n}::C^{n}:|0\rangle , \tag{26}$$

where n-th term of the sum

$$w_{n} = e^{B^{*}+B}\,\frac{1}{(n!)^{2}}\langle 0|:C^{+n}::C^{n}:|0\rangle \tag{27}$$

is the probability of creation of $n$ pairs from vacuum. The vacuum expectation value means that in $:C^{+n}::C^{n}:$ all operators must be paired in all possible ways, pairings being the Dyson ones (e.g., for the electron-positron field

$$\overline{\hat{\psi}_{\alpha}(x)\hat{\bar{\psi}}_{\beta}(y)} = \{\hat{\psi}^{(-)}_{\alpha}(x), \hat{\bar{\psi}}_{\beta}(y)\} = S^{(-)}_{\alpha\beta}(x-y), \quad \overline{\hat{\bar{\psi}}_{\beta}(y)\hat{\psi}_{\alpha}(x)} = \{\hat{\bar{\psi}}^{(-)}_{\beta}(y), \hat{\psi}_{\alpha}(x)\} = S^{(+)}_{\alpha\beta}(x-y)). \tag{28}$$

For example, the probabilities of creation of 0,1,... pairs are

$$w_{0} = e^{B^{*}+B} ,$$

$$w_{1} = e^{B^{*}+B}\ \overline{C^{+}C} ,$$

$$w_{2} = e^{B^{*}+B}\ \frac{1}{2}\left(\overline{C^{+}C}\,\overline{C^{+}C} + \overline{C^{+}C\,C^{+}C}\right),$$

$$w_{3} = e^{B^{*}+B}\left(\frac{1}{3!}\overline{C^{+}C}\,\overline{C^{+}C}\,\overline{C^{+}C} + \frac{1}{2}\overline{C^{+}C}\,\overline{C^{+}CC^{+}C} + \frac{1}{3}\overline{C^{+}C\,C^{+}C\,C^{+}C}\right),$$

$$w_{4} = e^{B^{*}+B}\Big(\frac{1}{4!}\overline{C^{+}C}\,\overline{C^{+}C}\,\overline{C^{+}C}\,\overline{C^{+}C} + \frac{1}{4}\overline{C^{+}C}\,\overline{C^{+}C}\,\overline{C^{+}CC^{+}C} +$$

$$+ \frac{1}{3}\overline{C^{+}C}\,\overline{C^{+}CC^{+}CC^{+}C} + \frac{1}{8}\overline{C^{+}CC^{+}C}\,\overline{C^{+}CC^{+}C} + \frac{1}{4}\overline{C^{+}CC^{+}CC^{+}CC^{+}C} ,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \tag{29}$$

To avoid entangling of the pairing lines we alternate C and $C^{+}$, although all $C^{+}$'s stand in fact to the left of all C's , and this defines correct Dyson pairings.

The sum in (3.4) can be represented as follows

$$\sum_{n=0}^{\infty}\frac{1}{(n!)^{2}}\langle 0|:C^{+n}::C^{n}:|0\rangle = e^{D} , \tag{30}$$

where $-B^{*}-B = D = \overline{C^{+}C} + \frac{1}{2}\overline{C^{+}CC^{+}C} + \frac{1}{3}\overline{C^{+}CC^{+}CC^{+}C} + \cdots =$

$$= \overline{G^{+}G} + \frac{1}{2}\overline{G^{+}GG^{+}G} + \frac{1}{3}\overline{G^{+}GG^{+}GG^{+}G} + \cdots . \tag{31}$$

The probability of creation of $n$ pairs is obtained from $e^{D}$

using the decomposition $e^{D}$ into a series in $D$ and singling out of all terms, which contain both C and $C^+$ $n$ times. If we introduce the quantity

$$D_\alpha = \alpha\, \overline{C^+C} + \frac{\alpha^2}{2}\, \overline{C^+CC^+C} + \frac{\alpha^3}{3}\, \overline{C^+CC^+CC^+C} + \cdots =$$
$$= \alpha\, \overline{G^+G} + \frac{\alpha^2}{2}\, \overline{G^+GG^+G} + \frac{\alpha^3}{3}\, \overline{G^+GG^+GG^+G} + \cdots \qquad (32)$$

then the generating function for $w_n$ is

$$\gamma(\alpha) = e^{B^*+B} e^{D_\alpha} = \sum_{n=0}^{\infty} \alpha^n w_n . \qquad (33)$$

