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Исключительные калибровочные теории в 3x3 матричном формализме

Предлагается матричная интерпретация конструкции Титса-Винберга для исключительных групп в терминах 3x3 матриц над алгеброй Розенфельда, в которую посредством введенного внешнего умножения включены автоморфизмы. Это позволяет представить исключительные калибровочные теории в компактном виде, что упрощает обращение с соответствующими моделями.

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Exceptional Gauge Theories in 3x3 Matrix Formalism

A matrix interpretation of Tits-Vinberg construction for the Lie algebras of exceptional groups is proposed. We use 3x3 matrices over Rosenfeld algebra completed with its automorphisms which are induced via external muliplication of imaginary units. This allows one to express exceptional gauge theories in a compact and comprehensible form suitable for purposes of model building.

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1. Recently, interesting attempts to unify strong, weak and electromagnetic interactions into a single gauge theory based upon exceptional group have been made by Feza Gürsey and other authors (Gürsey 1975, Gürsey et al. 1976 a,b, Sikivie and Gürsey 1977, Konstein et al. 1977, Ramond 1976, Ramond 1977). However, exceptional groups possess high dimensions (e.g., $\dim E_6=78, \dim E_7=133$) and a huge number of structure constants, respectively. Therefore in the standard gauge theory formulation where the fundamental representation of dimension N is described by an element of N-dimensional vector space, etc., all the calculations become tedious that makes the results to be hardly comprehensible. At the same time a tight connection is well known between all the exceptional groups and 3x3 matrices over the Cayley algebra of octaves. Their fundamental representations can be treated compactly in terms of such matrices and this fact was used by Gürsey (1975). However, their adjoint representations include also the automorphisms of the so-called Rosenfeld algebras.

In the present paper some external multiplication operation of imaginary units is introduced to represent these automorphisms. In such a way a purely matrix formulation of the Tits-Vinberg construction is achieved entirely in terms of 3x3 matrices. The formalism proposed gives a possibility to formulate the exceptional gauge theories in compact and adequate to group structure form. Both fundamental and adjoint representations are found to be generalized 3x3 matrices. The invariants take a simple form. All the calculations reduce to multiplications of these matrices and manipulations with the traces of their products. Reduction to physically interesting subgroups of initial grand group becomes trivial.

2. According to the famous Hurwitz theorem all the hypercomplex systems of numbers reduce to the algebras of real numbers (R), complex numbers (C), quaternions (Q) and octaves (O).

Table 1

algebra	R	C	Q	O
basis	1	$1, i$	$1, q_i, i=1,2,3$	$1, e_\alpha, \alpha=1\ldots7$
multiplication table	$1\cdot1=1$	$i^2=-1$	$q_i q_k = -\delta_{ik} + \epsilon_{ik\ell} q_\ell$	$e_\alpha e_\beta = -\delta_{\alpha\beta} + f_{\alpha\beta\gamma} e_\gamma$
involution	–	$i \overset{*}{\to} -i$, $1 \overset{*}{\to} 1$	$q \overset{*}{\to} -q$, $1 \overset{*}{\to} 1$	$e_\alpha \overset{*}{\to} -e_\alpha$, $1 \to 1$
continuous group of automorphisms	–	–	SO(3)	G_2

$\epsilon_{ik\ell} \equiv$ totally antisymmetric tensor of rank 3, $f_{\alpha\beta\gamma}$ are totally antisymmetric and $f_{\alpha\beta\gamma}=1$ for $\alpha\beta\gamma =$ (1 2 3) (5 1 6), (624), (435), (471), (673), (572). Let us define the quantities $e_{\alpha\beta}$ as to represent infinitesimal automorphisms of $\boldsymbol{O}$:

$$[e_{\alpha\beta}, e_\gamma] = L_{\alpha\beta,\gamma\delta}\, e_\delta \tag{1}$$

$$[e_{\alpha\beta}, e_{\gamma\delta}] = C_{\alpha\beta,\gamma\delta,\mu\nu}\, e_{\mu\nu}, \tag{2}$$

where $L_{\alpha\beta,\gamma\delta}$ is a realization of infinitesimal automorphism on imaginary units e_α

$$L_{\alpha\beta,\gamma\delta} = 3\delta_{\alpha\gamma}\delta_{\beta\delta} - 3\delta_{\alpha\delta}\delta_{\beta\gamma} - f_{\alpha\beta\epsilon} f_{\epsilon\gamma\delta}$$

and $C_{\alpha\beta,\gamma\delta,\mu\nu}$ are structure constants of group G_2 in such a realization.

