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# MATRIX AND ANALYTIC REPRESENTATIONS OF QUATERNIONS AND OCTONIONS.I.

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# MATRIX AND ANALYTIC REPRESENTATIONS OF QUATERNIONS AND OCTONIONS.I.

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Матричные и аналитические представления для кватернионов и октонионов. I.

Для кватернионов (матрицы Паули для спина 1/2) и октонионов (которые в последнее время пытаются использовать для объяснения кварковой структуры) построены очень простым способом матричные и аналитические представления в смысле теории представлений Дирака. Они предназначены для формализма матрицы плотности. Октонионные матрицы удовлетворяют модифицированной (по сравнению с алгеброй октонионов) ассоциативной алгебре. Аналитические представления аналогичны представлению Вигнера и представлению когерентных состояний для обычной квантовой механики бесспиновых частиц. Простейший вариант октонионной квантовой механики, включая уравнения движения для оператора плотности и наблюдаемых, сформулирован вначале прямо на языке октонионов, а затем преобразован в указанные представления.

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Matrix and Analytic Representations of Quaternions and Octonions. I.

Matrix and analytic representations (in the sense of the Dirac representation theory) are constructed in a very simple way for quaternions (Pauli spin 1/2 operators) and octonions. They are applicable in the density matrix formalism. The octonion matrices satisfy a modified (with respect to the octonion algebra) associative algebra. The analytic representations are similar to the Wigner and coherent state representations in quantum mechanics of spinless particles. A simplest case of octonion quantum mechanics is formulated, including equations of motion, at first, directly in terms of octonions, and then is transformed into the representations under consideration.

The investigation has been performed at the Laboratory of Theoretical Physics, JINR.

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## I. INTRODUCTION

The Dirac representation theory [1] deals usually with a description in terms of amplitudes. Representations of another kind are possible, which use the density matrix terms and, e.g., are close to the classical description of angular momentum, and relative to phase space representations in quantum mechanics of spinless particles, such as the Wigner and coherent state representations. Here we generate in a very simple way, along the line of the Dirac representation theory, similar matrix and analytic (continuous) representations for spin $\frac{1}{2}$, i.e. for quaternions (Sec. 2) and for octonions (Sec. 3). The algebra of octonion 8x8 matrix representatives certainly differs from the octonion algebra. One can say that the latter masks an additional term of the matrix algebra (see eqs. (73) and (74)) by means of the non-associativity (like the Jordan algebra does with the non-commutativity).

Note, that the mathematical theory of continuous representations was elaborated by J.R.Klauder [2]. Spin coherent states and generalizations on other classical groups were considered in papers [3,4]. Investigating the quark structure, F.Gürsey et al. [5-11] extensively develop now symmetry and quantum-mechanical aspects of exceptional groups based on the octonion algebra. In particular, 8x8 matrix representation of octonions was also considered [6], but approach and construction were different.

In Sec. 4 we apply the concepts of Sec. 3 to the simplest case of octonion quantum mechanics. The corresponding r-number system was proposed by P.Jordan[12], but it is not exceptional quantum mechanics of Jordan, Neumann and Wigner[13]. At first, we construct the octonion quantum mechanics directly in terms of octonions, and define the density operator and equations of motion for it and for other operators, i.e., analogs of Born-Jordan-Dirac and Neumann (Liouville) equations. The Jordan algebra is not used. All octonion units are treated alike (the unit $e_7$ is not isolated), and the usual imaginary unit $i$ is used to make "observables" $e_i$ Hermitian. Then we transform the octonion quantum mechanics into the representations under consideration, and, in particular, write Liouville equations for the quaternions and octonions.

## 2. REPRESENTATIONS OF QUATERNIONS (PAULI MATRICES).

We start with the completeness relation for Pauli spin $\frac{1}{2}$ matrices

$$|e_0\|\otimes\|e_0| = \frac{1}{2}\left(|e_0|\otimes\|e_0\| - |e_i|\otimes\|e_i\|\right) = \tag{1.a}$$

$$= \frac{4}{\Omega_4}\int d^4a\,\delta(a_\mu^2-1)\,|U(a)|\otimes\|\bar{U}(a)\|\,, \tag{1.b}$$

which is written in terms of quaternion units:

$$e_0^2 = e_0, \quad e_i^2 = -e_0, \quad e_ie_j = -\delta_{ij}e_0 + \varepsilon_{ijk}e_k \quad (i,j,k=1,2,3) \tag{2}$$

with quaternion conjugation $e_i \rightarrow \bar{e}_i = -e_i$. We can return to the Pauli matrices $\sigma_i$ by the substitution

$$e_j = -i\sigma_j \,. \tag{3}$$

In eq. (1.b) $\Omega_4 = 2\pi^2$ is the surface of the unit 4-dimensional sphere, $U(a)$ is the rotation matrix in the form of the "matrix, associated with the vector" $a_\mu = (a_0, \vec{a}),\ a_\mu^2 = a_\mu a_\mu = a_0^2 + \vec{a}^2$, $\mu = 0,1,2,3$,

$$U(a) = a_0e_0 + a_ie_i = a_\mu e_\mu \,. \tag{4}$$



In eqs. (1) we employ indexless matrix notation, using single and double vertical lines. It is easy to reproduce index notations, e.g., if $|e_i|\otimes\|e_i\|$ is translated to be $(e_i)_{\alpha\beta}(e_i)_{\gamma\delta}$, then $|e_0\|\otimes\|e_0|$ means $\delta_{\alpha\delta}\delta_{\gamma\beta}$. In eq. (1.b) it is explicitly stressed that we deal with integration over the group O(4). The Olinde Rodrigues parameters $a_\mu$ are in many relations more convenient than angles (the Euler angles and others[14,15]). The integral over $a_\mu$ is trivial $\left(\int d^4a\,\delta(a_\mu^2-1)a_\varkappa a_\lambda = \frac{1}{8}\delta_{\varkappa\lambda}\Omega_4\right)$, and eq. (1.b) reduces to eq. (1.a).

Usual derivation of the completeness relation begins with the proof that $\int dg\,U(g)F\bar{U}(g)$ or $e_\mu F\bar{e}_\mu$ is a constant multiple of the unit $e_0$ whatever the matrix $F = c_0e_0 + c_ie_i$ is. Note that this fact is a simple consequence of the identities

$$e_ie_0e_i = -3e_0\,, \qquad e_ie_je_i = e_j\,. \tag{5}$$

Really, the terms with $c_j$ cancel, and we have

$$e_0Fe_0 - e_iFe_i = 4c_0e_0 = 4\,\mathrm{Tr}\,F\,. \tag{6}$$

With such a normalization the trace turns out to be the real part of quaternion (Tr=Re) (The definition independent of dimensionality of representative matrices). Equation (1.a) follows from eq. (6) by substituting for $F$ all possible matrices with all but one zero entries.

In this simple way one can obtain analogs of eqs. (6) and (8) (main consequences of eq. (1.a)) for other complete matrix sets (e.g., for 4x4 $\gamma$ -matrices), and for some non-complete matrix sets (see, e.g. eqs. (17) and (18)), and also for the octonions (eqs. (62) and (66)).