If $\hat{N}$ is the particle number operator (number of particles plus number of antiparticles), and $\hat{n} = \frac{1}{2}\hat{N}$, then the mean number of created pairs equals

$$\bar{n} = \langle 0 | S^+ \hat{n} S | 0 \rangle = \sum_{n=0}^{\infty} n w_n = \sum_{n=0}^{\infty} n \alpha^{n-1} w_n \Big|_{\alpha=1} = e^{B^*+B} \frac{d}{d\alpha} e^{D_\alpha} \Big|_{\alpha=1} =$$
$$= \overline{C^+C} + \overline{C^+CC^+C} + \overline{C^+CC^+CC^+C} + \cdots \qquad (34)$$

The function $\gamma(\alpha)$ serves as a generating function both for the probabilities $w_n$ and for factorial moments

$$w_n = \frac{1}{n!} \frac{d^n \gamma}{d\alpha^n}(0), \qquad (35)$$

$$\mu_{[m]} = \overline{n^{[m]}} \equiv \overline{n(n-1)\cdots(n-m+1)} = \frac{d^m \gamma}{d\alpha^m}(1) . \qquad (36)$$

So, after differentiation we obtain the factorial moments

$$\bar{n} = e^{B^*+B} \frac{d}{d\alpha} e^{D_\alpha} \Big|_{\alpha=1} = D_1'$$

$$\overline{n(n-1)} = e^{B^*+B} \frac{d^2}{d\alpha^2} e^{D_\alpha} \Big|_{\alpha=1} = D_1'' + D_1'^2$$

$$\overline{n(n-1)(n-2)} = e^{B^*+B} \frac{d^3}{d\alpha^3} e^{D_\alpha} \Big|_{\alpha=1} = D_1''' + 3D_1'' D_1' + D_1'^3$$

$$\overline{n(n-1)(n-2)(n-3)} = e^{B^*+B} \frac{d^4}{d\alpha^4} e^{D_\alpha} \Big|_{\alpha=1} = D_1'''' + 4D_1''' D_1' + 3D_1''^2 + 6D_1'' D_1'^2 + D_1'^4$$
$$\cdots \qquad (37)$$

where

$$D_1' = \overline{C^+C} + \overline{C^+CC^+C} + \overline{C^+CC^+CC^+C} + \cdots =$$
$$= \overline{G^+G} + \overline{G^+GG^+G} + \overline{G^+GG^+GG^+G} + \cdots$$

$$
\begin{aligned}
D_1'' &= \overline{C^+CC^+C} + 2\,\overline{C^+CC^+CC^+C} + \cdots = \\
&= \overline{G^+GG^+G} + 2\,\overline{G^+GG^+GG^+G} + \cdots
\end{aligned}
$$

...

$$ \tag{38} $$

The characteristic function is given as

$$
\chi(\beta) = \gamma(e^{i\beta}) = \sum_{n=0}^{\infty} e^{i\beta n} w_n = \sum_{m=0}^{\infty} \frac{(i\beta)^m}{m!}\mu_m = e^{\sum_{\ell=0}^{\infty} \frac{(i\beta)^\ell}{\ell!}\varkappa_\ell}, \tag{39}
$$

where $\mu_m$ and $\varkappa_\ell$ are moments and cumulants

$$
\mu_m = \overline{n^m} = (-i)^m \frac{d^m \chi}{d\beta^m}(0), \qquad \varkappa_\ell = (-i)^\ell \frac{d^\ell \ln\chi}{d\beta^\ell}(0). \tag{40}
$$

The moments, factorial moments and cumulants are expressed in terms of each other in a well-known manner.

