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GENERALIZED DISPERSION RELATIONS ON PARABOLIC MANIFOLDS FOR PION-PION SCATTERING

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GENERALIZED DISPERSION RELATIONS ON PARABOLIC MANIFOLDS FOR PION-PION SCATTERING

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Обобщенные дисперсионные соотношения на параболических многообразиях для $\pi\pi$-рассеяния

Обсуждается возможность вывода обобщенных дисперсионных соотношений для полной амплитуды в скалярном случае с помощью параболических многообразий в (x,y) плоскости переменных Вандерса. Сформулированные дисперсионные соотношения дают новые представления для парциальных амплитуд типа Роя, справедливых в той же самой области, как и в случае линейных многообразий.

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Generalized Dispersion Relations on Parabolic Manifolds for Pion-Pion Scattering

The possibility is discussed to derive generalized dispersion relations for total amplitudes in the scalar case by using parabolic manifolds in the x-y plane of Wanders variables. The dispersion relations formulated will yield new partial wave relations of the Roy type being valid in the same region as in the case of linear manifolds.

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1. In recent years partial wave relations of the Roy-type [1] have extensively been used in the phenomenology of pion-pion elastic scattering [2] and in finding semi-phenomenological as well as rigorous absolute bounds for low energy parameters [3]. The original equations derived on the basis of fixed-t dispersion relations for the total amplitudes are valid in a restricted region of s-values and do not involve crossing symmetry. Within the axiomatic framework Mahoux, Roy and Wanders (in the following MRW) could prove completely crossing symmetric generalized dispersion relations and got a family of Roy-type equations with the extended validity domain $-28m_\pi^2 \le s \le 125m_\pi^2$ [4]. They used linear manifolds for this, where x and y are crossing symmetric combinations of the Mandelstam variables s,t,u [5].

$$x = -\frac{1}{16}(st + tu + us), \quad y = \frac{1}{64}stu. \tag{1}$$

(The mapping of the real Mandelstam plane onto the (x,y) plane is explained in ref. [4] and summarized in the appendix).

Later on, Auberson and Epele extended the above given domain up to $s_{max} = 164.7m_\pi^2$ by applying hyperbolic manifolds [6]. However, it is rather complicated to write down the corresponding dispersion relations explicitly. In order to get more restrictive bounds on scattering lengths and the total amplitudes than till now [7] it should be useful to extend the class of generalized dispersion relations which can be employed practically.

We emphasize that in general new dispersion relations derived on the basis of different curves will satisfy certain sum rules automatically. This circumstance is important when dispersion relations are phenomenologically used[8]. However, as is shown by Orlov and Shirkov [9], e.g., such sum rules can be obtained in the framework of representations proved for fixed t if appropriate assumptions about the existence of derivatives of the amplitudes at $s=4$, $t=0$ are made. One can hope that one-dimensional representations valid in the whole complex s plane will supply essentially new restrictions. But the derivation of such representations runs into mathematical difficulties, therefore the problem really to be solved consists in extending the validity range of the representations.

In this paper we formulate new dispersion representations on parabolic manifolds. The corresponding Roy-type relations are valid at least in the same domain as for linear manifolds.

2. For simplicity we regard the scalar case. The generalization to the charged case is straightforward by introducing appropriate combinations of amplitudes with definite isospin [10]. Furthermore we assume Mandelstam analyticity. In our context this means the following.

(i) The dynamical singularities of the total amplitude $F(s,t) \equiv \hat{F}(x,y)$ are determined by the zeroth of the denominator of the Mandelstam representation

$$M(\alpha, s, t) = (\alpha - s)(\alpha - t)(\alpha - u) \equiv \hat{M}(x,y) = \alpha^2(\alpha - 4) - 16\alpha x - 64y = 0 \tag{2}$$

for all $\alpha \geq 4$ $(m_\pi^2 = 1)$.

(ii) The convergence domain of the partial wave expansion in the s-channel, for instance, is given by the Lehmann ellipse the right extremity r of which is determined by the boundaries of double spectral regions, i.e.,

$$r(s) = 1 + \frac{2t_M(s)}{s-4}$$

$$t_M(s) = \begin{cases} \dfrac{16s}{s-4} & \text{for } 4 < s \leq 20 \\ \dfrac{4s}{s-16} & \phantom{\text{for }} 20 \leq s < \infty . \end{cases} \tag{3}$$

By supposing Mandelstam analyticity our results will not essentially deviate from those of the axiomatic framework. (The analyticity domain turns out to be slightly larger. Consequently the MRW relations will yield Roy-type equations being valid up to $s_{max} = 134$).

(iii) In the limit $s \to \infty$ and for fixed $t < 4$ the imaginary part of the amplitude is bounded by

$$|\mathrm{Im}\, F(s,t)| < c\, s^{1+\epsilon}, \quad \epsilon < 1 \tag{4}$$

(see ref. [11]).