Now we introduce an external multiplication of imaginary units

$$e_\alpha \vee e_\beta = -e_\beta \vee e_\alpha \equiv e_{\alpha\beta} \tag{3}$$

and supply the resulting set of elements $\xi = \{1, e_\alpha, e_{\alpha\beta}\}$ with structure of an algebra with involution. The latter must be in agreement with involution in $\{1, e_\alpha\}$. Then transformation properties of e_α and $e_{\alpha\beta}$ under the automorphisms group together with the involution " * "

$$e_\alpha \overset{*}{\to} -e_\alpha, \quad e_{\alpha\beta} \overset{*}{\to} -e_{\alpha\beta} \tag{4}$$

fix the anticommutators:

$$\{e_{\alpha\beta}, e_\gamma\} = 0 \tag{5}$$

$$\{e_{\alpha\beta}, e_{\gamma\delta}\} = L_{\alpha\beta,\gamma\delta} \tag{6}$$

(An arbitrary constant in the right-hand side of eq. (6) is chosen to be 1 by reasons of subsequent realization of Tits-Vinberg construction). Further definitions

$$e_{\alpha\beta} \vee e_{\gamma\delta} = \frac{1}{2}[e_{\alpha\beta}, e_{\gamma\delta}] \tag{7}$$

$$e_\alpha \vee e_{\beta\gamma} = \frac{1}{2}[e_\alpha, e_{\beta\gamma}] \tag{8}$$

give us the construction of $\xi_{\boldsymbol{O}}$ closed with respect to both usual and external multiplication. We may formally introduce an external multiplication in the rest of Hurwitz algebras setting it equal to usual commutator of imagniary units. In $\boldsymbol{R}$ and $\boldsymbol{C}$ it results identically in zero, and in Q it gives an element $\epsilon_{ijk} q_k$ because

$$[\epsilon_{ijk} q_k, q_\ell] = (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) q_\ell$$

realizes an infinitesimal automorphism of SO(3) on q_i. Now we have to consider an arbitrary Rosenfeld algebra $H^m_B \equiv A^m \times O$,

$$H^1_8 = R \times O \equiv O,\ H^2_8 = C \times O,\ H^4_8 = Q \times O,\quad H^8_8 = O \times O .$$

Denote by q^m_A imaginary units of A^m. Basic elements of H^m_8 are

$$1, e_\alpha, q^m_A, e_\alpha q^m_A, \quad A = 1,2\ldots,$$

involution is given by

$$e_\alpha \overset{*}{\to} -e_\alpha,\quad q^m_A \overset{*}{\to} -q^m_A \quad ; \quad e_\alpha q^m_A \overset{*}{\to} e_\alpha q^m_A .$$

The external multiplication may be introduced naturally in H^m_8 by (3), (7), (8) for q^m_A and e_α separately and by

$$q^m_A \vee e_\alpha = 0 \qquad (9)$$

$$(q^m_A e_\alpha) \vee (q^m_B e_\beta) \equiv \frac{1}{2}\{q^m_A q^m_B\}(e_\alpha \vee e_\beta) + \frac{1}{2}\{e_\alpha e_\beta\}(q^m_A \vee q^m_B). \qquad (10)$$

Basic elements of corresponding ξ-construction are

$$(1, q^m_A, q^m_{AB}, e_\alpha, e_{\alpha\beta}, e_\alpha q^m_A).$$

It is clear that

$$[q^m_{AB}, e_\alpha] = 0; \quad [e_{\alpha\beta}, q^m_A] = 0. \qquad (11)$$

Thus ξ-construction over algebra H^m_8 introduces in the latter its infinitesimal automorphisms. Using it one may simplify the Tits-Vinberg construction (Vinberg 1966) for the Lie algebras of exceptional groups and utilize the latter efficiency in gauge theory construction. Namely, an element of Lie al-