We obtain as central moments

$$
\begin{aligned}
\bar{n} &= D_1' \\
\overline{(n-\bar{n})^2} &= D_1'' + D_1' \\
\overline{(n-\bar{n})^3} &= D_1''' + 3D_1'' + D_1' \\
\overline{(n-\bar{n})^4} &= D_1'''' + 6D_1''' + 7D_1'' + D_1' + 3(D_1'' + D_1')^2 .
\end{aligned}
$$

...

$$ \tag{41} $$

Let us give the inverse relations

$$
\begin{aligned}
D_1' &= \bar{n} \\
D_1'' &= \overline{(n-\bar{n})^2} - \bar{n} \\
D_1''' &= \overline{(n-\bar{n})^3} - 3\overline{(n-\bar{n})^2} + 2\bar{n} \\
D_1'''' &= \overline{(n-\bar{n})^4} - 6\overline{(n-\bar{n})^3} + 11\overline{(n-\bar{n})^2} - 6\bar{n} - 3\left[\overline{(n-\bar{n})^2}\right]^2
\end{aligned}
$$

...

$$ \tag{42} $$

For the discussed distribution "cumulants" of $\gamma(\alpha)$ are the simplest constituents:

$$
\gamma(\alpha) = e^{\sum_{n=0}^{\infty} \alpha^n \lambda_n} \tag{43}
$$

$$\lambda_0 = B^* + B \ ,$$

$$\lambda_1 = \overline{C^+ C} = \overline{G^+ G} \ ,$$

$$\lambda_2 = \overline{C^+ C\, C^+ C} = \overline{G^+ G\, G^+ G} \ ,$$

$$\ldots\ldots\ldots\ldots \tag{44}$$

and all others are expressed via them.

We have considered the distribution for the number of pairs. One can obtain probabilities for finding pairs with given quantum numbers decomposing each pairing function $S^{(-)}$ and $S^{(+)}$ in eqs. (29) over suitable complete sets of one-particle states.

## APPENDIX A. FORMALISM OF OPERATOR $\mathcal{N}$

Let $|n\rangle$ be n-quantum states

$$|n\rangle = |i_1 \cdots i_n\rangle = a^+_{i_1} \cdots a^+_{i_n} |0\rangle \tag{A.1}$$

(A.1)

(for example, the electron-positron state in x-representation

$$|n\rangle = \hat{\bar{\psi}}^{(+)}_{\beta_1}(y_1) \ldots \hat{\bar{\psi}}^{(+)}_{\beta_q}(y_q)\, \hat{\psi}^{(+)}_{\alpha_1}(x_1) \ldots \hat{\psi}^{(+)}_{\alpha_p}(x_p)|0\rangle \ , \quad p+q=n \ ) \tag{A.2}$$

or any superposition of such states. The operator $\mathcal{N}$ for neutral and charged scalar fields and for a spinor one can be written, e.g., as follows

$$\mathcal{N} = \frac{i}{2} \int d^3x \ \hat{\varphi}^{(1)}(x) \overleftrightarrow{\partial_4} \hat{\varphi}(x) \ , \tag{A.3a}$$

$$\mathcal{N} = i \int d^3x\, \hat{\varphi}^{*(1)}(x) \overleftrightarrow{\partial_4} \varphi(x) \ , \tag{A.3b}$$

$$\mathcal{N} = i \int d^3x\, \hat{\bar{\psi}}^{(1)}(x) \gamma_4 \hat{\psi}(x) \ . \tag{A.3c}$$

The latter is true in the $\beta$ -formalism for both spin 0 and 1 fields too (after substitution $\gamma_4 \to \beta_4$ ). The particle number operator $\hat{N}$ (number of particles plus number of antiparticles) is written via $\mathcal{N}$ as follows

$$\hat{N} = :\mathcal{N}: \tag{A.4}$$

Then we have relations involving $\mathcal{N}$

$$
\begin{aligned}
&:\mathcal{N}:|n\rangle = n|n\rangle \\
&:\mathcal{N}^2:|n\rangle = n(n-1)|n\rangle \\
&\cdots\cdots\cdots\cdots \\
&:\mathcal{N}^m:|n\rangle = n^{[m]}|n\rangle \quad \text{for} \quad m \leq n \quad \left(n^{[m]} = n(n-1)\cdots(n-m+1)\right) \\
&:\mathcal{N}^n:|n\rangle = n!|n\rangle \\
&:\mathcal{N}^m:|n\rangle = 0 \quad \text{for} \quad m > n
\end{aligned}
\tag{A.5}
$$