Now we proceed to consider representations.

A)
$$\mathrm{Tr}(\bar{e}_\mu F)\,. \tag{7}$$

Any operator $F$ (2x2 matrix) is completely defined by the representative (7)("4-vector "), since according to eq. (1.a)

$$F = e_\mu\,\mathrm{Tr}(\bar{e}_\mu F) \tag{8}$$

(the reconstruction theorem). The multiplications of $F$ by $e_\ell$

on the left or right may be written, using (3), as operators, acting on the representative (7)

$$Tr(\bar e_\varkappa e_\ell F)=(\delta_{\varkappa\ell}\delta_{\mu 0}-\delta_{\varkappa 0}\delta_{\ell\mu}+\varepsilon_{0\varkappa\ell\mu})Tr(\bar e_\mu F)=(e^\ell_\ell)_{\varkappa\mu}Tr(\bar e_\mu F) \qquad (9)$$

$$Tr(\bar e_\varkappa F e_\ell)=(\delta_{\varkappa\ell}\delta_{\mu 0}-\delta_{\varkappa 0}\delta_{\ell\mu}-\varepsilon_{0\varkappa\ell\mu})Tr(\bar e_\mu F)=(e^r_\ell)_{\varkappa\mu}Tr(\bar e_\mu F). \qquad (10)$$

Hence the left and the right representatives of $e_\ell$ are found to be

$$(e^\ell_\ell)_{\varkappa\mu}=\delta_{\varkappa\ell}\delta_{\mu 0}-\delta_{\varkappa 0}\delta_{\ell\mu}+\varepsilon_{0\varkappa\ell\mu}=Tr(\bar e_\varkappa e_\ell e_\mu), \qquad (11)$$

$$(e^r_\ell)_{\varkappa\mu}=\delta_{\varkappa\ell}\delta_{\mu 0}-\delta_{\varkappa 0}\delta_{\ell\mu}-\varepsilon_{0\varkappa\ell\mu}=Tr(\bar e_\varkappa e_\mu e_\ell), \qquad (12)$$

where the last expressions follow from eqs. (9) and (10), putting $F=e_\mu$. Now any operator, constructed out of quaternions $e_i$, and any equations for such operators may be written in terms of these representatives (see Sec.4). The representative (7) may be used for density matrix $(F=\rho)$, and eqs. (11), (12) for operators (observables), acting on it (e.g., Hamiltonian). As a general rule for all representations, the left representatives are multiplied in the same order as $e_i$, but right ones in the inverse order.

The above representatives and the matrix $\eta=\frac{1}{2}(e_0+e^\ell_i e^r_i)$ may be written in an explicit matrix form as follows

$$e^{\ell}_0{}^{r}=\begin{pmatrix}1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix},\ e^{\ell}_1{}^{r}=\begin{pmatrix}\cdot&1&\cdot&\cdot\\ -1&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&\mp1\\ \cdot&\cdot&\pm1&\cdot\end{pmatrix},\ e^{\ell}_2{}^{r}=\begin{pmatrix}\cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&\pm1\\ -1&\cdot&\cdot&\cdot\\ \cdot&\mp1&\cdot&\cdot\end{pmatrix},\ e^{\ell}_3{}^{r}=\begin{pmatrix}\cdot&\cdot&\cdot&1\\ \cdot&\cdot&\mp1&\cdot\\ \cdot&\pm1&\cdot&\cdot\\ -1&\cdot&\cdot&\cdot\end{pmatrix},\ \eta=\begin{pmatrix}-1&\cdot&\cdot&\cdot\\ \cdot&1&\cdot&\cdot\\ \cdot&\cdot&1&\cdot\\ \cdot&\cdot&\cdot&1\end{pmatrix}, \qquad (13)$$

where dots denote zeros. The matrices $e^\ell_i$ satisfy the quaternion algebra (3), and the matrices $e^r_i$ the similar algebra, but with the change $\varepsilon_{ijk}\to-\varepsilon_{ijk}$. Both kinds of the representatives commute with each other

$$[e^\ell_i, e^r_j]=0 \qquad (14)$$

and are related as follows

$$e^r_i=-\eta e^\ell_i\eta, \qquad \eta=\frac{1}{2}(e_0+e^\ell_i e^r_i), \qquad \eta^2=e_0. \qquad (15)$$

In fact, $e^\ell_i$ and $e^r_i$ have much in common with familiar Dirac 4x4 matrices $\rho_i$ and $\sigma_i$ *). Only 16 matrices

*) However, $e^{\ell,r}_i$ are real and $e^{\ell,r\,2}_i=-e_0$, but some of $\rho_i$ and $\sigma_i$ are imaginary and $\rho_i^2=\sigma_i^2=e_0$.



$e_0$, $e^\ell_i$, $e^r_i$ and $e^\ell_i e^r_j$ together, i.e., $e^\ell_\varkappa e^r_\lambda$ with $\varkappa,\lambda=0,1,2,3$, form a complete set (but not $e^\ell_i$ or $e^r_i$ separately) with the completeness relation

$$4\,|e_0\|\otimes\|e_0|=|e^\ell_\varkappa e^r_\lambda|\otimes\|\bar e^r_\lambda\bar e^\ell_\varkappa\|. \qquad (16)$$

However, there exist analogs of usual consequences of eq. (16) for $e^\ell_i$ and $e^r_i$ separately;

$$e^\ell_0 F^\ell e^\ell_0-e^\ell_i F^\ell e^\ell_i=4\,Tr\,F^\ell \quad (Tr\equiv Re), \qquad (17)$$

$$F^\ell=e^\ell_i\,Tr(\bar e^\ell_i F^\ell), \qquad (18)$$

where $F^\ell=c_0 e^\ell_0+c_i e^\ell_i$, and the same is true for $e^r_i$. Equations (17) and (18) are derived along the line, indicated on p. 5.

Another identification of $e^\ell_i$. If we denote $1=\begin{pmatrix}1&0\\0&1\end{pmatrix}$, $i=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, then $e^\ell_i$ reduces to Pauli matrices as follows

$$e^\ell_1=i\begin{pmatrix}1&0\\0&-1\end{pmatrix}=i\sigma_3,\quad e^\ell_2=i\begin{pmatrix}0&-i\\i&0\end{pmatrix}=i\sigma_2,\quad e^\ell_3=i\begin{pmatrix}0&1\\1&0\end{pmatrix}=i\sigma_1. \qquad (19)$$

Note that

$$Tr(\bar e_\varkappa F)=F^\ell_{\varkappa\mu}Tr(\bar e_\mu)=F^\ell_{\varkappa 0}=F^r_{\varkappa 0}. \qquad (20)$$

If $\rho$ and $\hat F$ are a density matrix and any operator (both are quaternions), then the expectation value of $F$ is given by $2\,Tr(F\rho)$. Let us give possible representations of the latter in terms of our representatives.