gebra of exceptional group F_4, E_6, E_7, E_8 is represented by a sum of antihermitian traceless matrix and infinitesimal automorphism in H^1_8, H^2_8, H^4_8, H^8_8, respectively. $\xi_{H^m_8}$ being introduced, gives us a possibility of representing such an element as a single 3x3 antihermitian matrix over $\xi_{H^m_8}$, which is a sum of antihermitian traceless matrix over H^m_8 and some linear combination of elements q^m_{AB} and $e_{\alpha\beta}$

$$\tilde{\mathfrak{A}} = \mathfrak{A} + (a_{AB} q^m_{AB} + a'_{\alpha\beta} e_{\alpha\beta})\, E, \qquad (12)$$

where E is the 3x3 unit matrix.

The Lie multiplication takes a simple form and is given by a generalized commutator:

$$[\tilde{\mathfrak{A}}\tilde{\mathfrak{B}}]_g = [\tilde{\mathfrak{A}}\tilde{\mathfrak{B}}] - \frac{1}{3}\mathrm{Tr}\,[\tilde{\mathfrak{A}}\tilde{\mathfrak{B}}]E + \frac{1}{3}\mathrm{Tr}\,[\tilde{\mathfrak{A}} \vee \tilde{\mathfrak{B}}]E . \qquad (13)$$

Here $[\tilde{\mathfrak{A}},\tilde{\mathfrak{B}}]$ is a matrix commutator, $[\tilde{\mathfrak{A}} \vee \tilde{\mathfrak{B}}]$ is a matrix commutator in which matrix elements are multiplying externally. The last term in r.h.s. of (13) is needed for validity of the Jacobi identity and represents the corresponding term in the Tits-Vinberg construction in terms of $\xi_{H^m_8}$. Thus, we have a realization of adjoint representation of any exceptional group as corresponding antihermitian 3x3 matrix. For the purpose of gauge theory building we have to know the transformations of the fundamental representation. The following trick is convenient to obtain fundamental representations (especially having in mind 56 of E_7). Using the fact that

$$F_4 \subset E_6 \subset E_7 \subset E_8 \qquad (14)$$

with the help of decomposition of antihermitian in H^m_8 matrix $\mathfrak{A}$

$$\mathfrak{A} = A + q_A M_A, \qquad (15)$$

where A is antihermitian, M_A are hermitian 3x3 matrices over octaves, it is easy to extract an adjoint representation of lower group from the adjoint representation of any (except F_4) group in (14) and, using (13), to establish the transformational properties of reminder under transformations of this lower group. For example, after exclusion of 52 (adjoint representation of F_4) from 78 - adjoint representation of E_6 , 26 components remain, transforming according to the fundamental representation of F_4 . In such a way we get the known decompositions

$$\begin{array}{lll} E_6 & \text{with respect to } F_4 & 78=52+26, \\ E_7 & \text{with respect to } E_6 & 133=78+27+\overline{27}+1, \\ E_8 & \text{with respect to } E_7 & 248=133+56+56+1+1+1, \end{array} \quad (16)$$

and the following realizations of exceptional groups in fundamental representations given in table 2.

Now having all needed in hand, we proceed to build the gauge E_6 -theory on octonion formalism. In such a theory the fundamental fermions combine in 27-plet which is 3x3 matrix N over H_8^2 , such that $N^+ = N$. The gauge fields (vector mesons) form 78-plet and they are represented by antihermitian matrix $\tilde{\mathfrak{A}}_\mu$ over $\xi_{H_8^2}$. Covariant derivative is

$$D_\mu N = \partial_\mu N + e(\tilde{\mathfrak{A}}_\mu N + N\tilde{\mathfrak{A}}_\mu^+), \quad (17)$$

e ≡ gauge constant, spinor indices being suppressed. The covariant curl of vector field is

$$\tilde{\mathfrak{F}}_{\mu\nu} = \partial_\mu \tilde{\mathfrak{A}}_\nu - \partial_\nu \tilde{\mathfrak{A}}_\mu + e[\tilde{\mathfrak{A}}_\mu, \tilde{\mathfrak{A}}_\nu]_g . \quad (18)$$