Let $Q$ be any operator, which is an even function of annihilation and creation operators. Operators C, G and $\mathcal{N}$, and any functions of them satisfy this assumption. Then

$$:Q\mathcal{N}:|i_1\cdots i_n\rangle = a^+_{i_1}:Q:|i_2\cdots i_n\rangle - a^+_{i_2}:Q:|i_1 i_3\cdots i_n\rangle + \cdots + (-1)^{n-1} a^+_{i_n}:Q:|i_1 i_2 \cdots i_{n-1}\rangle$$

(n terms)

$$:Q\mathcal{N}^2:|i_1\cdots i_n\rangle = 2a^+_{i_1}a^+_{i_2}:Q:|i_3\cdots i_n\rangle - \cdots \quad + 2a^+_{i_{n-1}}a^+_{i_n}:Q:|i_1\cdots i_{n-2}\rangle$$

$(\frac{1}{2}n(n-1)$ terms)

$$
\begin{aligned}
&\cdots\cdots\cdots\cdots \\
&:Q\mathcal{N}^m:|i_1\cdots i_n\rangle = m!\,a^+_{i_1}\cdots a^+_{i_m}:Q:|i_{m+1}\cdots i_n\rangle + \cdots \pm m!\cdots \\
&\qquad (C^m_n \text{ terms}) \qquad\qquad \text{if} \quad m \leq n \\
&:Q\mathcal{N}^n:|i_1\cdots i_n\rangle = n!\,a^+_{i_1}\cdots a^+_{i_n}:Q:|0\rangle \\
&:Q\mathcal{N}^m:|i_1\cdots i_n\rangle = 0 \qquad \text{if} \quad m > n
\end{aligned}
\tag{A.6}
$$

These relations are clear from

$$
\begin{aligned}
:Q\mathcal{N}^m: &= i\int d^3x\, \hat{\bar{\psi}}^{(+)(+)}(x)\gamma_4 :Q\mathcal{N}^{m-1}:\hat{\psi}^{(-)}(x) - i\int d^3x\, \hat{\psi}^{(+)}(x):Q\mathcal{N}^{m-1}:\hat{\bar{\psi}}^{(+)(-)}(x)\gamma_4 = \\
&= \int d^3x\, \hat{\bar{\psi}}^{(+)}(x)\gamma_4 :Q\mathcal{N}^{m-1}:\hat{\psi}^{(-)}(x) + \int d^3x\, \hat{\bar{\psi}}^{(+)}_C(x)\gamma_4 :Q\mathcal{N}^{m-1}:\hat{\psi}^{(-)}_C(x)
\end{aligned}
\tag{A.7}
$$

and one can operate in this way with each $\mathcal{N}$ of remaining $\mathcal{N}^{m-1}$. Further,

$$:e^{-\mathcal{N}}:|n\rangle=\left(1-:\mathcal{N}:+\frac{1}{2!}:\mathcal{N}^2:-\dots+\frac{(-1)^n}{n!}:\mathcal{N}^n:\right)|n\rangle=(1-1)^n|n\rangle=$$

$$=\begin{cases}|0\rangle & \text{if } n=0\\ 0 & \text{if } n\neq 0\end{cases} \tag{A.8}$$

$$:Qe^{-\mathcal{N}}:|0\rangle=:Q:|0\rangle$$

$$:Qe^{-\mathcal{N}}:|i_1\rangle=:Q(1-\mathcal{N}):|i_1\rangle=[:Q:,a^+_{i_1}]|0\rangle$$

$$\dots\dots\dots\dots\dots\dots$$

$$:Qe^{-\mathcal{N}}:|i_1\dots i_n\rangle=:Q\left(1-\mathcal{N}+\frac{1}{2!}\mathcal{N}^2-\dots+\frac{(-1)^n}{n!}\mathcal{N}^n\right):|i_1\dots i_n\rangle=$$

$$=[\dots[\{[:Q:,a^+_{i_1}]a^+_{i_2}\}a^+_{i_3}]\dots a^+_{i_n}]_\pm|0\rangle \;.$$

Each of alternating commutators and anticommutators reduces the number of annihilation operators, entering into $Q$ , by unity.