a) Putting

$$\rho=e_\mu Tr(\bar e_\mu\rho) \qquad (21)$$

we get

$$Tr(F\rho)=Tr(e_\mu F)Tr(\bar e_\mu\rho)=Tr(\bar e_\mu\bar F)Tr(\bar e_\mu\rho)=\dots=(\bar F^\ell)_{\mu 0}(\rho^\ell)_{\mu 0}, \qquad (22)$$

where dots indicate a possibility of other dispositions of the symbol of conjugation. Here $\rho$ and $F$ are converted into column (row).

b) $$Tr(F\rho)=\alpha\,Tr(\bar e_\varkappa F\rho e_\varkappa) \quad \left(\alpha=\tfrac{1}{4}\right). \qquad (23)$$

Putting here

$$\rho e_{\varkappa} = e_{\mu} \operatorname{Tr}(\bar{e}_{\mu} \rho e_{\varkappa}) \tag{24}$$

we obtain (cf. eqs. (11) and (12)) the matrix realisation

$$\operatorname{Tr}(F\rho) = \alpha \operatorname{Tr}(\bar{e}_{\varkappa} F \rho e_{\varkappa}) = \alpha \operatorname{Tr}(\bar{e}_{\varkappa} F e_{\mu}) \operatorname{Tr}(\bar{e}_{\mu} \rho e_{\varkappa}) =$$

$$= \alpha (F^{\ell})_{\varkappa\mu} (\rho^{\ell})_{\mu\varkappa} e_0 , \tag{25}$$

where $\rho^{\ell} = \rho_0 e_0^{\ell} + \rho_i e_i^{\ell}$, $F^{\ell} = c_0 e_0^{\ell} + c_i e_i^{\ell}$.

B)

$$\operatorname{Tr}(\bar{U}(a) F) . \tag{26}$$

Due to eq. (1.b) we have the reconstruction theorem

$$F = \frac{8}{\Omega_4} \int d^4 a \, \delta(a_{\mu}^2 - 1) U(a) \operatorname{Tr}(\bar{U}(a) F) . \tag{27}$$

The representation (26) is a function of three continuous parameters (instead of the discrete index $\mu$ in A ):

$$U(a) = \frac{1}{\sqrt{a_{\nu}^2}} a_{\mu} e_{\mu} , \quad \left(\frac{a_0}{\sqrt{a_{\nu}^2}}\right)^2 + \left(\frac{\vec{a}}{\sqrt{a_{\nu}^2}}\right)^2 = 1 \tag{28}$$

(instead of $a_{\mu}$ , one can take the Euler angles).

The rotation matrices $U(a)$ form a basis like the density matrix basis. Consider properties of this basis:

$$\operatorname{Tr} \bar{U}(a) = \frac{a_0}{\sqrt{a_{\nu}^2}} , \tag{29}$$

$$\operatorname{Tr}(\bar{U}(a) e_i) = \frac{e_i}{\sqrt{a_{\nu}^2}} , \tag{30}$$

$$\operatorname{Tr}(\bar{U}(a) U(b)) = \frac{a_{\mu} b_{\mu}}{\sqrt{a_{\nu}^2}\sqrt{b_{\lambda}^2}} . \tag{31}$$

Thus, $a_i$ are like "expectation values" of matrices $e_i$ in "states" $U(a)$ (up to factor $1/\sqrt{a_{\nu}^2}$ ). Therefore, the "state" $U(a)$ is given in terms of these "expectation values".

Further, we must find left and right representatives for $e_i$ , such that

$$\operatorname{Tr}(\bar{U}(a) e_i F) = e_i^{\ell} \operatorname{Tr}(\bar{U}(a) F) , \tag{32}$$

$$\operatorname{Tr}(\bar{U}(a) F e_i) = e_i^{r} \operatorname{Tr}(\bar{U}(a) F) . \tag{33}$$



It is easy to check that

$$(a_0 e_0 - a_j e_j) e_i = a_0 e_i + a_i e_0 + \varepsilon_{ijk} a_j e_k = (a_i \partial_0 - a_0 \partial_i - \varepsilon_{ijk} a_j \partial_k)(a_0 e_0 - a_i e_i) , \tag{34}$$

$$e_i (a_0 e_0 - a_j e_j) = a_0 e_i + a_i e_0 - \varepsilon_{ijk} a_j e_k = (a_i \partial_0 - a_0 \partial_i + \varepsilon_{ijk} a_j \partial_k)(a_0 e_0 - a_i e_i) , \tag{35}$$

where $\partial_0 = \partial / \partial a_0$, $\partial_i = \partial / \partial a_i$ . Hence, left and right representatives of $e_i$ are given by

$$e_i^{\ell} = a_i \partial_0 - a_0 \partial_i - \varepsilon_{ijk} a_j \partial_k , \tag{36}$$

$$e_i^{r} = a_i \partial_0 - a_0 \partial_i + \varepsilon_{ijk} a_j \partial_k . \tag{37}$$

This representation is characterized by the same three continuous parameters as the rotation in classics (the Euler angles). Recall, that usually matrix $F$ is characterized by matrix elements between states with definite spin projections say, by $\langle m_{\frac{1}{2}} | F | m_{\frac{1}{2}}' \rangle$ , where exploited are two discrete variables having no analog in classics. The representation under consideration permits us to complete the Wigner description of spinless particles in terms of phase space, by the similar "classical" description of spin degrees of freedom.

The representations A and B are connected as follows

$$\operatorname{Tr}(\bar{e}_{\varkappa} F) = \frac{8}{\Omega_4} \int d^4 a \, \delta(a_{\mu}^2 - 1) \frac{a_{\varkappa}}{\sqrt{a_{\nu}^2}} \operatorname{Tr}(\bar{U}(a) F) , \tag{38}$$

$$\operatorname{Tr}(\bar{U}(a) F) = \frac{a_{\varkappa}}{\sqrt{a_{\nu}^2}} \operatorname{Tr}(\bar{e}_{\varkappa} F) . \tag{39}$$

The relationship between $\operatorname{Tr}(\bar{e}_{\varkappa} F)$ and $\operatorname{Tr}(\bar{U}(a) F)$ is similar to the one between a distribution and its characteristic function in mathematical statistics (they are connected via the Fourier transform). To emphasize this fact we write eqs. (38) and (39) in the other form

$$\mathcal{F}(\tfrac{1}{2} mn) = \frac{1}{8\pi^2} \int dg \, T^{\frac{1}{2}}_{mn}(g) \operatorname{Tr}(\bar{U}(g) F) , \tag{40}$$

$$\operatorname{Tr}(\bar{U}(g) F) = \sum_{mn} \bar{T}^{\frac{1}{2}}_{mn}(g) \mathcal{F}(\tfrac{1}{2} mn) , \tag{41}$$

where $T^{\frac{1}{2}}_{mn}$ are the generalized spherical harmonics of order $\frac{1}{2}$ /15/ , and $\bar{T}^{1/2}_{mn}$ are conjugated ones. The coefficients $\mathcal{F}(\frac{1}{2} mn)$ are, in fact, the traces $\operatorname{Tr}(\sigma_{mn} F)$ , where $\sigma_{mn}$ are the set of four matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ , i.e. $\frac{1}{2}(e_0 \mp i e_1)$, $\frac{1}{2}(e_2 \mp i e_3)$ . Thus, quantities $\mathcal{F}(\frac{1}{2} mn)$