3. We choose polynomially parametrized parabolic manifolds in the real (x,y) plane

$$x = \sum_{\nu=0}^{2} a_\nu \tau^\nu, \quad y = \sum_{\nu=0}^{2} b_\nu \tau^\nu, \tag{5}$$

where a_ν, b_ν are real parameters, and restrict them by the following conditions:

(a) The zeros τ_0 of the denominator of the Mandelstam representation have to be real for each $\alpha \geq 4$ (i.e., $x(a,\tau_0), y(b,\tau_0)$ are real).

(b) The curves are forbidden to cross those regions of the real (x, y) plane which are limited by the lines E_+ and E_-C_- (i.e., the points $(x(a,\tau_0), y(b,\tau_0))$ belong to the inner side of the Lehmann ellipse).

(c) The parameter set a_ν, b_ν varies in such a way that the points (x, y) for all values s of a certain interval $[s_{min}, s_{max}]$ cover the whole physical region $t \in [-\frac{s-4}{2}, 0]$.

According to conditions (a) and (b) the discontinuity of the amplitude can be expanded in terms of partial waves. With respect to condition (c) one can project the dispersion relation and gets Roy-type equations which hold for $s \in [s_{min}, s_{max}]$.

From condition (a) it can easily be seen that only those parabolic manifolds are allowed which intersect all straight lines $s = \text{const.} > 4$. Otherwise equation (2) would have complex roots for all $s = \text{const.} > 4$ which are not intersected by the parabola (5). If we take into account also condition (b) then $a_2 > 0, b_2 > -a_2$ necessary hold and the curve must intersect the line $y = \tilde{y}(x)$ (cf. appendix). Thus, we conclude that the Roy-type equations we shall finally get are valid up to $s_{max} = 134 m_\pi^2$ (as in the paper MRW [4]) for parabolas tending to straight lines.

Let us proceed now to the formulation of generalized dispersion relations. First, we clarify the necessary number of subtractions. From eq. (2) for $\alpha = s \geq 4$ we have two real roots corresponding to the two branches of the parabola

$$\tau = \tau_\pm(s,a,b) = \frac{1}{2(a_2 s + 4b_2)}\left(-(a_1 s + 4b_1) \pm \sqrt{R(s)}\right)$$
$$R(s) = (a_1 s + 4b_1)^2 - 4(a_0 s + 4b_0 - \frac{1}{16}s^2(s-4))(a_2 s + 4b_2). \qquad (6)$$

Using eqs. (5) and (6) we get the expression

$$t = t_\pm(s,a,b) = -\frac{s-4}{2}\left(1 - \sqrt{1 - \frac{4s}{s-4} + 64\frac{x_\pm(s,a,b)}{(s-4)^2}}\right). \qquad (7)$$

Therefore we have

$$\tau \xrightarrow[s\to\infty]{} \pm\frac{1}{4\sqrt{a_2}}s,$$
$$t \xrightarrow[s\to\infty]{} \text{const.} \leq 4. \qquad (8)$$

According to assumption (iii) it turns out that at most two subtractions are necessary.

It follows from eq. (2) there are in general four finite branch points in the complex τ plane corresponding to $\alpha = 4$ and $\alpha = \alpha_\pm$, where $\alpha_\pm$ are connected with the intersection of the parabola with the curve $y = \tilde{y}(x)$. For all branches the discontinuity $\frac{1}{2i}\Delta\hat{F}(x(a,\tau+i\bar{\epsilon}), y(b,\tau+i\bar{\epsilon}))$ may differ from $\operatorname{Im}F(s+i\epsilon, t_\pm(s+i\epsilon,a,b))$ only by the sign in accordance with the relation between the signs of the imaginary increments in $\tau \pm i\bar{\epsilon}$, $s \pm i\epsilon$ (use eq. (6)).

Considering this correspondence we get the representation

$$F(s,t_\pm(s,a,b)) = P(\tau_\pm(s,a,b) - \tau_0) + \frac{(\tau_\pm(s,a,b)-\tau_0)^2}{\pi}\int_4^\infty ds'\left[\frac{d\tau_-(s',a,b)}{ds'}\times\right.$$
$$\times\frac{\operatorname{Im}F(s',t_-(s',a,b))}{(\tau_-(s')-\tau_0)^2(\tau_-(s')-\tau_\pm(s))} + \qquad (9)$$
$$\left.+\frac{d\tau_+(s',a,b)}{ds'}\frac{\operatorname{Im}F(s',t_+(s',a,b))}{(\tau_+(s')-\tau_0)^2(\tau_+(s')-\tau_\pm(s))}\right],$$

where P represents a polynomial of first degree.

Concluding we remark that in the simplest case (conditions (a), (b)) the representation (9) is valid in the same region s as in the case of MRW. However, by leaving these restrictions on the manifolds (5), i.e. ,by studying the analytic properties in the complex (x,y) plane we shall be led to an extension of the validity domain obviously.