If $Q$ would be an odd function of annihilation and creation operators, this changes the signs in eqs. (A.6), e.g., in the r.h.s. of the first relation all terms would contain opposite signs, while signs in the second relation would conserve, etc. In eqs. (A.9) the commutators and anticommutators would be alternating again, but inner operation would be the anticommutator $\{:Q:,a^+_{i_1}\}$.

For the Bose-Einstein statistics all signs in relations, analogous to eqs. (A.6), are plus, and relations similar to eqs. (A.9) include commutators only (cf. [30], Appendix A).

If $Q$ contains $k$ creation and $\ell$ annihilation operators:

$$Q\sim\sum_{\substack{p_1\dots p_k\\ q_1\dots q_\ell}} c\; a^+_{p_1}\dots a^+_{p_k}a_{q_1}\dots a_{q_\ell} \tag{A.10}$$

then

$$\langle f_1\dots f_m|:Qe^{-\mathcal{N}}:|i_1\dots i_n\rangle\equiv\langle 0|a_{f_m}a_{f_{m-1}}\dots a_{f_1}:Qe^{-\mathcal{N}}:a^+_{i_1}a^+_{i_2}\dots a^+_{i_n}|0\rangle=$$

$$=\delta_{mk}\delta_{n\ell}\langle 0|a_{f_m}a_{f_{m-1}}\dots a_{f_1}:Q:a^+_{i_1}a^+_{i_2}\dots a^+_{i_n}|0\rangle= \tag{A.11}$$

$$=\delta_{mk}\delta_{n\ell}[a_{f_m}\dots[a_{f_2}[a_{f_1}[\dots\{[:Q:,a^+_{i_1}]a^+_{i_2}\}\dots a^+_{i_n}]_\pm]_\mp]_\pm\dots]_{(+\text{ or }-)}$$

The last operation will be anticommutator (commutator), if $Q$ is an even (odd) function of $a$ and $a^+$ .

Note, that all the preceding equations follow from the formulas

$$:Q\, e^{-\mathcal{N}}:\, a_i^+ = \begin{cases} :[:Q:, a_i^+]\, e^{-\mathcal{N}}: & \text{if } Q \text{ is even} \quad (A.12a) \\ :\{:Q:, a_i^+\}\, e^{-\mathcal{N}}: & \text{if } Q \text{ is odd} \quad (A.12b) \end{cases}$$

$$a_i :Q\, e^{-\mathcal{N}}: = \begin{cases} :[a_i, :Q:]\, e^{-\mathcal{N}}: & \text{if } Q \text{ is even} \quad (A.13a) \\ :\{a_i, :Q:\}\, e^{-\mathcal{N}}: & \text{if } Q \text{ is odd} \quad (A.13b) \end{cases}$$

$$:Q\, e^{-\mathcal{N}}: a_{i_1}^+ \cdots a_{i_n}^+ = \begin{cases} :[\cdots\{[:Q:, a_{i_1}^+]\, a_{i_2}^+\}\cdots a_{i_n}^+]_\mp\, e^{-\mathcal{N}}: & (A.14a) \\ :[\cdots[\{:Q:, a_{i_1}^+\}\, a_{i_2}^+]\cdots a_{i_n}^+]_\pm\, e^{-\mathcal{N}}: & (A.14b) \end{cases}$$

$$a_{i_n} \cdots a_{i_1} :Q\, e^{-\mathcal{N}}: = \begin{cases} :[a_{i_n} \cdots \{a_{i_2}[a_{i_1}, :Q:]\}\cdots]_\mp\, e^{-\mathcal{N}}: & (A.15a) \\ :[a_{i_n} \cdots [a_{i_2}\{a_{i_1}, :Q:\}]\cdots]_\pm\, e^{-\mathcal{N}}: & (A.15b) \end{cases}$$