For representations under consideration bilinear and trilinear invariants are

$$Tr\{N^*, M\}, \quad (19a)$$

$$Tr\{N^*, (\tilde{\mathfrak{A}} M + M\tilde{\mathfrak{A}}^+)\}, \quad (19b)$$

$$Tr(\tilde{\mathfrak{A}}\tilde{\mathfrak{B}} + \tilde{\mathfrak{B}}^*\tilde{\mathfrak{A}}^*). \quad (19c)$$

Table 2

Group	adjoint representation	dim	fundamental representation	dim	realization of group in fundamental representation
F_4	Antihermitian 3×3 matrix over $\xi_{H_8^1}$ (see (12)) $\tilde{\mathfrak{A}}_1$	52	J * Tr J=0	26	$\delta_{F_4} J = [\tilde{\mathfrak{A}}_1, J]$
E_6	Antihermitian 3×3 matrix over $\xi_{H_8^2}$ $\tilde{\mathfrak{A}}_2$	78	$N = J_1 + iJ_2$	27	$\delta_{E_6} N = \tilde{\mathfrak{A}}_2 N + N\tilde{\mathfrak{A}}_2^+$
				$\overline{27}$	$\delta_{E_6} N = \tilde{\mathfrak{A}}_2^* N + N\tilde{\mathfrak{A}}_2^{*+}$ **
E_7	Antihermitian 3×3 matrix over $\xi_{H_8^4}$ $\tilde{\mathfrak{A}}_4$ $\tilde{\mathfrak{A}}_4 = \tilde{\mathfrak{A}}_2 + wa^* + w^*a + q_3\phi E$ where $a = J_1 + iJ_2$ $w = \frac{q_1 + iq_2}{2}$	133	A set (X, Y, η, ξ) where η, ξ are complex numbers, X and Y are of form $J_1 + iJ_2$	56	$\delta_{E_7} X = \tilde{\mathfrak{A}}_2 X + X\tilde{\mathfrak{A}}_2^+ + 2a^* \times Y - -ia\eta - \frac{i}{3}\phi X$ $\delta_{E_7} Y = \tilde{\mathfrak{A}}_2^* Y + Y\tilde{\mathfrak{A}}_2^{*+} - 2a \times X + +ia\xi + \frac{i}{3}\phi Y$ *** $\delta_{E_7} \eta = -\frac{i}{2} Tr\{a^*, X\} - i\phi\eta$ $\delta_{E_7} \xi = \frac{i}{2} Tr\{a, Y\} + i\phi\xi$
E_8	Antihermitian 3×3 matrix over $\xi_{H_8^8}$ $\tilde{\mathfrak{A}}_8$	248	Coincides with adjoint representation $\tilde{\mathfrak{A}}_8$	248	See (13)

* Here a symbol J is used for hermitian 3×3 matrix over **O**.

** Here and thereafter a symbol "+" denotes hermitean conjugation in **O** while symbol "*" denotes replacement $i \to -i$.

*** Freudenthal product is : $a \times b \equiv \frac{1}{2}(\{a,b\} - Tr(a)b - Tr(b)a + (Tr(a)Tr(b) - \frac{1}{2}Tr\{a,b\})E)$.

Here N , M are 27-plets, $\tilde{\mathfrak{A}}$, $\tilde{\mathfrak{B}}$ are 78-plets. Using eq. (19) the invariant Lagrangian is written in the compact form

$$\mathcal{L} = \mathrm{Tr}\{N^*, \not{D} N\} + \frac{1}{4}\mathrm{Tr}(\tilde{\mathcal{F}}_{\mu\nu}\tilde{\mathcal{F}}^{\mu\nu} + \tilde{\mathcal{F}}^{\mu\nu *}\tilde{\mathcal{F}}^*_{\mu\nu}), \qquad (20)$$

$\not{D} \equiv \gamma_\mu D_\mu$, spinor indices being suppressed. Analogously it is easy to introduce the scalar field, transforming as <u>27</u> or <u>78</u> in the theory. The trilinear invariant

$$\mathrm{Tr}\{(\Phi \times \Phi), \Phi\}, \quad \Phi = \underline{27} \qquad (21)$$

is useful for constructing its Higgs self interaction together with (19a) and its square. This very compact and elegant form of the E_6 -gauge theory can be translated into the customary language of complex numbers using the so-called basis of the Cayley algebra:

$$u_a = \frac{e_a + ie_{a+3}}{2}; \qquad u^*_a = \frac{e_a - ie_{a+3}}{2} \qquad (22)$$

$$u_0 = \frac{1 + ie_7}{2}; \qquad u^*_0 = \frac{1 - ie_7}{2} \qquad (a = 1.2.3)$$

The advantages of such a basis were mentioned by Günaydin and Gürsey (1973) (see the corresponding multiplication table therein).