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APPENDIX

Mapping of the Mandelstam Plane s,t,u onto the Real (x,y) Plane

Each sector of the real Mandelstam plane bounded by the straight lines $u=t$, $s=t$ and $t=s$, $u=s$, etc., is mapped onto the same domain in the (x,y) plane as shown in the figure. The images of definite curves can be represented as follows.

$$s = \text{const}: \qquad x = \frac{1}{16}\, s(s-4) - 4\frac{y}{s} \qquad (*)$$

$$C_{\pm}: \qquad y = y_{\pm}(x) = \frac{1}{27}\left[1-3(3x+1) \pm 2(3x+1)^{3/2}\right], \quad x \geq -\frac{1}{3}$$

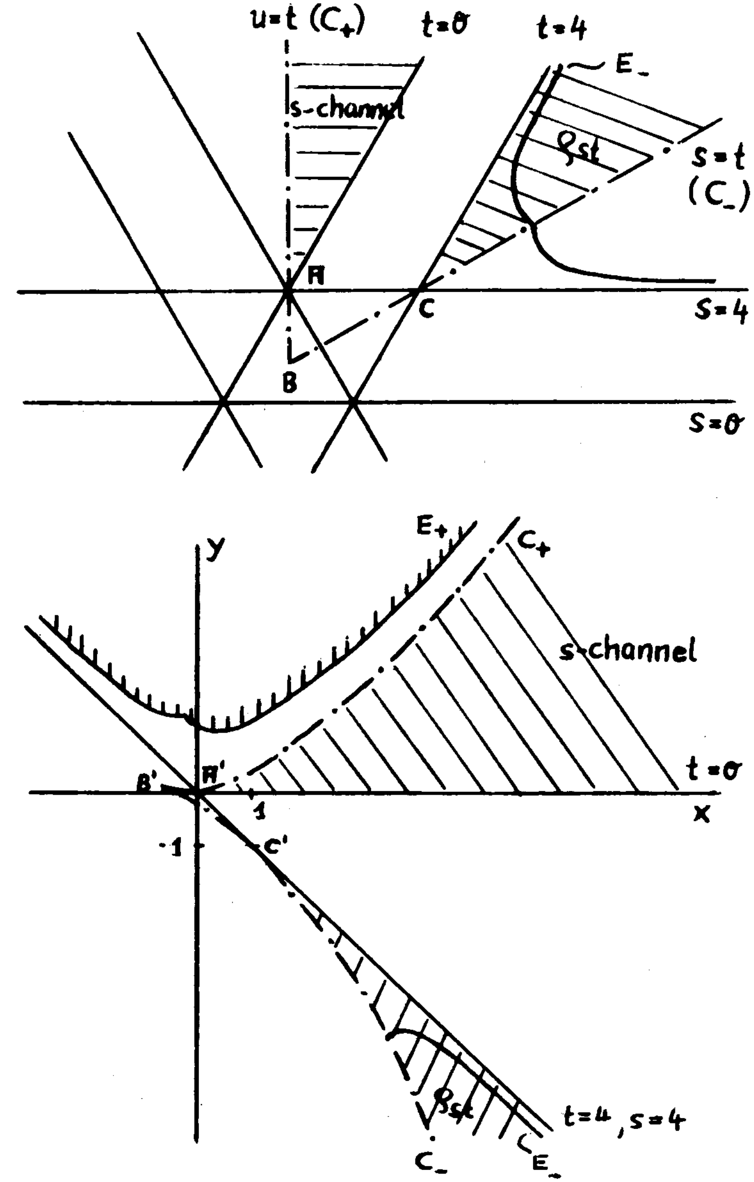
$$E_{-}: \qquad \left.\begin{aligned} y &= -\frac{1}{64}\, s\left(\frac{64}{s-16}+4\right)\left(\frac{64}{s-16}+s\right) \\ x &= \text{acc. to } (*) \end{aligned}\right\} \text{ for } s \geq 20$$

The real (x,y) plane additionally exhibits the image of the imaginary minor axis of the Lehmann ellipses.

$$E_{+}: \qquad \left.\begin{aligned} y &= \frac{1}{256}\, s\left[(s+12)^2 - 128\right]^2 (s-4)^{-2} \\ x &= \text{acc. to } (*) \end{aligned}\right\} \; 4 < s \leq 20$$

for

$$\left.\begin{aligned} y &= \frac{1}{256}\, s\left[(s-6)^2 + 28\right]^2 (s-16)^{-2} \\ x &= \text{acc. to } (*) \end{aligned}\right\} \; 20 \leq s < \infty$$



All straight lines $s = \text{const} \geq 4$ are tangents to C_-. Therefore, the region $s \geq 4$ is bounded by the curve

$$y = \tilde{y}(x) = \begin{cases} -x & x < 1 \\ & \text{for} \\ y_-(x) & x \geq 1 \,. \end{cases}$$

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