As an example let us derive eq. (A.12a):

$$:Q\, e^{-\mathcal{N}}: a_j^+ = :Q\left(1 - \mathcal{N} + \tfrac{1}{2!}\mathcal{N}^2 - \tfrac{1}{3!}\mathcal{N}^3 + \cdots\right): a_j^+ =$$

$$= \left(:Q: - a_i^+ :Q: a_i + \tfrac{1}{2!} a_{i_1}^+ a_{i_2}^+ :Q: a_{i_2} a_{i_1} - \tfrac{1}{3!} a_{i_1}^+ a_{i_2}^+ a_{i_3}^+ :Q: a_{i_3} a_{i_2} a_{i_1} + \cdots\right) a_j^+ =$$

$$= :Q: a_j^+ - a_i^+ :Q: \left(\{a_i a_j^+\} - a_j^+ a_i\right) + \tfrac{1}{2!} a_{i_1}^+ a_{i_2}^+ :Q: \left(a_{i_2}\{a_{i_1} a_j^+\} - \{a_{i_2} a_j^+\} a_{i_1} + a_j^+ a_{i_2} a_{i_1}\right)$$

$$- \tfrac{1}{3!} a_{i_1}^+ a_{i_2}^+ a_{i_3}^+ :Q: \left(a_{i_3} a_{i_2}\{a_{i_1} a_j^+\} - a_{i_3}\{a_{i_2} a_j^+\} a_{i_1} + \{a_{i_3} a_j^+\} a_{i_2} a_{i_1} - a_j^+ a_{i_3} a_{i_2} a_{i_1}\right) + \cdots$$

$$= [:Q:, a_j^+] + a_i^+ [:Q:, a_j^+] a_i + \tfrac{1}{2!} a_{i_1}^+ a_{i_2}^+ [:Q:, a_j^+] a_{i_2} a_{i_1} +$$

$$+ \tfrac{1}{3!} a_{i_1}^+ a_{i_2}^+ a_{i_3}^+ [:Q:, a_j^+] a_{i_3} a_{i_2} a_{i_1} + \cdots = :[:Q:, a_j^+]\, e^{-\mathcal{N}}: \quad (A.16)$$

When evaluating the matrix element (15), it is convenient to arrange the alternating commutators and anticommutators in the order (15.b) (but not (15.a)), i.e. firstly ones with $a_{i_1}^+, \cdots a_{i_m}^+$ then with $a_{\bar{j}_\ell}, \cdots a_{\bar{j}_1}$, further with $a_{j_k}, \cdots a_{j_1}$, and finally with $a_{\bar{i}_1}^+, \cdots a_{\bar{i}_n}^+$. One can easily see, that alternating commutators and anticommutators with $a_{i_1}^+, \cdots a_{i_m}^+$ and $a_{\bar{j}_\ell}, \cdots a_{\bar{j}_1}$ give the result

$$\pm(m+\ell)!\, :[G, a_{i_1}^+][G, a_{i_2}^+]\cdots[G, a_{i_m}^+][a_{\bar{j}_1}, G][a_{\bar{j}_2}, G]\cdots[a_{\bar{j}_\ell}, G]\, e^{-\mathcal{N}}: \quad (A.17)$$

and those with remaining operators lead us to eq. (15.c).

Note, that

$$:e^{-\mathcal{N}}: = |0\rangle\langle 0| , \quad :e^{-\mathcal{N}}::e^{-\mathcal{N}}: = :e^{-\mathcal{N}}: \qquad \text{(A.18)}$$

and that in terms of $\mathcal{N}$ the completeness relation, i.e., decomposition of unity into $0-$ , $1-$ , $2-$ $\cdots$ - quantum projectors, is written as

$$1 = :e^{\mathcal{N}}e^{-\mathcal{N}}: = :\left(1+\mathcal{N}+\frac{1}{2!}\mathcal{N}^2+\dots\right)e^{-\mathcal{N}}: =$$

$$= :e^{-\mathcal{N}}: + :\mathcal{N}e^{-\mathcal{N}}: + \frac{1}{2!}:\mathcal{N}^2e^{-\mathcal{N}}: + \cdots =$$

$$= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\sigma_1=-,+}\cdots\sum_{\sigma_n=-,+}\int d^3x_1\cdots d^3x_n\,\hat{\bar{\psi}}^{(+)}_{\sigma_1}(x_1)\gamma_4\cdots\hat{\bar{\psi}}^{(+)}_{\sigma_n}(x_n)\gamma_4|0\rangle\langle 0|\hat{\psi}^{(-)}_{\sigma_n}(x_n)\cdots\hat{\psi}^{(-)}_{\sigma_1}(x_1) , \qquad \text{(A.19)}$$

where $\psi_- \equiv \psi$ , $\psi_+ \equiv \psi_c$ so that, e.g., $\psi_-^{(-)}$ and $\psi_+^{(-)}$ are electron and positron annihilation operators.