are linear combinations of $\mathrm{Tr}(\bar{e}_\mu F)$. The identities (8) and (27) take the form

$$F = \frac{1}{8\pi^2}\int dg\, U(g)\,\mathrm{Tr}(\bar{U}(g)F) = \sum_{mn} \sigma_{mn}\,\mathcal{F}\left(\tfrac{1}{2}mn\right). \tag{42}$$

c)

$$\langle 0|\bar{U}(a)\,F\,U(a)|0\rangle = 2\,\mathrm{Tr}(\rho(a)F). \tag{43}$$

Any operator $F$ is completely defined by these expectation values (see eqs. (55)) in the "coherent states" /3,4/

$$|a\rangle = U(a)|0\rangle, \qquad \rho(a) = U(a)|0\rangle\langle 0|\bar{U}(a), \tag{44}$$

where $|0\rangle$ is any fixed vector (column). We adopt that $|0\rangle$ is an eigenvector of $\sigma_z = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ with eigenvalue +1 :

$$|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}, \qquad |0\rangle\langle 0| = \begin{pmatrix}1&0\\0&0\end{pmatrix} = \tfrac{1}{2}(e_0 + \sigma_z). \tag{45}$$

The coherent states are labelled also by three continuous parameters, characterizing U(a). From eq. (1.b) we immediately get the completeness relation for coherent states

$$\frac{4}{\Omega_4}\int d^4a\,\delta(a_\mu^2 - 1)\,U(a)|0\rangle\langle 0|\bar{U}(a) = e_0. \tag{46}$$

Now it is convenient to use the Cayley parameters

$$U(a) = \frac{1}{\sqrt{|u|^2+|v|^2}}\begin{bmatrix}u & -v^*\\ v & u^*\end{bmatrix}, \quad \bar{U}(a) = \frac{1}{\sqrt{|u|^2+|v|^2}}\begin{pmatrix}u^* & v^*\\ -v & u\end{pmatrix}. \tag{47}$$

The expectation values of the matrices $\sigma_i$ are

$$\begin{aligned}
\langle a|\sigma_1|a\rangle &= \frac{v^*u+u^*v}{|u|^2+|v|^2} = \frac{2(a_0a_2+a_1a_3)}{a_0^2+\vec{a}^2} = \tau_{13}\\
\langle a|\sigma_2|a\rangle &= \frac{i(v^*u-u^*v)}{|u|^2+|v|^2} = \frac{2(a_2a_3-a_0a_1)}{a_0^2+\vec{a}^2} = \tau_{23}\\
\langle a|\sigma_3|a\rangle &= \frac{|u|^2-|v|^2}{|u|^2+|v|^2} = \frac{a_0^2+a_3^2-a_1^2-a_2^2}{a_0^2+\vec{a}^2} = \tau_{33},
\end{aligned} \tag{48}$$

where the relations

$$\langle 0|\sigma_1|0\rangle = \langle 0|\sigma_2|0\rangle = 0, \qquad \langle 0|\sigma_3|0\rangle = 1 \tag{49}$$

$$\bar{U}(a)\,\sigma_i\,U(a) = \tau_{ik}\sigma_k \tag{50}$$



were used. Since the vector rotation matrix $\|\tau_{ij}\|$ is orthogonal,

$$\tau_{i3}\tau_{i3} = 1 \tag{51}$$

only two of $\tau_{13}, \tau_{23}, \tau_{33}$ are independent, and we cannot use them instead of $a_\mu$ to characterize the coherent state $|a\rangle$, unlike the usual coherent state representation. The non-normalized states $|a\rangle' = \sqrt{|u|^2+|v|^2}\,|a\rangle$ and $'\langle a|$ depend on independent variables

$$'\langle a| = (u^*\ v^*), \qquad |a\rangle' = \begin{pmatrix}u\\v\end{pmatrix} \tag{52}$$

and we easily get, e.g., the left representatives as follows

$$'\langle a|\sigma_1 = \left(v^*\frac{\partial}{\partial u^*} + u^*\frac{\partial}{\partial v^*}\right){}'\langle a|, \quad {}'\langle a|\sigma_1F|a\rangle' = \left(v^*\frac{\partial}{\partial u^*} + u^*\frac{\partial}{\partial v^*}\right){}'\langle a|F|a\rangle' \tag{53}$$

and after returning to the normalized states

$$\begin{aligned}
\langle a|\sigma_1F|a\rangle &= \left(v^*\frac{\partial}{\partial u^*} + u^*\frac{\partial}{\partial v^*} + \langle a|\sigma_1|a\rangle\right)\langle a|F|a\rangle \equiv \sigma_1^\ell\langle a|F|a\rangle\\
\langle a|\sigma_2F|a\rangle &= \left(iv^*\frac{\partial}{\partial u^*} - iu^*\frac{\partial}{\partial v^*} + \langle a|\sigma_2|a\rangle\right)\langle a|F|a\rangle \equiv \sigma_2^\ell\langle a|F|a\rangle\\
\langle a|\sigma_3F|a\rangle &= \left(u^*\frac{\partial}{\partial u^*} - v^*\frac{\partial}{\partial v^*} + \langle a|\sigma_3|a\rangle\right)\langle a|F|a\rangle \equiv \sigma_3^\ell\langle a|F|a\rangle.
\end{aligned} \tag{54}$$

The right representatives are compex conjugate to the left ones. The construction of above representatives is similar to that in the Wigner or in the coherent state representations: a differential operator plus expectation value. (see, e.g.,/16/).

Now we can express any operator $F$ via its coherent state expectation values as follows

$$F = \frac{2}{\Omega_4}\int d^4a\,\delta(a_\mu^2-1)\,e_\nu\hat{\bar{e}}_\nu\,\langle 0|\bar{U}(a)FU(a)|0\rangle = \tag{55.a}$$

$$= \frac{16}{\Omega_4^2}\int d^4a\,\delta(a_\mu^2-1)\int d^4b\,\delta(b_\nu^2-1)\,U(b)\,\hat{\bar{U}}(b)\,\langle 0|\bar{U}(a)FU(a)|0\rangle, \tag{55.b}$$

where $\hat{e}_\nu$ with $\nu=0$ equals to unity and with $\nu\neq 0$ to all $e_\nu^\ell$ from eqs. (54) (or equivalently to all $e_\nu^r$ ), acting on variables $a_\mu$. The same substitution also transforms $\bar{U}(b)$ into $\hat{\bar{U}}(b)$. One can easily check eqs. (55), using the completeness relations (1).

The connection with the previous representations A and B is clear from (55), in particular,

$$\mathrm{Tr}(e_\varkappa F)=\frac{2}{\Omega_4}\int d^4a\,\delta(a_\mu^2-1)\,\hat{e}_\varkappa\langle 0|\bar{U}(a)FU(a)|0\rangle \tag{56}$$

$$\mathrm{Tr}(U(a)F)=\frac{2}{\Omega_4}\int d^4b\,\delta(b_\mu^2-1)\,\hat{U}(a)\langle 0|\bar{U}(b)FU(b)|0\rangle\,. \tag{57}$$

This coherent state representation also uses the parameters of "classical nature".