We should remark only, that u_a and u^*_a transform as a triplet and antitriplet, respectively under $SU^c(3)$. The latter is the subgroup of the automorphism group of octaves which leaves e_7 invariant and which is identified with the colour symmetry in theories under consideration. Introducing the split-basis in fundamental and adjoint representations one comes to the following decompositions of matrices over octaves

$$N = Mu^*_0 + M^T u_0 + M_a u^*_a + N_a u_a, \qquad (23)$$

where M is a usual (complex) matrix of general form, "T" denotes transposition of matrix, N_a and M_a are complex matrices of the form

$$\begin{pmatrix} 0 & \omega_a & \mu_a \\ -\omega_a & 0 & \nu_a \\ -\mu_a & -\nu_a & 0 \end{pmatrix} \qquad (24)$$

$$\tilde{\mathfrak{A}} = Bu^*_0 + \tilde{B}u_0 + A_a u^*_a + A^+_a u_a + (G_{\substack{ik \\ i,k \neq 7}} e_{ik} + \theta_k e_{7k})E;$$

$$i, k = 1, \ldots, 7.,$$

where B and $\tilde{B}$ are antihermitian traceless complex matrices A_a is a traceless complex matrix of general form, and G_{ik}, θ_k are real, "+" with respect to usual matrices means hermitean conjugation. Known decompositions of E_6 with respect to maximal subgroup SU(3) × SU(3) × SU(3)

$$\underline{27} = (3.\bar{3}.1^c) + (\bar{3}.1.\bar{3}^c) + (1.3.3^c)$$

$$\underline{78} = (8.1.1^c) + (1.8.1^c) + (3.3.\bar{3}^c) + (\bar{3}.\bar{3}.3^c) + \qquad (25)$$

$$+ (1.1.8^c)$$

are derived immediately from definitions (13) and Table 2, provided the generators of remaining after extracting of $SU^c(3)$ groups $SU(3)_{1(2)}$ are

$$SU(3)_1 : \frac{e_7 + i}{2}\lambda_a = iu^*_0\lambda_a \qquad (\lambda_a - \text{Gell–Mann matrices } a = 1.2\ldots8). \qquad (26)$$

$$SU(3)_2 : \frac{e_7 - i}{2}\lambda_a = -iu_0\lambda_a$$

According to these decompositions

M transforms as $(3.\bar{3}.1^c)$

M_a – as $(\bar{3}.1.\bar{3}^c)$, N_a – as $(1.3.3^c)$, B – as $(8.1.1^c)$

$\tilde{B}$ – as $(1.8.1^c)$, G_{ik} – as $(1.1.8^c)$,

$\tilde{A}_a$ – as $(3.3.\bar{3}^c)$ and $\tilde{A}^+_a$ – as $(\bar{3}.\bar{3}.3^c)$,

$$\tilde{A}_a \equiv A_a + \frac{i\theta_a}{3}E,$$

respectively. Here

$\theta_k e_{7k} \equiv \tilde{\theta}_\alpha v^*_\alpha + \tilde{\theta}^*_\alpha v_\alpha$, $\tilde{\theta}_\alpha \equiv \theta_\alpha + i\theta_{\alpha+3}$, $v_\alpha \equiv e_{7\alpha} + ie_{7\alpha+3}$
$\alpha = 1,2,3$.