## APPENDIX B

In the decomposition of T-product $T\, e^{iL}$ into N-products only the following connected pairings occur

| | | | | |
|---|---|---|---|---|
| LL | 2! | LL | 1! | |
| LLL | 3! | LLL | 2! | |
| ..... | ... | ..... | ... | (B.1) |
| LL...L | n! | LL...L | (n-1)! | |

where multiplicities are indicated. Decomposing T $(L^n)$ into N-products, we obtain

$$T(L^n) = \sum c\, \underbrace{L\dots L}_{k_1}\ \underbrace{LL\dots LL}_{k_2}\ \underbrace{LLL\dots LLL}_{k_3}\dots :\underbrace{LL\dots L}_{\ell_1}\ \underbrace{LL\dots LL}_{\ell_2}\ \underbrace{LLL\dots LLL}_{\ell_3}\dots: \qquad \text{(B.2)}$$

where the sum runs over all $k_1, k_2, k_3, \dots, \ell_1, \ell_2, \ell_3, \dots$, which satisfy

$$n = k_1 + \ell_1 + 2(k_2+\ell_2) + 3(k_3+\ell_3) + \cdots \qquad \text{(B.3)}$$

The coefficient c of the summand is

$$c = \frac{1}{k_1!k_2!k_3!\cdots}\,\frac{1}{\ell_1!\ell_2!\ell_3!\cdots}\left(\frac{n!}{\underbrace{1!\ldots1!}_{k_1+\ell_1}\;\underbrace{2!\ldots2!}_{k_2+\ell_2}\;\underbrace{3!\ldots3!}_{k_3+\ell_3}\cdots}\right)\cdot$$

$$\cdot(1!)^{k_1}(1!)^{k_2}(2!)^{k_3}\ldots(1!)^{\ell_1}(2!)^{\ell_2}(3!)^{\ell_3}\ldots=$$

$$=\frac{n!}{k_1!2^{k_2}k_2!3^{k_3}k_3!\ldots\ell_1!\ell_2!\ell_3!\ldots} \tag{B.4}$$

In the first expression in parentheses there is embraced the polynomial coefficient for the decomposition of n multipliers L into individuals, pairs, triads, etc. If some pairing (or individual) enters k times, we must divide by $k!$. These factors precede the polynomial coefficient. Additionally we need to multiply by the multiplicity of each connected pairing. They follow the polynomial coefficient.

In decomposition of $:e^{C^+}::e^{C}:$ into N-products closed connected lines (e.g., $C^+C$) and non-closed ones (e.g., $C^+C$, $C\,C^+$) enter. The closed lines contain equal number of $C^+$ and C, and the non-closed ones either equal or different by unity. Multiplicities of all possible pairings are the following

| | | |
|---|---|---|
| $C^+C$ | | $1$ |
| $C^+C\ C^+C$ | | $1!\,2!$ |
| $C^+C\ C^+C\ C^+C$ | | $2!\,3!$ |
| ........... | | ... |
| $C^+C\ C^+C\ldots C^+C$ | (both C and $C^+$ n times) | $(n-1)!\,n!$ |
| $C^+C$, $C\,C^+$ | | $1$ |
| $C^+C\ C^+C$, $C\ C^+C\ C^+$ | | $(2!)^2$ |
| $C^+C\ C^+C\ C^+C$, $C\ C^+C\ C^+C\ C^+$ | | $(3!)^2$ |
| ....................... | | .... |
| $C^+C\ C^+C\ldots C^+C$, $C\ C^+C\ C^+\ldots C\ C^+$ | (both C and $C^+$ n times) | $(n!)^2$ |
| $C^+C\ C^+$, $C\ C^+C$ | | $1!\,2!$ |
| $C^+C\ C^+C\ C^+$, $C\ C^+C\ C^+C$ | | $2!\,3!$ |
| .................... | | ... |
| $C^+C\ldots C\ C^+$ (C n-I times, $C^+$ n times), $C\ C^+\ldots C^+C$ (C n times, $C^+$ n-I times) | | $(n-1)!\,n!$ |