D) $\langle 0|F|0\rangle,\ \langle 0|[e_iF]|0\rangle,\ \langle 0|e_iFe_j+e_jFe_i|0\rangle\,.$ (58)

These expectation values form one more matrix representation. Note only that it is connected with the coherent state representation like a distribution with characteristic function, in particular,

$$\langle 0|\bar{U}(a)FU(a)|0\rangle=\frac{a_0^2}{a_\nu^2}\langle 0|F|0\rangle-\frac{a_0a_i}{a_\nu^2}\langle 0|[e_iF]|0\rangle-\frac{a_ia_j}{a_\nu^2}\langle 0|e_iFe_j|0\rangle \tag{59}$$

This equation is similar to eq. (39) or (41), but now with $T^0_{mn}, T^1_{mn}$ and $T^2_{mn}$ instead of $T^{\frac{1}{2}}_{mn}$. An inverse relation follows straightforward from eqs. (55.a) and (49).

## 3. REPRESENTATIONS FOR OCTONIONS

The current discussion by F.Gürsey et al.[5-10] of symmetry and quantum-mechanical aspects of the exceptional groups makes it of interest to introduce the representations, similar to the above ones, for octonions. It is possible, in spite of the absence of the matrix realization (in the usual sense) and corresponding completeness relation. Now we assume the trace for octonions to be also the real part $(\mathrm{Tr}\equiv\mathrm{Re})$. Further we find from the octonion algebra

$$e_ie_j=-\delta_{ij}e_0+\varepsilon_{ijk}e_k\quad(i,j,k=1,2,\dots 7) \tag{60}$$

the analogs of the identities (5), (6) and (8):

$$e_ie_0e_i=-7e_0\,,\quad e_ie_je_i=5e_j \tag{61}$$

$$\frac{1}{12}(5e_0Fe_0-e_iFe_i)=c_0e_0=\mathrm{Tr}F \tag{62}$$

(where $F$ is an arbitrary octonion $F=c_0e_0+c_ie_i$) and eq. (66).



However $\quad e_0Fe_0-e_iFe_i=8c_0e_0-4c_ie_i\neq\mathrm{Tr}F\,.$ (63)

With the definition adopted for the trace $(\mathrm{Tr}=\mathrm{Re})$ we get the associativity and corresponding cyclic interchangeability under the trace sign in the situations *)

$$\mathrm{Tr}(ab),\ \mathrm{Tr}(abc),\ \mathrm{Tr}(abcb),\ \mathrm{Tr}(abca)\,, \tag{64}$$

where a,b,c are arbitrary octonions. No additional brackets are needed here.

Now introduce representations.

A) $\qquad\mathrm{Tr}(\bar{e}_\mu F)$ (65)

$$F=e_\mu\mathrm{Tr}(\bar{e}_\mu F)\,. \tag{66}$$

Left and right representatives have the form (11) and (12), and in the explicit matrix form read

$$e_1^{\overset{l}{r}}=\begin{pmatrix}\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\end{pmatrix}\quad e_2^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\end{pmatrix}\quad e_3^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\end{pmatrix}$$

$$e_4^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}\quad e_5^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}\quad e_6^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\pm1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\pm1\\\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\mp1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot&\cdot\end{pmatrix}$$

$$e_7^{\overset{l}{r}}=\begin{pmatrix}\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\\\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\mp1&\cdot\\\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\pm1&\cdot&\cdot&\cdot&\cdot\\-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\end{pmatrix}\quad e_0^{\overset{l}{r}}=\begin{pmatrix}1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}\quad \eta=\begin{pmatrix}-1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1&\cdot\\\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&1\end{pmatrix}$$

$$e_i^r=-\eta e_i^l\eta\,,\quad \eta=\frac{1}{6}(e_0+e_i^le_i^r)\,,\quad \eta^2=e_0 \tag{67, 68}$$

*) $\mathrm{Tr}$ of any associator $=0$ (associator is always pure imaginary). The associator anticommutes with imaginary part of each its element: $\{\mathrm{Im}\,a,(a,b,c)\}=0$ .

As a matter of fact, these matrices do not satisfy the octonion algebra. Instead of eq. (60) now we get

$$e_i^\ell e_j^\ell = -\delta_{ij} e_0 + \varepsilon_{ijk} \tilde{e}_k^\ell , \qquad (69)$$

where $\tilde{e}_k^\ell$ differs from $e_k^\ell$ only in sign of entries, being dependent on $i$ and $j$ .

Left and right representatives always commute in any other forms of quantum mechanics and quantum field theory/16/. But here

$$[e_i^\ell , e_j^r] \neq 0 \qquad (70)$$

because of non-associativity. Nevertheless, eqs. (9)-(12) and (20)-(25) remain valid for the octonion representatives, except for $\alpha$ which now equals $1/8$ . Equations (61)-(63) and (66) are also valid for them.

Using eq. (9) twice we get

$$Tr(\bar{e}_\varkappa (e_i (e_j F))) = (e_i^\ell e_j^\ell)_{\varkappa\mu} Tr(\bar{e}_\mu F) , \qquad (71)$$

and thus the usual matrix product $e_i^\ell e_j^\ell$ corresponds to the "ordered" product $(e_i(e_j F))$ . We get the representative of the product of two octonions as follows

$$Tr(\bar{e}_\varkappa ((e_i e_j)F)) = Tr(\bar{e}_\varkappa (e_i(e_j F) + (e_i , e_j , F))) =$$

$$= (e_i^\ell e_j^\ell + [e_i^\ell e_j^r])_{\varkappa\mu} Tr(\bar{e}_\mu F) = (e_i^\ell e_j^\ell + [e_i^r e_j^\ell])_{\varkappa\mu} Tr(\bar{e}_\mu F) \qquad (72)$$

Thus obtaining the relations

$$(e_i e_j)^\ell = e_i^\ell e_j^\ell + [e_i^\ell e_j^r] = e_i^\ell e_j^\ell + [e_i^r e_j^\ell] = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k^\ell \qquad (73)$$

$$(e_i e_j)^r = e_j^r e_i^r + [e_j^\ell e_i^r] = -\delta_{ij} e_0 + \varepsilon_{ijk} e_k^r \qquad (74)$$

as a matrix form of the octonion algebra. The additional term $[e_i^\ell e_j^r]$ arises here. The consequences of eqs. (73) and (74) are

$$[e_i^\ell e_j^r] = [e_i^r e_j^\ell] = -\tfrac{1}{4}[e_i^\ell e_j^\ell] - \tfrac{1}{4}[e_i^r e_j^r] + \tfrac{1}{2}\varepsilon_{ijk}(e_k^\ell - e_k^r) \qquad (75)$$

$$[e_i^\ell e_i^r] = 0 \qquad \text{( no summation )} \qquad (76)$$

$$e_i^\ell e_j^\ell - e_i^r e_j^r = \varepsilon_{ijk}(e_k^\ell + e_k^r) \qquad (77)$$

$$\{e_i e_j\}^\ell = \{e_i^\ell e_j^\ell\} = -2\delta_{ij} e_0 \qquad (78)$$

$$[e_i e_j]^\ell = [e_i^\ell , e_j^\ell + 2e_j^r] = [e_i^\ell + 2e_i^r , e_j^\ell] . \qquad (79)$$

The way like that in eqs. (9), (10), (71), (72) permits us to obtain many other relations. It is, in fact, the matrix realization of a more abstract mathematical formalism of the left and right multiplications /17/ . The same is valid for other our representations.