The Lagrangian of interaction in terms of these decompositions is written as

$$\mathcal{L}_{int} = e\,\mathrm{Tr}\,(M^+ \mathbb{B} M + M^* \tilde{\mathbb{B}} M^T + M^+_\alpha A_\alpha M^T + N^+_\alpha \mathbb{A}^+_\alpha M^T -$$
$$- M^*_\alpha \mathbb{B} M_\alpha - N^*_\alpha \tilde{\mathbb{B}} N_\alpha - \frac{1}{2} M^*_\alpha \mathbb{G}_A \Lambda^A_{\alpha\beta} M_\beta - \qquad (27)$$
$$- \frac{1}{2} N^*_\alpha \mathbb{G}_A \Lambda^A_{\alpha\beta} N_\beta - M^*_\alpha \tilde{\mathbb{A}}^+_\beta N_\gamma \epsilon_{\alpha\beta\gamma} +$$
$$+ N^*_\alpha \tilde{\mathbb{A}}_\beta M_\gamma \epsilon_{\alpha\beta\gamma}) + h.c.,$$

where we defined

$G_{ik} e_{ik} u_\alpha \equiv G_A \Lambda^A_{\alpha\beta} u_\beta$,
$A = 1.2 \ldots 8$

$\Lambda^A_{\alpha\beta}$ is a realization of $SU^c(3)$ over u_α, " $*$ " over matrix means complex conjugation, $\mathbb{B} \equiv \gamma_\mu B_\mu$, etc. If electric charge in such a theory is defined as

$$Q \equiv Q_1 + Q_2 , \qquad (28)$$

where $Q_{1(2)}$ is the electric charge operator in $SU(3)_{1(2)}$ (cf. Gürsey 1975), then matrix M in 27-plet of fermions will represent leptons, N_α and M_α are quarks and antiquarks. Among vector mesons we find intermediate bosons of weak interaction B_μ and $\tilde{B}_\mu$, gluons $G_{A\mu}$ which mediate strong interaction and leptoquarks $A_{\alpha\mu}$ whose interactions do not conserve baryon number. The structure of all interactions is evident from (27). Now we shall discuss briefly the E_7-theory, which is most interesting in view of the applications. Note that the known decompositions with respect to maximal subgroup $SU(6) \times SU^c(3)$

can be easily obtained directly from (13) and Table 2.

$$\underline{56} = (20.1^c) + (6.3^c) + (\overline{6}.\overline{3}^c)$$
$$\underline{133} = (35.1^c) + (15.\overline{3}^c) + (\overline{15}.3^c) + (1.8^c). \qquad (29)$$

In comparison with the above E_6-scheme E_7 theory contains a richer spectrum of leptons, additional quarks, leptoquarks and intermediate bosons. The kinetic term of the theory is built with the help of invariant

$$\xi^*\xi + \eta^*\eta + \mathrm{Tr}\frac{1}{2}\{X^*X\} + \mathrm{Tr}\frac{1}{2}\{Y^*Y\}. \qquad (30)$$

The unitary transformation, connecting representation 56* and 56, can be easily found

$$\underline{56}^* \to (-Y^*, X^*, -\xi^*, \eta^*) = \underline{56}. \qquad (31)$$

(This reflects the pseudoreality of this representation). Hence, another form of this invariant exists:

$$\xi_1 \eta_2 - \xi_2 \eta_1 + \mathrm{Tr}\frac{1}{2}\{X_1 Y_2\} - \mathrm{Tr}\frac{1}{2}\{X_2 Y_1\} \qquad (32)$$

$(X_i Y_i$, $\xi_i \eta_i)$, $i = 1.2$ are two 56-plets. An interaction is given by invariant

$$\xi^* \Delta_{E_7} \xi + \eta^* \Delta_{E_7} \eta + \mathrm{Tr}\frac{1}{2}\{X^*, \Delta_{E_7} X\} + \mathrm{Tr}\frac{1}{2}\{Y^*, \Delta_{E_7} Y\} + h.c. \qquad (33)$$

where Δ_{E_7} is the E_7-transformation given in Table 2. in which all parameters are replaced by corresponding gauge fields which are multiplied by γ-matrices. The kinetic term and interaction for vector fields are built with the help of invariant

$$\mathrm{Tr}\,(\tilde{\mathfrak{A}}_4 \tilde{\mathfrak{B}}_4 + \tilde{\mathfrak{B}}^\#_4 \tilde{\mathfrak{A}}^\#_4) \qquad (34)$$

(# denotes quaternion conjugation), in complete analogy with E_6-theory. Again, the introduction of scalar 56-plet or 133-plet is possible. Its self-interaction is given in the first case by the invariant

$$\mathrm{Tr}\frac{1}{2}\{X\times X, Y\times Y\} - \xi\,\mathrm{Tr}\frac{1}{2}\{X\times X, X\} - \eta\,\mathrm{Tr}\frac{1}{2}\{Y\times Y, Y\} - \frac{1}{4}\left(\mathrm{Tr}\frac{1}{2}\{XY\} - \xi\eta\right)^2 \qquad (35)$$

(cf. Jacobson 1971).