(B.5)

The decomposition $: c^{+m} :: c^{n} :$ is

$$: c^{+m} :: c^{n} : = \sum c \; \underbrace{c^+c \ldots c^+c}_{k_1} \; \underbrace{c^+c\, c^+c \ldots c^+c\, c^+c}_{k_2} \ldots : \underbrace{c^+ \ldots c^+}_{\ell_{10}} \; \underbrace{c \ldots c}_{\ell_{01}}$$

$$\underbrace{c^+c \ldots c^+c}_{\ell_{11}} \; \underbrace{c\, c^+ \ldots c\, c^+}_{\ell'_{11}} \; \underbrace{c^+c\, c^+ \ldots c^+c\, c^+}_{\ell_{21}} \; \underbrace{c\, c^+c \ldots c\, c^+c}_{\ell_{12}}$$

$$\underbrace{c^+c\, c^+c \ldots c^+c\, c^+c}_{\ell_{22}} \; \underbrace{c\, c^+c\, c^+ \ldots c\, c^+c\, c^+}_{\ell'_{22}} \ldots : \tag{B.6}$$

where the sum runs over $k_1, k_2, \ldots, \ell_{10}, \ell_{01}, \ldots$, which satisfy

$$\begin{aligned} m &= (k_1+\ell_{10}+\ell_{11}+\ell'_{11}+\ell_{12}) + 2(k_2+\ell_{21}+\ell_{22}+\ell'_{22}+\ell_{23}) + \cdots \\ n &= (k_1+\ell_{01}+\ell_{11}+\ell'_{11}+\ell_{21}) + 2(k_2+\ell_{12}+\ell_{22}+\ell'_{22}+\ell_{32}) + \cdots \end{aligned} \tag{B.7}$$

and coefficient $c$ equals

$$c = \frac{1}{k_1!\,k_2!\,k_3!\cdots} \; \frac{1}{\ell_{10}!\,\ell_{01}!\,\ell_{11}!\,\ell'_{11}!\,\ell_{12}!\,\ell_{21}!\,\ell_{22}!\,\ell'_{22}!\cdots} \cdot$$

$$\cdot\left( \frac{m!}{\underbrace{1 \cdots\cdots 1}_{k_1+\ell_{10}+\ell_{11}+\ell'_{11}+\ell_{12}} \; \underbrace{2! \cdots\cdots 2!}_{k_2+\ell_{21}+\ell_{22}+\ell'_{22}+\ell_{23}} \cdots \underbrace{i! \cdots\cdots i!}_{k_i+\ell_{i\,i-1}+\ell_{ii}+\ell'_{ii}+\ell_{i\,i+1}} \cdots} \right) \cdot$$

$$\cdot\left( \frac{n!}{\underbrace{1 \cdots\cdots 1}_{k_1+\ell_{01}+\ell_{11}+\ell'_{11}+\ell_{21}} \; \underbrace{2! \cdots\cdots 2!}_{k_2+\ell_{12}+\ell_{22}+\ell'_{22}+\ell_{32}} \cdots \underbrace{j! \cdots\cdots j!}_{k_j+\ell_{j-1\,j}+\ell_{jj}+\ell'_{jj}+\ell_{j+1\,j}}} \right) \cdot$$

$$(1!2!)^{k_2}(2!3!)^{k_3}\cdots(1!2!)^{\ell_{21}+\ell_{12}}(2!2!)^{\ell_{22}+\ell'_{22}}(2!3!)^{\ell_{23}+\ell_{32}}(3!3!)^{\ell_{33}+\ell'_{33}}\cdots =$$

$$= \frac{m!\,n!}{k_1!\,2^{k_2}k_2!\,3^{k_3}k_3!\cdots\ell_{10}!\,\ell_{01}!\,\ell_{11}!\,\ell'_{11}!\,\ell_{12}!\,\ell_{21}!\,\ell_{22}!\,\ell'_{22}!\cdots} \tag{B.8}$$

In the first expression in parentheses polynomial coefficients are embraced: the first one for decomposition of $c^{+m}$ into individuals, pairs, triads,... and the second one for similar decomposition of $c^n$. As above, they are divided by factorials (which precede them) and multiplied by multiplicity of each connected pairing (which follow them).

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