B)
$$Tr(\bar{U}(a)F) , \qquad (80)$$

$$F = \frac{16}{\Omega_8}\int d^8 a \, \delta(a_\mu^2 - 1) U(a) Tr(\bar{U}(a)F) . \qquad (81)$$

Here $U(a)$ has again the form (4) or (28), but $i$ and $\mu$ run over 1,2,...7 and 0,1,2,...7, respectively; $\Omega_8 = \pi^4/3$ is the surface of the unit 8-dimensional sphere. Equation (81) is a consequence of the identity

$$\frac{16}{\Omega_8}\int d^8 a \, \delta(a_\mu^2 - 1) \, U(a) \otimes \bar{U}(a) = e_0 \otimes e_0 - e_i \otimes e_i . \qquad (82)$$

Left and right operator representatives for octonions are defined also by eqs. (32) and (33) and are given by eqs. (36) and (37). The connection between the representations A and B are such as for the quaternions.

C)
$$Tr(V(a)\rho_0 \bar{V}(a)F) = Tr(\rho(a)F) . \qquad (83)$$

Here $\rho(a) = V(a)\rho_0 \bar{V}(a)$ is an analog of "coherent state". It is not convenient to choose $V(a) = U(a)$ since, e.g., for $\rho_0 = \frac{1}{2}(e_0 - ie_1)$

$$\frac{16}{\Omega_8}\int d^8 a \, \delta(a_\mu^2 - 1) Tr(U(a)\rho_0 \bar{U}(a) F) = Tr((4e_0 + 2ie_1)F) , \qquad (84)$$

but not $Tr F$ . Define

$$V(a) = N^{-1}(\sqrt{5}\, a_0 e_0 + a_i e_i), \quad N^2 = 5a_0^2 + a_i a_i , \qquad (85)$$

so that

$$\frac{1}{12}\frac{16}{\Omega_8}\int d^8 a \, \delta(a_\mu^2 - 1) N^2 V(a) \otimes \bar{V}(a) = \frac{1}{12}(5\, e_0 \otimes e_0 - e_i \otimes e_i) . \qquad (86)$$

The reconstruction theorem may be written as follows

$$F = \frac{1}{6}\frac{16}{\Omega_8}\int d^8 a \, \delta(a_\mu^2 - 1) N^2 \, e_\nu \hat{\bar{e}}_\nu Tr(\rho(a)F) = \qquad (87.a)$$

$$= \frac{1}{6}\left(\frac{16}{\Omega_8}\right)^2 \int d^8 a \, \delta(a_\mu^2 - 1) N^2 \int d^8 b \, \delta(b_\nu^2 - 1) U(b) \hat{\bar{U}}(b) Tr(\rho(a)F) . \qquad (87.b)$$

Connections with the representations A and B are

$$\mathrm{Tr}(e_{\varkappa}F) = \frac{1}{6}\,\frac{16}{\Omega_8}\int d^8a\,\delta(a_\mu^2-1)\,N^2\,\hat{e}_{\varkappa}\mathrm{Tr}(\rho(a)F), \quad (88)$$

$$\mathrm{Tr}(U(b)F) = \frac{1}{6}\,\frac{16}{\Omega_8}\int d^8a\,\delta(a_\mu^2-1)\,N^2\,\hat{U}(b)\mathrm{Tr}(\rho(a)F). \quad (89)$$

It is implied in eqs. (87)-(89) that $\hat{e}_\mu$ are left or right operator representatives, and $\hat{U}(b)$ is constructed out of them (cf. p. 11 ). If we represent *)

$$\begin{aligned}N^2V(a)(e_0-ie_1)\overline{V}(a) &= |u|^2(e_0-ie_1)+(|v|^2+|w|^2+|z|^2)(e_0-ie_1)-\\ &-uv^*(e_2-ie_3)+u^*v(e_2+ie_3)-uw^*(e_4-ie_7)+u^*w(e_4+ie_7)-\\ &-uz^*(e_5+ie_6)+u^*z(e_5-ie_6),\end{aligned} \quad (90)$$

we find, e.g., the left repesentatives for non-normalized states as follows

$$\begin{aligned}&N^2(V(a)(e_0-ie_1)\overline{V}(a))e_1=\\ &=i\Big(u^*\frac{\partial}{\partial u^*}-v^*\frac{\partial}{\partial v^*}-w^*\frac{\partial}{\partial w^*}-z^*\frac{\partial}{\partial z^*}\Big)N^2V(a)(e_0-ie_1)\overline{V}(a)\end{aligned} \quad (91.a)$$

$$\begin{aligned}N^2(V(a)(e_0-ie_1)\overline{V}(a))e_2 &= \Big(v^*\frac{\partial}{\partial u^*}-u^*\frac{\partial}{\partial v^*}+\frac{|w|^2+|z|^2}{u^*}\frac{\partial}{\partial v}+\frac{izu^*}{u}\frac{\partial}{\partial w^*}-\\ &-\frac{v^*w+iuz^*}{u^*}\frac{\partial}{\partial w}-\frac{iu^*w}{u}\frac{\partial}{\partial z^*}-\frac{zv^*-iuw^*}{u^*}\frac{\partial}{\partial z}\Big)N^2V(a)(e_0-ie_1)\overline{V}(a),\end{aligned} \quad (91.b)$$

etc. Representatives for normalized states are obtained after commutation of the above operators with $N^2$. The right operators are again complex conjugate to the left ones. It seems possible to replace the divisions in eq. (91.b) by second derivatives. The representative (91.b) and further ones lack simplicity, and one can suppose that a more adequate definition of coherent states and corresponding representation may be given in terms of the automorphism group of the octonion algebra /10/ :

$$\rho(a)\to\rho(a,b)=(\overline{U}(b)\overline{U}(a))\big(U(b)(U(a)\rho_0\overline{U}(a))\overline{U}(b)\big)(U(a)U(b)). \quad (92)$$

D) $\mathrm{Tr}(\rho_0F),\ \mathrm{Tr}([e_i\rho_0]F),\ \mathrm{Tr}((e_i\rho_0e_j+e_j\rho_0e_i)F),$ (93)

where $e_i\rho_0e_j+e_j\rho_0e_i=(e_i\rho_0)e_j+(e_j\rho_0)e_i=e_i(\rho_0e_j)+e_j(\rho_0e_i)$ is implied. We shall not go into further details.