For understanding the structure of the theory we need the Lagrangian of interaction in a reduced form. Making use of decompositions (notation is the same as in Table 2)

$$\begin{aligned} X &= M u_0^* + M^T u_0 + M_\alpha u_\alpha^* + N_\alpha u_\alpha \\ Y &= P u_0^* + P^T u_0 + P_\alpha u_\alpha^* + T_\alpha u_\alpha \\ a &= a_0 u_0^* + a_0^T u_0 + a_\alpha u_\alpha^* + \tilde{a}_\alpha u_\alpha \end{aligned} \qquad (36)$$

we obtain the Lagrangian of interactions as a sum of terms (27) for X and Y separately and the following expression carrying new peculiar E interactions

$$\begin{aligned} \mathcal{L}_{\mathrm{int}\,E_7} = e\,\mathrm{Tr}[\, & M^+ a_0^+ P + M^* a_0^* P^T - M^* a_\alpha^* P_\alpha - M^+ \tilde{a}_\alpha^* T_\alpha - \\ & - M_\alpha^* a_0^+ P_\alpha - M_\alpha^* \tilde{a}_\alpha^* P^T + \epsilon_{\alpha\beta\gamma} M_\alpha^* a_\beta^* T_\gamma - N_\alpha^* a_0^* T_\alpha - \\ & - N_\alpha^* a_\alpha^* P + \epsilon_{\alpha\beta\gamma} N_\alpha^* \tilde{a}_\beta^* P_\gamma - \mathrm{Tr}(M) a_0^+ P + \\ & + \frac{1}{2}\mathrm{Tr}(M) a_\alpha^* P_\alpha + \frac{1}{2}\mathrm{Tr}(M)\tilde{a}_\alpha^* T_\alpha + \mathrm{Tr}(M^*)\mathrm{Tr}(a_0^*) P - \\ & - M^+ P\,\mathrm{Tr}(a_0^+) + \frac{1}{2} M_\alpha^* P_\alpha \mathrm{Tr}(a_0^+) + \frac{1}{2} N_\alpha^* T_\alpha \mathrm{Tr}(a_0^+) - \\ & - M^+ a_0^+ \mathrm{Tr}(P) - \frac{1}{2} M_\alpha^* \tilde{a}_\alpha^* \mathrm{Tr}(P) - \frac{1}{2} N_\alpha^* a_\alpha \mathrm{Tr}(P) - \\ & - \frac{i\phi}{3} M^+ M + \frac{i}{6}\phi (M_\alpha^* M_\alpha + N_\alpha^* N_\alpha) - \\ & - i\eta(\{M^+ a_0\} - (M_\alpha^* a_\alpha + N_\alpha^* \tilde{a}_\alpha))] + \\ & + (X \to Y,\ a^* \to -a,\ \eta \to -\xi) + \\ & + i\bar{\xi}\phi\xi + \frac{i}{2}\bar{\xi}\,\mathrm{Tr}(\{a_0 P\} - (\tilde{a}_\alpha P_\alpha - a_\alpha T_\alpha)) - \\ & - i\bar{\eta}\phi\eta - \frac{i}{2}\bar{\eta}\,\mathrm{Tr}(\{a_0^+ M\} - (a_\alpha^* M_\alpha + \tilde{a}_\alpha^* N_\alpha) + \\ & + \mathrm{h.c.} \end{aligned} \qquad (37)$$

Here $(X \to Y,\ a^* \to -a, \eta \to -\xi)$ denotes terms, obtained from the previous ones by these replacements. The Lagrangian (37) has a clear structure and fixed all the new couplings in the theory in comparison with E_6 one. Its length reflects a richer symmetry in the E_7 -case.

3. Thus, it has been shown that the 3x3 matrix formalism gives the exceptional gauge theories in compact and clear form and simplifies essentially their treatment. The evident structure of the theory in such a formalism makes the task of physical model building much more easy and clarifies the problem of assignment. These subjects will be discussed elsewhere.

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