*) $u=\sqrt{5}a_0+ia_1,\ v=a_2-ia_3,\ w=a_4-ia_7,\ z=a_5+ia_6.$



## 4. ON OCTONION QUANTUM MECHANICS

Now we consider as a model the quantum mechanics, which uses the octonions as operators, but is not exceptional one /13/. This r-number system was proposed by P.Jordan /12/. Really, it is equivalent to the set of above matrices $e_0^\ell, e_i^\ell$ (or $e_0^r, e_i^r$ ) with the Jordan product $A\circ B=\frac{1}{2}\{AB\}$ and simple multiplication table: $e_0^{\ell 2}=e_0^\ell,\ e_i^\ell\circ e_j^\ell=-\delta_{ij}e_0^\ell$ . We make no use of the Jordan algebra in what follows.

Eigenvalues and eigenstates. Let us solve the eigenvalue problem for the quaternion or octonion $e_1$

$$e_1\rho=\lambda_1\rho,\quad \rho e_1=\mu_1e_1, \quad (94)$$

where $\rho=\rho_0e_0+\rho_ie_i$ is unknown, and $\lambda_1$ and $\mu_1$ are eigenvalues. We have as solutions

$$\begin{array}{ll}\lambda_1=\mu_1=\pm i & \frac{1}{2}(e_0\mp ie_1)\\ \lambda_1=-\mu_1=\pm i & \left\{\begin{array}{l}\frac{1}{2}(e_2\mp ie_3)\\ \frac{1}{2}(e_4\mp ie_7)\\ \frac{1}{2}(e_5\pm ie_6)\end{array}\right.\end{array}\quad \left(\begin{array}{l}=\left\{\begin{array}{l}|\lambda_1=i\rangle\langle\mu_1=i|\\ |\lambda_1=-i\rangle\langle\mu_1=-i|\end{array}\right.\\ =\left\{\begin{array}{l}|\lambda_1=i\rangle\langle\mu_1=-i|\\ |\lambda_1=-i\rangle\langle\mu_1=i|\end{array}\right. \text{for quaternions}\end{array}\right). \quad (95)$$

There exist only four first solutions in the case of the quaternions. The first two of them are orthogonal to each other density operators. The latter two represent the non-diagonal direct products of bras and kets. In the case of octonions we have also two such groups. First of them contains also two solutions, two mutually orthogonal density operators. However, the second group includes the last six solutions.

As the eigenstates of $e_2$ one can found the density operators

$$\rho(\lambda_2=\pm i)=\frac{1}{2}(e_0\mp ie_2) \quad (96)$$

and in the case of an arbitrary axis $\vec{c}$

$$\rho(\lambda_c=\pm i)=\frac{1}{2}(e_0\mp ic_je_j),\quad c_jc_j=1. \quad (97)$$

The system of operators (95) is complete and orthogonal, when the operation Tr = Re is used as a scalar product. The system of density operators $\frac{1}{2}(e_0\mp ie_k)$ (k = 1,2,3 or 1,2,...7) is complete *), but not orthogonal.

*) One can decompose any other operator (quaternion or octonion) into a linear combination of them.

Any probability for finding one state in another is*)

$$w(\lambda_b=i,\lambda_a=i)=2\,Tr[\tfrac{1}{2}(e_0-i\vec{b}\vec{e})\tfrac{1}{2}(e_0-i\vec{a}\vec{e})]=\tfrac{1}{2}(1+\vec{a}\vec{b}),$$

$$w(\lambda_b=i,\lambda_a=i)+w(\lambda_b=-i,\lambda_a=i)=\tfrac{1}{2}(1+\vec{a}\vec{b})+\tfrac{1}{2}(1-\vec{a}\vec{b})=1, \qquad (98.a)$$

where 2 is a normalization factor. For example, $w(\lambda_1=i,\lambda_1=i)=1$,

$$w(\lambda_1=-i,\lambda_1=-i)=1,\ w(\lambda_1=-i,\lambda_1=i)=0,\ w(\lambda_1=i,\lambda_c=i)=\tfrac{1}{2}(1+c_1),\ w(\lambda_1=-i,\lambda_c=i)=\tfrac{1}{2}(1-c_1). \qquad (98.b)$$

These are valid for both quaternion and octonion cases. In the quaternion one, the latter two probabilities are in fact the well-known Pauli result.

Equations (98.a) prove that transition probabilities $w$ are always positive, $0\leqslant w\leqslant 1$, and their sum is equal to unity. These properties are conserved in the course of time evolution, since any transformation of the automorphism group preserves the form (97) ( $\vec{c}\to\vec{c}'$ with $\vec{c}'\vec{c}'=1$ ). Any similarity transformation acts analogously $(U(a)(e_0-i\vec{c}\vec{e})\overline{U}(a)=e_0-i\vec{c}'\vec{e}$, $\vec{c}'\vec{c}'=1$ ). For example, the probabilities for finding the states with definite $\lambda_1$ and $\lambda_2$ in the "coherent state" $\rho(a)=V(a)\frac{1}{2}(e_0-ie_1)\overline{V}(a)$ are given for octonions

$$w(\lambda_1=i,a)=2\,Tr[\tfrac{1}{2}(e_0-ie_1)\rho(a)]=\frac{|u|^2}{|u|^2+|v|^2+|w|^2+|z|^2},$$

$$w(\lambda_1=-i,a)=2\,Tr[\tfrac{1}{2}(e_0+ie_1)\rho(a)]=\frac{|v|^2+|w|^2+|z|^2}{|u|^2+|v|^2+|w|^2+|z|^2},$$

$$w(\lambda_1=i,a)+w(\lambda_1=-i,a)=1,$$

$$w(\lambda_2=i,a)=2\,Tr[\tfrac{1}{2}(e_0-ie_2)\rho(a)]=\frac{1}{2}-\frac{i}{2}\,\frac{uv^*-u^*v}{|u|^2+|v|^2+|w|^2+|z|^2},$$

$$w(\lambda_2=-i,a)=2\,Tr[\tfrac{1}{2}(e_0+ie_2)\rho(a)]=\frac{1}{2}+\frac{i}{2}\,\frac{uv^*-u^*v}{|u|^2+|v|^2+|w|^2+|z|^2},$$

$$w(\lambda_2=i,a)+w(\lambda_2=-i,a)=1. \qquad (98.c)$$

<u>The expectation value of an operator</u> $F$ is as usual

$$2\,Tr(\rho F). \qquad (99)$$

For example, for $F=e_1$ ($=i\sigma_2$ for quaternions) and $\rho=\frac{1}{2}(e_0-ie_1)$

$$2\,Tr(e_1\tfrac{1}{2}(e_0-ie_1))=i. \qquad (100)$$

<u>Only quantities</u> $ie_j$ serve as Hermitian operators.

*)The traces of the products of other solutions have another meaning, e.g., for quaternions the trace $2\,Tr[\frac{1}{2}(e_2-ie_3)\frac{1}{2}(-e_2-ie_3)]$ means in terms of amplitudes: $Sp(|i\rangle\langle -i|-i\rangle\langle i|)=\langle i|i\rangle\langle -i|-i\rangle=1$.



<u>Equations of motion for density operator and for "observables" $F$.</u> In the case of quaternions evolution of these operators is governed by the Neumann (Liouville) and Born-Jordan-Dirac equations

$$\frac{\partial}{\partial t}\rho(t)=-[\gamma,\rho(t)]\ \ (a),\qquad \frac{\partial}{\partial t}F(t)=[\gamma,F(t)]\ \ (b), \qquad (101)$$

with the solutions

$$\rho(t)=e^{-\gamma t}\rho(0)\,e^{\gamma t}\ \ (a),\qquad F(t)=e^{\gamma t}F(0)\,e^{-\gamma t}\ \ (b). \qquad (102)$$

The Hamiltonian $\gamma$ is pure imaginary quaternion. The evolution laws (102) are transformations of the one-parameter subgroup of the authomorphism group of the quaternion algebra.

The evolution law of octonion operators is universal*)

$$Tr\big((F_1(0)\cdots F_n(0))\,\rho(t)\big)=Tr\big((F_1(t)\cdots F_n(t))\,\rho(0)\big) \qquad (103)$$

if it is also taken to be one-parameter subgroup of the automorphism group of the octonion algebra**):

$$\rho(t)=e^{-t\,ad}\rho(0)\ \ (a),\qquad F(t)=e^{t\,ad}F(0)\ \ (b), \qquad (104)$$

where $ad\,\rho(0)\equiv[[\alpha\beta]\rho(0)]-3(\alpha,\beta,\rho(0))$. As equations of motion in the octonion case we have the Lie group equations

$$\frac{\partial}{\partial t}\rho=-[[\alpha\beta]\rho]+3(\alpha,\beta,\rho), \qquad (105.a)$$

$$\frac{\partial}{\partial t}F=[[\alpha\beta]F]-3(\alpha,\beta,F), \qquad (105.b)$$

where $\alpha$ and $\beta$ are two imaginary octonions, which together play the role of Hamiltonian. The equations (105) generalize the Neumann (Liouville) and Born-Jordan-Dirac equations to the octonion case.

The condition of conservation in time is

$$[[\alpha\beta]\rho]=3(\alpha,\beta,\rho). \qquad (106)$$

In terms of our representatives the equations of motion for the quaternions and octonions may be written as follows

$$\frac{\partial}{\partial t}\tilde{\rho}=(-\gamma^{\ell}+\gamma^{r})\tilde{\rho}\ \ (a),\qquad \frac{\partial}{\partial t}\tilde{F}=(\gamma^{\ell}-\gamma^{r})\tilde{F}\ \ (b), \qquad (107)$$

$$\frac{\partial}{\partial t}\tilde{\rho}=(-[\alpha\beta]^{\ell}+[\alpha\beta]^{r}+3[\alpha^{\ell}\beta^{r}])\tilde{\rho}\ (a)\ \frac{\partial}{\partial t}\tilde{F}=([\alpha\beta]^{\ell}-[\alpha\beta]^{r}-3[\alpha^{\ell}\beta^{r}])\tilde{F}\ (b)\, \qquad (108)$$

*)If eqs. (102) are assumed, $Tr(F(0)\rho(t))=Tr(F(t)\rho(0))$, but $Tr(F_1(0)F_2(0)\,\rho(t))\neq Tr(F_1(t)F_2(t)\,\rho(0))$.

**)Such a form was not met by the author in literature.

where $\widetilde{\rho}$ is either $\mathrm{Tr}(\overline{e}_{\varkappa}\rho)$ or $\mathrm{Tr}(\overline{U}(\alpha)\rho)$ or any other of our representatives (for $F$ analogously). These equations are matrix ones and their solutions have now the usual for the Lie groups exponential form

$$\widetilde{\rho}(t)=e^{-(\gamma^{\ell}-\gamma^{r})t}\,\widetilde{\rho}(0)\ \text{(a)},\quad \widetilde{F}(t)=e^{(\gamma^{\ell}-\gamma^{r})t}\,\widetilde{F}(0)\ \text{(b)} \qquad (109)$$

$$\widetilde{\rho}(t)=\exp\Big(\big(-[\alpha\beta]^{\ell}+[\alpha\beta]^{r}+3[\alpha^{\ell}\beta^{r}]\big)t\Big)\widetilde{\rho}(0) \qquad (110.a)$$

$$\widetilde{F}(t)=\exp\Big(\big([\alpha\beta]^{\ell}-[\alpha\beta]^{r}-3[\alpha^{\ell}\beta^{r}]\big)t\Big)\widetilde{F}(0) \qquad (110.b)$$

Equations of motion (109.a) and (110.a) for quaternions and octonions are analogs of the statistical mechanics Liouville equation. In usual quantum mechanics and quantum field theory, quantum Liouville-like equations arise analogously (see, e.g. /16/). In all these cases, as in the case of the quaternions, left and right representatives always commute with each other and "Liouvillians" split into the sum of commuting left and right Hamiltonians. This fact permits us to split the density operator into products of amplitudes (spinor for quaternions) and to adopt the Schrödinger equation for them as the equation of motion.

However, for the octonions left and right representatives do not commute because of non-associativity, and $\exp\big((-[\alpha\beta]^{\ell}+[\alpha\beta]^{r}+3[\alpha^{\ell}\beta^{r}])t\big)$ does not split. This peculiarity indicates the impossibility to introduce the amplitudes (see /7/ for other arguments).

Such Liouville-like equations have however similarity with the Schrödinger equation. Recall that in the representation A $\widetilde{\rho}$ (and $\widetilde{F}$ too) is not a matrix, but column of $\mathrm{Tr}(\overline{e}_{\varkappa}\rho)$. For example, the solutions (95) read

$$\begin{pmatrix}1\\ \mp i\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 1\\ \mp i\\ 0\\ 0\\ 0\\ 0\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\\ 0\\ 1\\ 0\\ 0\\ \mp i\end{pmatrix},\ \begin{pmatrix}0\\ 0\\ 0\\ 0\\ 0\\ 1\\ \pm i\\ 0\end{pmatrix}, \qquad (111)$$

where the first pair represents density operators.

Conservation of the probability. From eqs. (101.a) and (107.a) or (105.a) and (108.a) there follows

$$\mathrm{Tr}\,\dot{\rho}(t)=0 \ \text{(a)}, \qquad \dot{\widetilde{\rho}}_0(t)=0 \ \text{(in repres. A) (b)}, \qquad (112)$$



the latter due to the fact, that matrices $\gamma^{\ell}-\gamma^{r}$ and $[\alpha\beta]^{\ell}-[\alpha\beta]^{r}-3[\alpha^{\ell}\beta^{r}]$ have only zero entries in zero rows and columns (both matrices $[\alpha\beta]^{\ell}-[\alpha\beta]^{r}=2\varepsilon_{ijk}\alpha_i\beta_j(e^{\ell}_k-e^{r}_k)$ and $[\alpha^{\ell}\beta^{r}]$ are such separately). Thus, the total probability conserves

$$\mathrm{Tr}\,\rho(t)=\mathrm{Tr}\,\rho(0) \ \text{(a)}, \qquad \widetilde{\rho}_0(t)=\widetilde{\rho}_0(0) \ \text{(in repres. A) (b)}. \qquad (113)$$

In other representations analogously.

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